Electronic Journal of Differential Equations, Vol. 2009(2009), No. 103, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND MULTIPLICITY RESULTS FOR SINGULAR ϕ-LAPLACIAN BVPS ON THE POSITIVE HALF-LINE

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Abstract. This work proves the existence and multiplicity of positive solutions for a second-order nonlinear three-point ϕ-Laplacian boundary-value problem posed on the positive half-line. The nonlinearity depends on the solution and its derivative and may exhibit a time singularity at the origin. Existence of single and multiple nontrivial positive solutions is proved using fixed point index theory.

1. Introduction

This paper concerns the existence of positive solutions to the following three-point ϕ-Laplacian boundary value problem posed on the positive half-line:

$$\begin{gathered} -(\phi(y'))'(t) = m(t) f(t, y(t), y'(t)), \quad t \in I \\ y(0) = \alpha y'(\eta), \quad \lim_{t \to +\infty} y'(t) = 0, \end{gathered} \tag{1.1}$$

where $\alpha \geq 0$ and $\eta \in (0, \infty)$ are given real numbers. The interval $I := (0, +\infty)$ denotes the set of positive real numbers and $\mathbb{R}^+ := [0, +\infty)$. The function $f : I \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous; the function $m : I \to \mathbb{R}^+$ is continuous but is allowed to have a singularity at $t = 0$. ϕ is a nonlinear operator of derivation generalizing the p-Laplacian operator.

Boundary value problems on the half-line arise in many applications of physical phenomena and in chemistry and biology. A general survey of existence theory is well developed in the books by Agarwal et al [2, 3]. In case of second-order differential equations corresponding to $\phi = I_d$, Problem (1.1) has been extensively studied in the literature. Using the theory of fixed point index on cones of Banach spaces, the authors obtained in [8, 9] some existence results for the generalized Fisher equation $-y'' + cy' + \lambda y = f(t, y(t), y'(t))$ $(c, \lambda > 0)$ subject to Dirichlet or Neumann limit condition at positive infinity; see also [23] for the case of a multi-point condition at the origin. Indeed, since the pioneer works of Gupta [15, 16, 25], much attention has been devoted to three-point and more generally to multi-point boundary value problems (see [12, 18]). When the derivative operator is generalized

2000 *Mathematics Subject Classification.* 34B10, 34B15, 34B16, 34B18, 34B40.
Key words and phrases. ϕ-Laplacian; three point BVPs; positive solutions; singularity; fixed point in a cone.

Submitted May 14, 2009. Published August 27, 2009.

to $(q(t)x'(t))'$ and f depends on the first derivative, the Mawhin coincïdence theory is applied in [20] to get the existence of at least one solution.

However, some interesting recent works have been developed for the case of the so-called p-Laplacian operator $\phi_p(s) = |s|^{p-1}s$ $(p > 1)$. While the theory is well developed for p-Laplacian problems on bounded intervals (see e.g., [4, 5] and the references therein), less results are known for BVPs posed on infinite intervals. By means of the three-functional fixed point theorem, existence of three positive solutions are obtained in [13] when the nonlinearity depends on the first derivative and in [19] when it does not; some local growth conditions are assumed on the nonlinearity. Using the upper and lower solution technique, existence results of single and double solutions are obtained in [10]. The nonlinear alternative of Leray and Schauder has been recently employed in [24] to prove existence of positive solutions for a multi-point boundary-value problem associated with the equation

$$\big(\rho(t)|x'(t)^{p-2}|x'(t)\big)' + f(t, x(t), x'(t)) = 0$$

on $(0, +\infty)$ when $\rho \in C[0, +\infty) \cap C^1(0, +\infty)$ is positive and satisfies

$$\int_0^{\infty} \phi_p^{-1}(1/\rho(t))dt < \infty.$$

In this work, we assume that the nonlinear map $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ and

$$|\phi^{-1}(x)| \le \phi^{-1}(|x|), \quad \forall x \in \mathbb{R}. \tag{1.2}$$

In the second part of this paper, we further assume that ϕ is sub-multiplicative; i.e.,

$$\forall\, \alpha, \beta \in \mathbb{R}^+, \quad \phi(\alpha\beta) \le \phi(\alpha)\phi(\beta). \tag{1.3}$$

