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BLOW-UP OF SOLUTIONS FOR A SYSTEM OF NONLINEAR WAVE EQUATIONS WITH NONLINEAR DAMPING

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Abstract. We study the initial-boundary value problem for a system of nonlinear wave equations, involving nonlinear damping terms, in a bounded domain Ω with the initial and Dirichlet boundary conditions. The nonexistence of global solutions is discussed under some conditions on the given parameters. Estimates on the lifespan of solutions are also given.

1. Introduction

In this article we shall consider the following initial-boundary value problem for a system of nonlinear wave equations:

$$\Box u + |u_t|^{p-1}u_t + m_1^2 u = 4\lambda(u+\alpha v)^3 + 2\beta uv^2 \quad \text{in } \Omega\times[0,T), \tag{1.1}$$

$$\Box v + |v_t|^{q-1}v_t + m_2^2 v = 4\alpha\lambda(u+\alpha v)^3 + 2\beta vu^2 \quad \text{in } \Omega\times[0,T), \tag{1.2}$$

with initial conditions

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega, \tag{1.3}$$

$$v(x,0)=v_0(x), \quad v_t(x,0)=v_1(x), \quad x\in\Omega, \tag{1.4}$$

and boundary conditions

$$u(x,t)=0, \quad x\in\partial\Omega,\ t>0, \tag{1.5}$$

$$v(x,t)=0, \quad x\in\partial\Omega,\ t>0, \tag{1.6}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \Delta$, $\Delta = \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2}$ and $\Omega\subset\mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$ so that Divergence theorem can be applied and λ,β and α are real numbers, and p, $q\geq 1$, $T>0$.

The initial-boundary value problem for a single wave equation:

$$u_{tt} - \Delta u(t) + a|u_t(t)|^{p-1}u_t(t) = f(u), \tag{1.7}$$

where $a>0$, $p\geq 1$, was considered by many authors. For $f(u)=|u|^{m-1}u$, $m>1$, this model was first studied by Levine [7, 8] in the linear case ($p=1$). He showed that solutions with negative initial energy blow up in finite time. When $p=1$, Ikehata [5] proved that for sufficiently small initial data, the trajectory $(u(t),v(t))$ goes to $(0,0)$ in $H_0^1(\Omega)\times L^2(\Omega)$ as $t\to\infty$. Georgiev and Todorova [2] extended

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Levine's result to nonlinear case ($p > 1$). They showed that solutions continue to exist globally if $p \geq m$ and blow up in finite time if $p < m$ with sufficiently negative initial energy, that is, in the L^∞−norm for suitable large initial data. Later, Ikehata [4] showed that (1.7) admits a global solution for sufficiently small initial data for $p > 1$. In unbounded domain, for $f(u) = -\lambda(x)^2 u + |u|^{m-1}u$, $m > 1$, here $\lambda(x)$ satisfies some decay conditions, there are some results about global existence and asymptotic behavior in [14]. Aassila [1] treated (1.7) for $f(u) = -u + |u|^{m-1}u$, $m > 1$, and gave the global existence and energy decay property. Reed [15] proposed this interesting problem of (1.1)-(1.6) without damping terms in (1.1) and (1.2). As a model it describes the interaction of scalar fields u, v of mass m_1, m_2 respectively. This system defines the motion of charged mesons in an electromagnetic field which was first introduced by Segal [16]. Later, Jörgens [6], Makhankov [11], and Medeiros and Menzala [12] studied such systems to find the existence of weak solutions of the mixed problem in a bounded domain. Further generalizations are also given in [12,13] by Galerkin method. Recently, the existence of global and nonglobal solutions of a system of semilinear wave equations without dissipative terms were discussed in [9, 10].

In this paper we are interested in the blow-up behavior of solutions for a system (1.1)-(1.6) in a bounded domain Ω in $\mathbb{R}^3$. This work improves an earlier work [10], in which similar results have been established for (1.1)-(1.6) in the absence of the damping terms. The paper is organized as follows. In section 2, we give some lemmas which will be used later, and we mention the local existence Theorem 2.4. In section 3, we first define an energy function $E(t)$ by (3.1) and show that it is a nonincreasing function of t. Then, we discuss the blow-up properties of (1.1)-(1.6) in two cases. In first case, $p = q = 1$, the main result is given in Theorem 3.4, which contains the estimates of upper bound of the blow-up time. In second case, $1 < p$, $q < 3$, the nonexistence of global solutions is given in Theorem 3.6. Moreover, estimates for the blow-up time T are also given.