Note that if ϕ is sub-multiplicative, then the converse ϕ^{-1} is super-multiplicative, that is

$$\forall\, \alpha, \beta \in \mathbb{R}^+, \quad \phi^{-1}(\alpha\beta) \ge \phi^{-1}(\alpha)\phi^{-1}(\beta). \tag{1.4}$$

Clearly, ϕ is an extension of the usual p-Laplacian nonlinear operator which is sub-multiplicative and super-multiplicative, hence multiplicative.

Finally, recall that various physiological processes are modelled by singular differential equations. For instance, the electrical potential in an isolated neutral atom is governed by the following problem derived in 1927 by L.H. Thomas [22] and Fermi [11],

$$\begin{gathered} y'' = \sqrt{y^3/t} \\ y(0) = 1, \quad y(+\infty) = 0. \end{gathered}$$

A more general survey on singular boundary value problems can be found in [21]. This is our main motivation of considering the case when the factor m is time-singular. Our objective is then to prove some existence results of nontrivial positive solutions for Problem (1.1) under suitable conditions on the functions f and m. Throughout, by a solution we mean a positive solution $y \in C^1[0, +\infty)$ such that $\phi(y') \in C^1(0, +\infty)$ with $y(t) \ge 0$ on $[0, +\infty)$ and the equation in (1.1) is satisfied. Some preliminaries including the main assumptions, the problem transformation and a compactness criterion are gathered together in Section 2. Section 3 is devoted to proving two existence theorems, one of a single positive solution and the other one of three positive solutions. An example of application with a nonlinear operator of derivation ends this paper in Section 4.

2. Preliminaries

2.1. **Functional framework.** In this section, we present some definitions and lemmas we need in the proofs of the main results. For some real parameter $\theta > 0$, consider the space

$$X = \big\{y \in C^1([0,\infty),\mathbb{R}) : \lim_{t\to+\infty} \frac{y(t)}{e^{\theta t}} \text{ exists and } \lim_{t\to+\infty} y'(t) = 0\big\}$$

with the norm

$$\|y\|_\theta = \max\big\{\|y\|_1, \|y\|_2\big\},$$

where

$$\|y\|_1 = \sup_{t\in[0,\infty)} \frac{|y(t)|}{e^{\theta t}}, \quad \|y\|_2 = \sup_{t\in[0,\infty)} |y'(t)|\,.$$

Remark 2.1. Clearly X is a Banach space. Moreover, if $y \in X$ is such that $y(0) = \alpha y'(\eta)$, then

$$\begin{aligned}
\frac{y(t)}{e^{\theta t}} &= e^{-\theta t}\big\{\int_0^t y'(s)\,ds + y(0)\big\}\\
&= e^{-\theta t}\big\{\int_0^t y'(s)\,ds + \alpha y'(\eta)\big\}\\
&\le e^{-\theta t}\big\{t\|y\|_2 + \alpha\|y\|_2\big\}\\
&= \frac{t+\alpha}{e^{\theta t}}\|y\|_2, \quad \forall\, t \in \mathbb{R}^+.
\end{aligned}$$

Hence $\|y\|_1 \le K\|y\|_2$ where $K = \sup_{t\in\mathbb{R}^+}\gamma(t)$ with $\gamma(t) = \frac{t+\alpha}{e^{\theta t}}$; more precisely,

$$K = \begin{cases} \alpha, & \text{if } \theta\alpha \ge 1\\ \gamma\big(\frac{1-\theta\alpha}{\theta}\big), & \text{if } 0 \le \theta\alpha < 1. \end{cases}$$

As a consequence

$$\|y\|_2 \le \|y\|_\theta \le \max\{1, K\}\|y\|_2$$

and $\lim_{t\to+\infty} \frac{y(t)}{e^{\theta t}} = 0$.

2.2. **Integral formulation.** In order to transform (1.1) into a fixed point problem, we need the following auxiliary lemma the proof of which is omitted.