2. Preliminary results

In this section, we will give some lemmas and the local existence result in Theorem 2.4.

Lemma 2.1 (Sobolev-Poincaré inequality). *If $2 \leq p \leq 6$, then*

$$\|u\|_p \leq C(\Omega, p)\|\nabla u\|_2,$$

for $u \in H_0^1(\Omega)$, where

$$C(\Omega, p) = \sup\Big\{\frac{\|u\|_p}{\|\nabla u\|_2} : u \in H_0^1(\Omega), u \neq 0\Big\},$$

and $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$.

Lemma 2.2 ([9]). *Let $\delta > 0$ and $B(t) \in C^2(0, \infty)$ be a nonnegative function satisfying*

$$B''(t) - 4(\delta + 1)B'(t) + 4(\delta + 1)B(t) \geq 0. \tag{2.1}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{2.2}$$

with $r_2 = 2(\delta+1) - 2\sqrt{(\delta + 1)\delta}$, then $B'(t) > K_0$ for $t > 0$, where K_0 is a constant.

Lemma 2.3 ([9]). *If $J(t)$ is a nonincreasing function on $[t_0, \infty)$ and satisfies the differential inequality*

$$J'(t)^2 \geq a + bJ(t)^{2+\frac{1}{\delta}}, \quad \text{for } t \geq t_0, \tag{2.3}$$

where $a > 0, b \in \mathbb{R}$, then there exists a finite time T^* such that

$$\lim_{t \to T^{*-}} J(t) = 0\,.$$

Upper bounds for T^* are estimated as follows:

(i) *If $b < 0$, then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-a/b}}{\sqrt{-a/b} - J(t_0)}.$$

(ii) *If $b = 0$, then*

$$T^* \leq t_0 + \frac{J(t_0)}{J'(t_0)}.$$

(iii) *If $b > 0$, then*

$$T^* \leq \frac{J(t_0)}{\sqrt{a}} \quad \text{or} \quad T^* \leq t_0 + 2^{(3\delta+1)/(2\delta)} \frac{\delta c}{\sqrt{a}} \{1 - [1 + cJ(t_0)]^{-1/(2\delta)}\},$$

where $c = (\frac{a}{b})^{2+\frac{1}{\delta}}$.

Now, we state the local existence result which is proved in [19].

Theorem 2.4 (Local solution). *Let $p,\ q \geq 1$, and $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, then there exists a unique local solution (u, v) of (1.1)-(1.6) satisfying $(u, v) \in Y_T$, where*

$$\begin{aligned} Y_T = \big\{ & w = (u, v) : w \in C([0, T]; H_0^1(\Omega) \times H_0^1(\Omega)),\ w_t \in C([0, T]; L^2(\Omega) \times L^2(\Omega)), \\ & u_t \in L^{p+1}(\Omega \times (0, T)),\ v_t \in L^{q+1}(\Omega \times (0, T)) \big\}. \end{aligned}$$

3. Blow-up property

In this section, we will discuss the blow up phenomena of two problems, where $p = q = 1$ in subsection 3.1 and $1 < p, q < 3$ in subsection 3.2. Let (u, v) be a solution of (1.1)-(1.6), we define the energy functional

$$\begin{aligned} E(t) = \frac{1}{2} \int_\Omega [&|\nabla u|^2 + |\nabla v|^2 + u_t^2 + v_t^2 + m_1^2 u^2 + m_2^2 v^2 \\ &- 2\lambda(u + \alpha v)^4 - 2\beta u^2 v^2]\, dx, \quad \text{for } t \geq 0. \end{aligned} \tag{3.1}$$

Lemma 3.1. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$\frac{d}{dt} E(t) = -\|u_t\|_{p+1}^{p+1} - \|v_t\|_{q+1}^{q+1}. \tag{3.2}$$

Proof. Multiplying (1.1) by u_t and (1.2) by v_t, and integrating them over Ω. Then, adding them together, and integrating by parts, we obtain

$$E(t) - E(0) = -\int_0^t (\|u_t\|_{p+1}^{p+1} + \|v_t\|_{q+1}^{q+1}) dt \quad \text{for } t \geq 0.$$

Being the primitive of an integrable function, $E(t)$ is absolutely continuous and equality (3.2) is satisfied. ∎

3.1. **Case** $p = q = 1$**.** In this subsection we consider (1.1),(1.2) with $p = q = 1$:

$$\Box u + u_t + m_1^2 u = 4\lambda(u + \alpha v)^3 + 2\beta u v^2 \quad \text{in } \Omega \times [0, T), \tag{3.3}$$

$$\Box v + v_t + m_2^2 v = 4\alpha\lambda(u + \alpha v)^3 + 2\beta v u^2 \quad \text{in } \Omega \times [0, T). \tag{3.4}$$

Assumption:

(A1) $m_1^2\xi^2 + m_2^2\eta^2 - 2\lambda(\xi + \alpha\eta)^4 - 2\beta\xi^2\eta^2 < 0$, for all $\xi, \eta \in \mathbb{R}$.

Definition: A solution $w(t) = (u(t), v(t))$ of (3.3), (3.4), and (1.3)-(1.6) is called blow-up if there exists a finite time T^* such that

$$\lim_{t \to T^{*-}} \Big\{ \int_\Omega (u^2 + v^2)\, dx + \int_0^t (\|u\|_2^2 + \|v\|_2^2) dt \Big\} = \infty.$$

Let

$$a(t) = \int_\Omega (u^2 + v^2)\, dx + \int_0^t \int_\Omega (u^2 + v^2)\, dx ds, \quad \text{for } t \geq 0. \tag{3.5}$$

Lemma 3.2. *Assume* (A1)*, and that* $0 < \delta \leq 1/2$*, then we have*

$$\begin{aligned} &a''(t) - 4(\delta + 1) \int_\Omega (u_t^2 + v_t^2)\, dx \\ &\geq (-4 - 8\delta)E(0) + (4 + 8\delta) \int_0^t (\|u_t\|_2^2 + \|v_t\|_2^2) dt. \end{aligned} \tag{3.6}$$

Proof. Form (3.5), we have

$$a'(t) = 2\int_\Omega (u u_t + v v_t)\, dx + \|u\|_2^2 + \|v\|_2^2. \tag{3.7}$$

By (3.3), (3.4) and Divergence theorem, we get

$$\begin{aligned} a''(t) = 2\int_\Omega (u_t^2 + v_t^2)\, dx &- 2(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|m_1 u\|_2^2 + \|m_2 v\|_2^2) \\ &+ 8\lambda\|u + \alpha v\|_4^4 + 8\beta\|uv\|_2^2. \end{aligned} \tag{3.8}$$

By (3.2), we have from (3.8)

$$\begin{aligned} &a''(t) - 4(\delta + 1) \int_\Omega (u_t^2 + v_t^2)\, dx \\ &= (-4 - 8\delta)E(0) + (4 + 8\delta) \int_0^t (\|u_t\|_2^2 + \|v_t\|_2^2) ds \\ &\quad + [4\delta(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + 2(\|m_1 u\|_2^2 + \|m_2 v\|_2^2)] \\ &\quad + (4\delta - 2)[\|m_1 u\|_2^2 + \|m_2 v\|_2^2 - 2\lambda\|u + \alpha v\|_4^4 - 2\beta\|uv\|_2^2]. \end{aligned}$$

Therefore, from (A1), we obtain (3.6). ∎

We remark that (A1) is automatically true if $E(0) \leq 0$. Now, we consider three different cases on the sign of the initial energy $E(0)$.

(1) If $E(0) < 0$, then from (3.6), we have

$$a'(t) \geq a'(0) - 4(1 + 2\delta)E(0)t, \quad t \geq 0.$$

Thus we get $a'(t) > \|u_0\|_2^2 + \|v_0\|_2^2$ for $t > t^*$, where

$$t^* = \max\Big\{ \frac{a'(0) - (\|u_0\|_2^2 + \|v_0\|_2^2)}{4(1 + 2\delta)E(0)}, 0 \Big\}. \tag{3.9}$$

(2) If $E(0)=0$, then $a''(t)\geq 0$ for $t\geq 0$. If $a'(0)>\|u_0\|_2^2+\|v_0\|_2^2$, then we have $a'(t)>\|u_0\|_2^2+\|v_0\|_2^2$, $t\geq 0$.
(3) For the case that $E(0)>0$, we first note that

$$2\int_0^t\int_\Omega uu_t\,dx\,dt=\|u\|_2^2-\|u_0\|_2^2. \tag{3.10}$$

By Hölder inequality and Young's inequality, we have from (3.10),

$$\|u\|_2^2\leq\|u_0\|_2^2+\int_0^t\|u\|_2^2dt+\int_0^t\|u_t\|_2^2dt. \tag{3.11}$$