Lemma 2.1. *Let $v \in L^1(I)$. Then $y \in C^1(I)$ is a solution of*

$$\begin{gathered}
-(\phi(y'))'(t) = v(t), \quad t \in I\\
y(0) = \alpha y'(\eta), \quad \lim_{t\to+\infty} y'(t) = 0
\end{gathered} \tag{2.1}$$

if and only if

$$y(t) = C + \int_0^t \phi^{-1}\Big(\int_s^{+\infty} v(\tau)\,d\tau\Big)ds, \quad t \in I, \tag{2.2}$$

where $C = \alpha\phi^{-1}\Big(\int_\eta^{+\infty} v(\tau)\,d\tau\Big)$.

2.3. **General assumptions and a fixed point operator.** Assume first that the nonlinearity satisfies the following hypotheses:

(H1) The function $f : I\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ is continuous and when y, z are bounded, $f(t, e^{\theta t}y, z)$ is bounded on $[0, +\infty)$.

(H2) The function $m : I \to R^+$ is continuous and does not vanish identically on any subinterval of I. It may be singular at $t = 0$ but satisfies
$$A := \int_0^{+\infty} m(s)ds < \infty. \tag{2.3}$$

The first hypothesis means that f is bounded in term of the variable t and is justified by the fact the unboundedness of the nonlinearity is carried by the singular factor m. The second hypothesis means that the singularity is integrably bounded.

For a bounded subset $\Omega\subset X$, define the integral operator
$$Fy(t) = C + \int_0^t \phi^{-1}\Big(\int_s^{+\infty} m(\tau)f(\tau, y(\tau), y'(\tau))d\tau\Big)ds, \quad t\in I, \tag{2.4}$$
where $C = \alpha\phi^{-1}\big(\int_\eta^{+\infty} m(\tau)f(\tau, y(\tau), y'(\tau))d\tau\big)$. By Lemma 2.1, all solutions of (1.1) are fixed points of F on X and conversely. Define the cone
$$P := \big\{y \in X : y(t) \geq 0 \text{ on } \mathbb{R}^+ \text{ with } \ y(0) = \alpha y'(\eta)\big\}.$$
We have

Lemma 2.2. *Under Assumptions* (H1), (H2)*, F maps the set $\overline{\Omega}\cap P$ into P.*

Proof. First we show that $F : \overline{\Omega}\cap P \to X$ is well defined. Let $y \in \overline{\Omega}\cap P$. Then, there exists $M > 0$ such that $\|y\|_\theta \leq M$. By Assumption (H1), let
$$S_M = \sup\{f(t, e^{\theta t}y, z),\ t\in I, (y, |z|) \in [0, M]^2\}.$$
Since, for any $t \geq 0$, $0 \leq y(t)e^{-\theta t} \leq M$ and $|y'(t)| \leq M$, Assumption (H2) implies
$$\int_0^{+\infty} m(\tau)f(\tau, y(\tau), y'(\tau))d\tau = \int_0^{+\infty} m(\tau)f\big(\tau, e^{\theta\tau}y(\tau)e^{-\theta\tau}, y'(\tau)\big)d\tau \leq AS_M.$$
Hence for any fixed $t \in (0, +\infty)$,
$$\int_0^t \phi^{-1}\Big(\int_s^{+\infty} m(\tau)f(\tau, y(\tau), y'(\tau))d\tau\Big)ds \leq \int_0^t \phi^{-1}(AS_M)ds < \infty.$$
In addition, we can easily prove that for each $y \in \overline{\Omega}\cap P$,
$$Fy \in C^1([0,\infty),\mathbb{R}),\ Fy(t) \geq 0,\ t \in I,$$
$$Fy(0) = C = \alpha\phi^{-1}\Big(\int_\eta^{+\infty} m(\tau)f(\tau, y(\tau), y'(\tau))d\tau\Big) = \alpha(Fy)'(\eta),$$
and
$$\lim_{t\to+\infty}(Fy)'(t) = \lim_{t\to+\infty}\phi^{-1}\Big(\int_t^{+\infty} m(\tau)f(\tau, y(\tau), y'(\tau))d\tau\Big) = \phi^{-1}(0) = 0.$$
By Remark 2.1, we obtain
$$\begin{aligned} 0 &\leq \lim_{t\to+\infty}\frac{Fy(t)}{e^{\theta t}} \leq \lim_{t\to+\infty}\gamma(t)\sup_{t\in[0,\infty)}|(Fy)'(t)| \\ &\leq \lim_{t\to+\infty}\gamma(t)\phi^{-1}(AS_M) = 0. \end{aligned}$$
$\square$