Similarly,

$$\|v\|_2^2\leq\|v_0\|_2^2+\int_0^t\|v\|_2^2dt+\int_0^t\|v_t\|_2^2dt. \tag{3.12}$$

By Hölder inequality, Young′s inequality and then using (3.11) and (3.12), we have from (3.7),

$$a'(t)\leq a(t)+\|u_0\|_2^2+\|v_0\|_2^2+\int_\Omega(u_t^2+v_t^2)\,dx+\int_0^t(\|u_t\|_2^2+\|v_t\|_2^2)dt. \tag{3.13}$$

Hence by (3.6) and (3.12), we obtain

$$a''(t)-4(\delta+1)a'(t)+4(\delta+1)a(t)+K_1\geq 0,$$

where

$$K_1=(4+8\delta)E(0)+4(\delta+1)(\|u_0\|_2^2+\|v_0\|_2^2)\,.$$

Let

$$b(t)=a(t)+\frac{K_1}{4(1+\delta)},\quad t>0.$$

Then $b(t)$ satisfies (2.1). By Lemma 2.3 we see that if

$$a'(0)>r_2\big[a(0)+\frac{K_1}{4(1+\delta)}\big]+(\|u_0\|_2^2+\|v_0\|_2^2), \tag{3.14}$$

then $a'(t)>(\|u_0\|_2^2+\|v_0\|_2^2)$, $t>0$, where r_2 is given in Lemma 2.2. Consequently, we have the following result.

Lemma 3.3. *Assume* (A1) *and that either one of the following statements is satisfied:*

(i) $E(0)<0$,
(ii) $E(0)=0$ *and* $a'(0)>\|u_0\|_2^2+\|v_0\|_2^2$,
(iii) $E(0)>0$ *and* (3.14) *holds.*

Then, $a'(t)>\|u_0\|_2^2+\|v_0\|_2^2$ *for* $t>t_0$*, where* $t_0=t^*$ *is given by* (3.9) *in case (i) and* $t_0=0$ *in cases (ii) and (iii).*

Now, we find an estimate for the life span of $a(t)$. Let

$$J(t)=\big[a(t)+(T_1-t)(\|u_0\|_2^2+\|v_0\|_2^2)\big]^{-\delta},\quad\text{for } t\in[0,T_1], \tag{3.15}$$

where $T_1>0$ is a certain constant which will be specified later. Then we have

$$\begin{gathered}J'(t)=-\delta J(t)^{1+\frac{1}{\delta}}(a'(t)-\|u_0\|_2^2-\|v_0\|_2^2),\\ J''(t)=-\delta J(t)^{1+\frac{2}{\delta}}V(t),\end{gathered} \tag{3.16}$$

where

$$\begin{aligned} V(t) &= a''(t)\left[a(t)+(T_1-t)(\|u_0\|_2^2+\|v_0\|_2^2)\right] \\ &\quad -(1+\delta)(a'(t)-\|u_0\|_2^2-\|v_0\|_2^2)^2. \end{aligned} \tag{3.17}$$

For simplicity of calculation, we denote

$$\begin{gathered} P_u=\int_\Omega u^2\,dx,\quad P_v=\int_\Omega v^2\,dx,\\ Q_u=\int_0^t \|u\|_2^2dt,\quad Q_v=\int_0^t \|v\|_2^2dt,\\ R_u=\int_\Omega u_t^2\,dx,\quad R_v=\int_\Omega v_t^2\,dx,\\ S_u=\int_0^t \|u_t\|_2^2dt,\quad S_v=\int_0^t \|v_t\|_2^2dt. \end{gathered}$$

From (3.7), (3.10), and Hölder inequality, we get

$$\begin{aligned} a'(t) &= 2\int_\Omega (uu_t+vv_t)\,dx+\|u_0\|_2^2+\|v_0\|_2^2+2\int_0^t\int_\Omega (uu_t+vv_t)\,dx\,dt \\ &\le 2(\sqrt{R_uP_u}+\sqrt{Q_uS_u}+\sqrt{R_vP_v}+\sqrt{Q_vS_v})+\|u_0\|_2^2+\|v_0\|_2^2. \end{aligned} \tag{3.18}$$