2.4. **A compactness criterion.** To investigate the compactness of the operator F, we appeal to the following result.

Lemma 2.3 ([6]). *Let $M \subseteq C^1_\infty(\mathbb{R}^+,\mathbb{R})$. Then the set M is relatively compact in $C^1_\infty(\mathbb{R}^+,\mathbb{R})$ if the following conditions hold:*

(a) *M is uniformly bounded in $C^1_\infty(\mathbb{R}^+,\mathbb{R})$.*
(b) *The functions belonging to the sets*
$$A = \{y : y(t) = \frac{x(t)}{e^{\theta t}},\ x \in M\} \text{ and } B = \{z : z(t) = x'(t),\ x \in M\}$$
are almost equi-continuous on $\mathbb{R}^+$.
(c) *The functions from A and B are equi-convergent at $+\infty$.*

Next, we state two compactness results depending on whether or not the function m is singular at the origin.

Lemma 2.4 (The regular case). *Assume $m : [0,\infty) \to [0,\infty)$ is continuous. Then, the mapping $F : \overline{\Omega} \cap P \to P$ is completely continuous.*

Proof. **Claim 1.** F is continuous on P. Assume $\lim_{n\to+\infty} y_n \to y$ in P; then there exists $N > 0$ independent of n such that $\max\{\|y\|_\theta, \sup_{n\geq 1} \|y_n\|_\theta\} \leq N$. Letting
$$S_N = \sup\{f(t, e^{\theta t}y, z),\ t \in [0,+\infty), (y, |z|) \in [0,N]^2\},$$
we get
$$\int_0^{+\infty} m(s)(f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s)))ds \leq 2AS_N.$$
Then, the Lebesgue's dominated convergence theorem both with the continuity of f and ϕ^{-1} imply
$$\begin{aligned}
&|(Fy_n)'(t) - (Fy)'(t))| \\
&= \Big|\phi^{-1}\Big(\int_t^{+\infty} m(\tau)f(\tau, y_n(\tau), y_n'(\tau))d\tau\Big) - \phi^{-1}\Big(\int_t^{+\infty} m(\tau)f(\tau, y(\tau), y'(\tau))d\tau\Big)\Big| \\
&\to 0, \quad \text{as } n \to +\infty.
\end{aligned}$$
Consequently,
$$\|Fy_n - Fy\|_\theta \leq \max\{1, K\}\|Fy_n - Fy\|_2 \to 0, \quad \text{as } n \to +\infty,$$
yielding our claim.

Claim 2. F is compact provided it maps bounded sets into relatively compact sets. Let Ω be any bounded subset of X; then there exists $M > 0$ such that $\|y\|_\theta \leq M$ for all $y \in \overline{\Omega} \cap P$. On one hand
$$\|Fy\|_2 = \phi^{-1}\Big(\int_0^{+\infty} m(\tau)f(\tau, y(\tau), y'(\tau))\,d\tau\Big) \leq \phi^{-1}(AS_M),\ \forall\, y \in \overline{\Omega} \cap P.$$
Hence $\|Fy\|_\theta \leq \max\{1, K\}\|Fy\|_2 \leq \max\{1, K\}\phi^{-1}(AS_M)$ which proves that $F(\overline{\Omega} \cap P)$ is uniformly bounded. On the other hand, for any $y \in \overline{\Omega} \cap P$, any $T \in (0,+\infty)$