By (3.6), we have

$$a''(t)\ge(-4-8\delta)E(0)+4(1+\delta)(R_u+S_u+R_v+S_v). \tag{3.19}$$

Thus, from (3.18), (3.19), (3.17) and (3.15), we obtain

$$\begin{aligned} V(t) &\ge \left[(-4-8\delta)E(0)+4(1+\delta)(R_u+S_u+R_v+S_v)\right]J(t)^{-1/\delta} \\ &\quad -4(1+\delta)(\sqrt{R_uP_u}+\sqrt{Q_uS_u}+\sqrt{R_vP_v}+\sqrt{Q_vS_v})^2. \end{aligned}$$

And by (3.15) and (3.5), we have

$$\begin{aligned} V(t) &\ge (-4-8\delta)E(0)J(t)^{-1/\delta} \\ &\quad +4(1+\delta)[(R_u+S_u+R_v+S_v)(T_1-t)(\|u_0\|_2^2+\|v_0\|_2^2)+\Theta(t)], \end{aligned}$$

where

$$\begin{aligned} \Theta(t) &= (R_u+S_u+R_v+S_v)(P_u+Q_u+P_v+Q_v) \\ &\quad -(\sqrt{R_uP_u}+\sqrt{Q_uS_u}+\sqrt{R_vP_v}+\sqrt{Q_vS_v})^2. \end{aligned}$$

By Schwarz inequality, $\Theta(t)$ is nonnegative. Hence, we have

$$V(t)\ge(-4-8\delta)E(0)J(t)^{-1/\delta},\quad t\ge t_0. \tag{3.20}$$

Therefore, by (3.16) and (3.20), we get

$$J''^{1+\frac{1}{\delta}},\quad t\ge t_0. \tag{3.21}$$

Note that by Lemma 3.3, $J'(t)<0$ for $t>t_0$. Multiplying (3.21) by $J'(t)$ and integrating it from t_0 to t, we get

$$J'^2\ge\alpha+\beta J(t)^{2+\frac{1}{\delta}}\quad\text{for } t\ge t_0,$$

where

$$\alpha=\delta^2J(t_0)^{2+\frac{2}{\delta}}\left[(a'(t_0)-\|u_0\|_2^2-\|v_0\|_2^2)^2-8E(0)J(t_0)^{\frac{-1}{\delta}}\right], \tag{3.22}$$

$$\beta=8\delta^2E(0). \tag{3.23}$$

We observe that

$$\alpha > 0 \quad \text{if and only if} \quad E(0) < \frac{(a'(t_0) - \|u_0\|_2^2 - \|v_0\|_2^2)^2}{8\big[a(t_0) + (T_1 - t_0)(\|u_0\|_2^2 + \|v_0\|_2^2)\big]}.$$

Then by Lemma 2.3, there exists a finite time T^* such that $\lim_{t\to T^{*-}} J(t) = 0$ and the upper bound of T^* is estimated respectively according to the sign of $E(0)$. This means that

$$\lim_{t\to T^{*-}} \Big\{ \int_\Omega (u^2 + v^2)\,dx + \int_0^t (\|u\|_2^2 + \|v\|_2^2)dt \Big\} = \infty. \tag{3.24}$$

Theorem 3.4. *Assume that* (A1) *and that either one of the following statements is satisfied:*

(1) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $a'(0) > (\|u_0\|_2^2 + \|v_0\|_2^2)$
(iii) $0 < E(0) < \frac{(a'(t_0) - \|u_0\|_2^2 - \|v_0\|_2^2)^2}{8\big[a(t_0) + (T_1 - t_0)(\|u_0\|_2^2 + \|v_0\|_2^2)\big]}$ *and* (3.14) *holds.*

Then the solution $(u(t), v(t))$ *blows up at finite time* T^* *in the sense of* (3.24)*. In case (i),*

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\{1, \sqrt{-\alpha/\beta}\}$*, we have*

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{-\alpha/\beta}}{\sqrt{-\alpha/\beta} - J(t_0)}.$$

In case (ii),

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)} \quad \text{or} \quad T^* \le t_0 + \frac{J(t_0)}{\sqrt{\alpha}}.$$

In case (iii),

$$T^* \le \frac{J(t_0)}{\sqrt{\alpha}} \quad \text{or} \quad T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha}} \big\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\big\},$$

where $c = (\alpha/\beta)^{2+\frac{1}{\delta}}$*, here* α *and* β *are given in* (3.22), (3.23)*. Note that in case (i),* $t_0 = t^*$ *is given in* (3.9) *and* $t_0 = 0$ *in case (ii) and (iii).*

We remark that the choice of T_1 in (3.15) is possible under some conditions as in [17, 18].

3.2. **Case $1 < p$, $q < 3$.** In this subsection we consider (1.1), (1.2) with $1 < p$, $q < 3$:

$$\Box u + |u_t|^{p-1}u_t + m_1^2 u = 4\lambda(u + \alpha v)^3 + 2\beta uv^2 \quad \text{in } \Omega \times [0, T),$$
$$\Box v + |v_t|^{q-1}v_t + m_2^2 v = 4\alpha\lambda(u + \alpha v)^3 + 2\beta vu^2 \quad \text{in } \Omega \times [0, T).$$

Definition: A solution (u, v) of (1.1)-(1.6) is called blow-up if there exists a finite time $T > 0$ such that

$$\lim_{t\to T^-} \Big[\int_\Omega (|\nabla u|^2 + |\nabla v|^2)\,dx \Big] = \infty.$$

Lemma 3.5. *For all* $\lambda > 1$, $\alpha \ne 0$*, there exists* $\beta > 0$ *such that*

$$\xi^4 + \alpha^4\eta^4 \le \lambda(\xi + \alpha\eta)^4 + \beta\xi^2\eta^2, \quad \text{for all } \xi, \eta \in \mathbb{R}. \tag{3.25}$$

Proof. If $\eta = 0$, then (3.25) is true for $\lambda > 1$, $\xi \in \mathbb{R}$. Now, let $x = \frac{\xi}{\eta}$, where $\xi, \eta \in \mathbb{R}$, and $\eta \neq 0$. Then to show (3.25) is equivalent to claim that for all $\lambda > 1, \alpha \neq 0$, there exists $\beta > 0$ such that $h(x) \leq \beta x^2$, here $h(x) = x^4 + \alpha^4 - \lambda(x+\alpha)^4$, $x \in \mathbb{R}$. Since $h(x)$ is a continuous function, $h(0) = \alpha^4 - \lambda\alpha^4 < 0$, and $h(\pm\infty) = -\infty$, there exists a finite number M such that $M = \sup_{x\in\mathbb{R}} h(x)$. If $M \leq 0$, we could choose any $\beta > 0$. If $M > 0$, since $h(0) < 0$, there exists $\delta > 0$ such that $h(x) < 0$ for $|x| < \delta$. Thus, we could choose $\beta = M$ in this interval. For $|x| \geq \delta$, $h(x) \leq M = \frac{M}{\delta^2}\delta^2 \leq \frac{M}{\delta^2}x^2$. Therefore, from above discussion, we can take $\beta = \max\{\frac{M}{\delta^2}, M\}$, and we have $h(x) \leq \beta x^2$, for $x \in \mathbb{R}$. □

Theorem 3.6 (Nonexistence of global solutions)**.** *If $1 < p$, $q < 3$, $E(0) < 0$ and (3.25) holds, then the solutions of (1.1)-(1.6) blow up at a finite time T, $0 < T \leq \frac{z(0)^{1-r}}{c_7(1-r)}$, where $z(0) = k_1(-E(0))^{1-\alpha_1} + \int_\Omega (u_1u_0 + v_1v_0)\,dx$, here k_1, α_1, and r are certain positive constants given in the proof, and c_7 is given in (3.43).*