and $t_1, t_2 \in [0,T]$, we have

$$
\begin{aligned}
&|\frac{Fy(t_2)}{e^{\theta t_2}} - \frac{Fy(t_1)}{e^{\theta t_1}}| \\
&= |C(e^{-\theta t_2} - e^{-\theta t_1}) + e^{-\theta t_2}\int_0^{t_2}\phi^{-1}\Big(\int_s^{+\infty} m(\tau)f(\tau,y(\tau),y'(\tau))d\tau\Big)ds \\
&\quad - e^{-\theta t_1}\int_0^{t_1}\phi^{-1}\Big(\int_s^{+\infty} m(\tau)f(\tau,y(\tau),y'(\tau))d\tau\Big)ds| \\
&\le \Big(C + \int_0^{t_2}\phi^{-1}(AS_M)ds\Big)|e^{-\theta t_2} - e^{-\theta t_1}| + e^{-\theta t_1}|\int_{t_2}^{t_1}\phi^{-1}(AS_M)ds|
\end{aligned}
$$

which tends to 0 as $|t_1 - t_2| \to 0$. Also, by the continuity of ϕ^{-1},

$$
\begin{aligned}
&|(Fy)'(t_2) - (Fy)'(t_1)| \\
&= \Big|\phi^{-1}\Big(\int_{t_2}^{+\infty} m(s)f(s,y(s),y'(s))ds\Big) - \phi^{-1}\Big(\int_{t_1}^{+\infty} m(s)f(s,y(s),y'(s))ds\Big)\Big| \\
&= \Big|\phi^{-1}\Big(\int_{t_2}^{t_1} m(s)f(s,y(s),y'(s))ds + \int_{t_1}^{+\infty} m(s)f(s,y(s),y'(s))ds\Big) \\
&\quad - \phi^{-1}\Big(\int_{t_1}^{+\infty} m(s)f(s,y(s),y'(s))ds\Big)\Big|
\end{aligned}
$$

tends to 0 as $|t_1 - t_2| \to 0$. This proves that $F(\overline{\Omega}\cap P)$ is equi-continuous. Since (H2) yields

$$
\lim_{t\to+\infty}\int_t^{+\infty} m(s)\,ds = 0,
$$

the Lebesgue dominated convergence theorem implies

$$
\begin{aligned}
\lim_{t\to+\infty}|\frac{Fy(t)}{e^{\theta t}} - \lim_{s\to+\infty}\frac{Fy(s)}{e^{\theta s}}| &\le \lim_{t\to\infty}\gamma(t)\sup_{t\in[0,\infty)}|(Fy)'(t)| \\
&\le \lim_{t\to\infty}\gamma(t)\phi^{-1}\Big(S_M\int_0^{+\infty} m(s)\,ds\Big) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{t\to+\infty}|(Fy)'(t) - \lim_{s\to+\infty}(Fy)'(s)| &= \lim_{t\to+\infty}|\phi^{-1}\Big(\int_t^{+\infty} m(\tau)f(\tau,y(\tau),y'(\tau))\,d\tau\Big)| \\
&\le \phi^{-1}\Big(S_M\lim_{t\to+\infty}\int_t^{+\infty} m(s)\,ds\Big) = 0.
\end{aligned}
$$

This means that $F(\overline{\Omega}\cap P)$ is equi-convergent at ∞. By Lemma 2.3, $F(\overline{\Omega}\cap P)$ is relatively compact. □

Lemma 2.5 (The singular case)**.** *Let m be singular at $t=0$. Then, the mapping F given by* (2.4) *is completely continuous.*

Proof. For each $n \ge 1$, define the approximating operator F_n on $\overline{\Omega}\cap P$ by

$$
F_n y(t) = C + \int_{\frac{1}{n}}^{t}\phi^{-1}\Big(\int_s^{+\infty} m(\tau)f(\tau,y(\tau),y'(\tau))d\tau\Big)ds, \quad t \in I.
$$

Thus it suffices to prove that F_n converges uniformly to F on $\overline{\Omega}\cap P$. For any $t\in I$ and $y\in\overline{\Omega}\cap P$ satisfying $\|y\|_\theta\leq M$, by (H1) the following estimates hold

$$\begin{aligned}|F_ny(t)-Fy(t))|e^{-\theta t} &= \Big|\int_0^{\frac{1}{n}} e^{-\theta t}\phi^{-1}\Big(\int_s^{+\infty} m(\tau)f(\tau,y(\tau),y'(\tau))d\tau\Big)ds\Big| \\ &\leq \frac{1}{n}e^{-\theta t}\phi^{-1}\big(AS_M\big),\end{aligned}$$

and

$$|(F_ny)'(t)-(Fy)'(t))| = 0, \quad \forall\, n\in\mathbb{N}.$$

Consequently, Assumption (H2) both with the Cauchy criterion for convergent integrals imply that

$$\|F_ny-Fy\|_\theta = \max\{\|F_ny-Fy\|_1, \|F_ny-Fy\|_2\}\to 0, \quad \text{as } n\to+\infty.$$

Since from Lemma 2.4, the operator $F_n:\overline{\Omega}\cap P\to P$ is completely continuous for each $n\geq 1$ and F_n converges to F uniformly on any closed, bounded subset of $\overline{\Omega}\cap P$, the uniform limit operator F is completely continuous, proving the lemma. $\square$

3. Existence results

3.1. **One positive solution.** The following Lemma is needed in this section. Detailed properties of the fixed point index on cones of Banach spaces may be found in [7, 26].

Lemma 3.1 ([1, 7, 14, 26])**.** *Let Ω be a bounded open subset of a real Banach space E, P a cone of E, $\theta\in\Omega$ and $A:\overline{\Omega}\cap P\to P$ a completely continuous operator. Suppose that*

$$Ax\neq\lambda x, \quad \forall\, x\in\partial\Omega\cap P,\ \lambda\geq 1.$$

Then the index $i(A,\Omega\cap P,P)=1$.

The main existence result in this section is

Theorem 3.1. *Assume* (H1), (H2) *and*

(H3) *for all $(t,y,z)\in I\times\mathbb{R}^+\times\mathbb{R}$,*

$$0\leq f(t,y,z)\leq a(t)\phi(e^{-\theta t}y)+b(t)\phi(|z|)+c(t),$$

where $a,b,c\in C^0(\mathbb{R}^+)$, $mb, mc\in L^1(I)$, and there exists $R>0$, such that

$$\phi^{-1}\left((|ma|_{L_1}+|mb|_{L_1})\phi(R)+|mc|_{L_1}\right) < \frac{R}{\max\{K,1\}}\,. \tag{3.1}$$

Then (1.1) *has at least one nonnegative, concave, and nondecreasing solution $y\in P\cap B_\theta(0,R)$ where $B_\theta(0,R)$ is the open ball centered at the origin with radius R in the $\theta-$weighted space X.*

If further $\min_{I\times[0,R]\times[0,R]} f(t,e^{\theta t}y,z)\geq 1$, then $y(t)\geq w(t)$ for all $t\in I$, where

$$\omega(t):=\alpha\phi^{-1}\Big(\int_\eta^{+\infty} m(\tau)d\tau\Big)+\int_0^t\phi^{-1}\Big(\int_s^{+\infty} m(\tau)d\tau\Big)\,ds, \quad t\in I. \tag{3.2}$$

Remark 3.1. The properties of ϕ and m imply that $\omega\geq 0$ and ω is well defined. Moreover ω is the unique solution of Problem (2.1) for $v\equiv m$.

Proof. Consider the open ball $\Omega := \{y \in X : \|y\|_\theta < R\}$. From Lemmas 2.4 and 2.5, the mapping $F : \overline{\Omega} \cap P \to P$ is completely continuous.

Claim 1. $Fy \neq \lambda y$, for any $y \in \partial\Omega \cap P$ and $\lambda \geq 1$.