Proof. Let
$$a(t) = \frac{1}{2}\int_\Omega (u^2 + v^2)\,dx, \quad \text{for } t \geq 0. \tag{3.26}$$
By differentiating, we obtain
$$a'(t) = \int_\Omega (u_tu + v_tv)\,dx,$$
$$a''(t) = \int_\Omega (u_t^2 + u_{tt}u + v_t^2 + v_{tt}v)\,dx, \quad \text{for } t \geq 0.$$
By using (1.1), (1.2) and (3.2), we obtain
$$\begin{aligned} a''(t) &= 2\int_\Omega (u_t^2 + v_t^2)\,dx - 2E(t) + 2B(t) \\ &\quad - \int_\Omega |u_t|^{p-1}u_tu\,dx - \int_\Omega |v_t|^{q-1}v_tv\,dx, \end{aligned} \tag{3.27}$$
where
$$B(t) = \lambda\|u + \alpha v\|_4^4 + \beta\|uv\|_2^2. \tag{3.28}$$
By Hölder inequality, we observe that
$$|\int_\Omega |u_t|^{p-1}u_tu\,dx| \leq |\Omega|^{\frac{3-p}{4(p+1)}}\|u_t\|_{p+1}^p\|u\|_4.$$
Then from (3.25),
$$|\int_\Omega |u_t|^{p-1}u_tu\,dx| \leq |\Omega|^{\frac{3-p}{4(p+1)}}\|u_t\|_{p+1}^p B(t)^{\frac{1}{4}}. \tag{3.29}$$
Noting that from (3.2) and (3.1), we have
$$B(t) \geq -E(t) \geq -E(0) > 0. \tag{3.30}$$
Thus, from (3.29), (3.30), and $1 < p < 3$, we obtain
$$\big|\int_\Omega |u_t|^{p-1}u_tu\,dx\big| \leq \|u_t\|_{p+1}^p|\Omega|^{\frac{3-p}{4(p+1)}}B(t)^{\frac{1}{p+1}}(-E(t))^{\frac{1}{4}-\frac{1}{p+1}}. \tag{3.31}$$
Then, by Young's inequality,
$$\big|\int_\Omega |u_t|^{p-1}u_tu\,dx\big| \leq \Big[\varepsilon_1^{p+1}B(t) + c(\varepsilon_1)^{-\frac{p+1}{p}}|\Omega|^{\frac{3-p}{4p}}\|u_t\|_{p+1}^{p+1}\Big](-E(t))^{\frac{1}{4}-\frac{1}{p+1}}, \tag{3.32}$$

here ε_1 is a positive constant to be specified later. Letting $0 < \alpha_1 < \min\{\frac{1}{p+1} - \frac{1}{4}, \frac{1}{q+1} - \frac{1}{4}\}$, and by (3.32) and (3.30), we have

$$\begin{aligned} |\int_\Omega |u_t|^{p-1}u_t u\,dx| &\le c(\varepsilon_1)^{-\frac{p+1}{p}}|\Omega|^{\frac{3-p}{4p}}(-E(0))^{\alpha_1+\frac14-\frac{1}{p+1}}(-E(t))^{-\alpha_1}(-E'(t)) \\ &\quad + \varepsilon_1^{p+1}B(t)(-E(0))^{\frac14-\frac{1}{p+1}}. \end{aligned} \tag{3.33}$$

In the same way, we have

$$\begin{aligned} \Big|\int_\Omega |v_t|^{q-1}v_t v\,dx\Big| &\le c(\varepsilon_2)^{-\frac{q+1}{q}}|\Omega|^{\frac{3-q}{4q}}(-E(0))^{\alpha_1+\frac14-\frac{1}{q+1}}(-E(t))^{-\alpha_1}(-E'(t)) \\ &\quad + \varepsilon_2^{q+1}B(t)(-E(0))^{\frac14-\frac{1}{q+1}}, \end{aligned} \tag{3.34}$$

here ε_2 is a positive constant. Now, we define

$$Z(t) = k_1(-E(t))^{1-\alpha_1} + a'(t), \quad t \ge 0, \tag{3.35}$$

where $k_1 > -a''^{1-\alpha_1}$ is a positive number to be chosen later. From (3.35), we see

$$Z'(t) = k_6(1-\alpha_1)(-E(t))^{-\alpha_1}(-E'(t)) + a''(t), \quad t \ge 0.$$

By (3.27), (3.33) and (3.34), we get

$$\begin{aligned} Z'(t) &\ge \mu(-E(t))^{-\alpha_1}(-E'(t)) + (-2E(t)) + 2\int_\Omega (u_t^2 + v_t^2)\,dx \\ &\quad + \big[2 - \varepsilon_1^{p+1}(-E(0))^{\frac14-\frac{1}{p+1}} - \varepsilon_2^{q+1}(-E(0))^{\frac14-\frac{1}{q+1}}\big]B(t), \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} \mu &= k_1(1-\alpha_1) - c(\varepsilon_1)^{-\frac{p+1}{p}}|\Omega|^{\frac{3-p}{4p}}(-E(0))^{\alpha_1+\frac14-\frac{1}{p+1}} \\ &\quad - c(\varepsilon_2)^{-\frac{q+1}{q}}|\Omega|^{\frac{3-q}{4q}}(-E(0))^{\alpha_1+\frac14-\frac{1}{q+1}}. \end{aligned}$$

We choose

$$\varepsilon_1^{p+1} = \frac12(-E(0))^{\frac{1}{p+1}-\frac14}, \quad \varepsilon_2^{q+1} = \frac12(-E(0))^{\frac{1}{q+1}-\frac14},$$

and k_1 is sufficiently large such that $\mu > 0$ and $Z(0) > 0$. Then (3.36) becomes

$$Z'(t) \ge \big[-2E(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + B(t)\big]. \tag{3.37}$$