Let $y \in \partial\Omega \cap P$. By Assumption (H3), the following estimates hold

$$\begin{aligned}
|(Fy)'(t)| &= \phi^{-1}\Big(\int_t^{+\infty} m(\tau) f(\tau, y(\tau), y'(\tau))\, d\tau\Big) \\
&\leq \phi^{-1}\Big(\int_0^{+\infty} m(\tau)\left(a(\tau)\phi(e^{-\theta\tau} y(\tau)) + b(\tau)\phi(|y'(\tau)|) + c(\tau)\right)\, d\tau\Big) \\
&\leq \phi^{-1}\Big((|ma|_{L_1}\phi(\|y\|_1) + |mb|_{L_1}\phi(\|y\|_2)) + |mc|_{L_1}\Big) \\
&\leq \phi^{-1}\left((|ma|_{L_1} + |mb|_{L_1})\,\phi(\|y\|_\theta) + |mc|_{L_1}\right) \\
&\leq \phi^{-1}\left((|ma|_{L_1} + |mb|_{L_1})\,\phi(R) + |mc|_{L_1}\right) \\
&< \frac{R}{\max\{1, K\}} = \frac{\|y\|_\theta}{\max\{1, K\}}\,.
\end{aligned}$$

Passing to the supremum over t, we infer that

$$\|Fy\|_2 < \frac{1}{\max\{1, K\}}\|y\|_\theta, \quad \forall\, y \in \partial\Omega \cap P.$$

Hence,

$$\|Fy\|_\theta \leq \max\{1, K\}\|Fy\|_2 < \|y\|_\theta, \quad \forall\, y \in \partial\Omega \cap P. \tag{3.3}$$

As a consequence

$$Fy \neq \lambda y, \quad \forall\, y \in \partial\Omega \cap P,\ \forall\, \lambda \geq 1. \tag{3.4}$$

Indeed, on the contrary there would exist some $y_0 \in \partial\Omega \cap P$ and $\lambda_0 \geq 1$ such that $Fy_0 = \lambda_0 y_0$. Thus

$$\|Fy_0\|_\theta = \lambda_0\|y_0\|_\theta \geq \|y_0\|_\theta = R,$$

contradicting (3.3). This implies that (3.4) holds. Therefore, Lemma 3.1 yields

$$i\,(F, \Omega \cap P, P) = 1. \tag{3.5}$$

Hence (3.5) and the solution property of the fixed point index imply that the operator F has a fixed point y which belongs to $\Omega \cap P$. Moreover, we have that $(\phi(y'))'(t) = (\phi((Fy)'))'(t) = -m(t)f(t, y(t), y'(t)) \leq 0$, $t \in I$, which implies that y is concave on I. Next, we show that y is a nontrivial solution.

Claim 2. For this fixed point y, we claim that $Fy(t) \geq \omega(t)$ on I, where ω is as given by (3.2). Otherwise, we have

$$\sup_{t \in I}\{\omega(t) - Fy(t)\} > 0.$$

Now, we distinguish between two cases.

Case 1. $\lim_{t\to+\infty}\{\omega(t) - Fy(t)\} = \sup_{t\in I}\{\omega(t) - Fy(t)\} > 0$. Under the assumption that $\min_{I\times[0,R]\times[0,R]} f(t, e^{\theta t}y, z) \geq 1$ and using the fact that ϕ^{-1} is nondecreasing, we get

$$\begin{aligned}\lim_{t\to+\infty}\{\omega(t)-Fy(t)\} &= \alpha\phi^{-1}\Big(\int_\eta^{+\infty} m(\tau)\,d\tau\Big)+\int_0^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} m(\tau)\,d\tau\Big)ds\\ &\quad -\alpha\phi^{-1}\Big(\int_\eta^{+\infty} m(\tau)f(\tau,y(\tau),y'(\tau))\,d\tau\Big)\\ &\quad -\int_0^{+\infty}\phi^{-1}\Big(\int_s^{+\infty} m(\tau)f(\tau,y(\tau),y'(\tau))\,d\tau\Big)ds\le 0.\end{aligned}$$

This is a contradiction to the assumption in Case 1.