Hence $Z(t) > 0$ for $t \ge 0$. Note that $r = 1/(1-\alpha_1) > 1$, from (3.35), and by Young's inequality and Hölder inequality, it follows that

$$\begin{aligned} Z(t)^r &\le 2^{2(r-1)}\Big[k_1^r(-E(t)) + \Big|\int_\Omega u_t u\,dx\Big|^r + \Big|\int_\Omega v_t v\,dx\Big|^r\Big] \\ &\le 2^{2(r-1)}[k_1^r(-E(t)) + \|u_t\|_2^r\|u\|_2^r + \|v_t\|_2^r\|v\|_2^r]. \end{aligned} \tag{3.38}$$

On the other hand, using Hölder inequality, we have

$$\|u_t\|_2^r\|u\|_2^r \le c_1\|u_t\|_2^r\|u\|_4^r,$$

here $c_1 = |\Omega|^{r/4}$. And by Young's inequality, we obtain

$$\|u_t\|_2^r\|u\|_2^r \le c_2(\|u_t\|_2^{r\beta_1} + \|u\|_4^{r\beta_2}), \tag{3.39}$$

where $\frac{1}{\beta_1}+\frac{1}{\beta_2}=1$, $c_2=c_2(c_1,\,\beta_1,\,\beta_2)>0$. In particular, we take $r\beta_1=2$; that is, $\beta_1=2(1-\alpha_1)$. Therefore, for α_1 small enough, the numbers β_1 and β_2 are close to 2. For $0<\alpha_1<\min\{\frac{1}{p+1}-\frac{1}{4},\frac{1}{q+1}-\frac{1}{4}\}$, by (3.25) and (3.30), we have

$$\begin{aligned}\|u\|_4^{r\beta_2}\big(\|u\|_4^4\big)^{r\beta_2/4} &\le B(t)^{r\beta_2/4}\\ &= (\frac{1}{-E(0)}B(t))^{r\beta_2/4}(-E(0))^{r\beta_2/4}\\ &\le c_3B(t)\end{aligned}$$

because

$$r\beta_2=\frac{2}{1-2\alpha_1}<4,$$

where $c_3=(-E(0))^{\frac{r\beta_2}{4}-1}$. Then, by (3.25), we obtain

$$\|u_t\|_2^r\|u\|_2^r\le c_4\big(\|u_t\|_2^2+B(t)\big). \tag{3.40}$$

Similarly, we also get

$$\|v_t\|_2^r\|v\|_2^r\le c_5\big(\|v_t\|_2^2+B(t)\big), \tag{3.41}$$

here $c_4=c_2\max(1,\ c_3)$, and c_5 is some positive constant. Then, from (3.38), (3.40) and (3.41), we deduce that

$$Z(t)^r\le 2^{2(r-1)}c_6[-2E(t)+\|u_t\|_2^2+\|v_t\|_2^2+B(t)], \tag{3.42}$$

where $c_6=\max\{\frac{k_1^r}{2},c_4+c_5\}$. Therefore, by (3.37) and (3.42), we have

$$Z'(t)\ge c_7Z(t)^r, \tag{3.43}$$

$c_7=\frac{1}{2^{2(r-1)}c_6}$. A simple integration of (3.43) over $(0,t)$ yields

$$Z(t)\ge\big(Z(0)^{1-r}-c_7(r-1)t\big)^{-\frac{1}{\alpha_1-1}}. \tag{3.44}$$

Since $Z(0)>0$, (3.44) shows that Z becomes infinite in a finite time $T\le\frac{Z(0)^{1-r}}{c_7(r-1)}$. From (3.1), we have

$$-2E(t)+\|u_t\|_2^2+\|v_t\|_2^2\le 2B(t). \tag{3.45}$$

Thus, by (3.37) and (3.45), we get

$$Z(t)^r\le 3B(t). \tag{3.46}$$

By Poincaré inequality and Hölder inequality, we have

$$B(t)\le c_8(\|\nabla u\|_2+\|\nabla v\|_2)^4, \tag{3.47}$$

$c_8=c_8(\alpha,\beta,\Omega)>0$. Hence, from (3.46) and (3.47), we obtain

$$Z(t)^r\le 3c_8(\|\nabla u\|_2+\|\nabla v\|_2)^4.$$

Therefore, the proof is complete. □

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