Case 2. There exists a real number $t_1\ge 0$ such that

$$\omega(t_1)-Fy(t_1)=\sup_{t\in\mathbb{R}^+}\{\omega(t)-Fy(t)\}>0.$$

Arguing as in case 1, we can easily check that $\omega(t_1)-Fy(t_1)\le 0$ and a contradiction is reached. Therefore y is a nonnegative, concave, and nondecreasing solution to Problem (1.1) and satisfies

$$0\le\|y\|_\theta<R,\quad y(t)\ge\omega(t),\quad \forall\, t\in I.$$

□

3.2. **A multiplicity result.** The following Lemma is needed in this section.

Lemma 3.2 ([1, 7, 14]). *Let Ω be a bounded open set in a real Banach space E, P a cone of $E, \theta\in\Omega$ and $A:\overline{\Omega}\cap P\to P$ a completely continuous mapping. Assume that*

$$Ax\not\le x,\quad \forall\, x\in\partial\Omega\cap P.$$

Then the index $i(A,\Omega\cap P,P)=0$.

Theorem 3.2. *Assume* (H1)–(H3)*, and, instead of (3.1), there exist two constants $0<R_1<R_2$ such that*

$$\phi^{-1}\left((|ma|_{L_1}+|mb|_{L_1})\phi(R_i)+|mc|_{L_1}\right)<\frac{R_i}{\max\{1,K\}},\quad \text{for } i=1,2. \tag{3.6}$$

- (H4) *Assume that for some $0<\gamma<\delta$,*

$$f(t,e^{\theta t}y,z)\ge g(t,y),\quad \forall\,(t,y,z)\in[\gamma,\delta]\times\mathbb{R}^+\times\mathbb{R},$$

where $g\in C([\gamma,\delta]\times\mathbb{R}^+)$ and there exists $\eta>R_1$ such that

$$g(t,y)>\frac{\phi(1/\gamma)}{\int_\gamma^\delta m(\tau)\,d\tau}\,\phi(e^{\theta t}y),\quad \text{for } y\in[0,\eta],\ t\in[\gamma,\delta]. \tag{3.7}$$

Then for each constant $R_1<r<\min(R_2,\eta)$, Problem (1.1) *has at least three nonnegative, concave, and nondecreasing solutions $y_1,y_2,y_3\in P$ satisfying $0\le\|y_1\|_\theta<R_1<\|y_2\|_\theta<r<\|y_3\|_\theta<R_2$. If further $\min_{I\times[0,R_1]\times[0,R_1]}f(t,e^{\theta t}y,z)\ge 1$, then*

$$y_1(t)\ge w(t),\quad \forall\, t\in I.$$

Proof. **Claim 1.** Consider the open balls $\Omega_{R_i}:=\{y\in X:\|y\|_\theta<R_i\}$, $i=1,2$. Arguing as in Claim 1 of Theorem 3.1 and using (3.6), we can check that

$$i(F,\Omega_{R_i}\cap P,P)=1,\quad i=1,2. \tag{3.8}$$

Claim 2. Let $\ell := \phi(1/\gamma)/\int_\gamma^\delta m(\tau)\,d\tau$, $R_1 < r < \min(R_2,\eta)$ and consider the open ball $\Omega_r := \{y \in X : \|y\|_\theta < r\}$. We claim that $Fy \nleq y$, for any $y \in \partial\Omega_r \cap P$. Otherwise, let $y_0 \in \partial\Omega_r \cap P$ be such that

$$Fy_0 \le y_0. \tag{3.9}$$

Then $0 \le e^{-\theta t}y_0(t) \le r < \eta$, $\forall\, t \in [\gamma,\delta]$. Moreover, by virtue of (H4), (3.7) and (3.9) together with the property (1.4) of ϕ^{-1} and the definition of ℓ, we obtain successively the following estimates: for every $t \in [\gamma,\delta]$,

$$\begin{aligned}
y_0(t) &\ge C + \int_0^t \phi^{-1}\Big(\int_s^{+\infty} m(\tau)f(\tau,y_0(\tau),y_0'(\tau))d\tau\Big)ds\\
&\ge \int_0^\gamma \phi^{-1}\Big(\int_s^{+\infty} m(\tau)f(\tau,y_0(\tau),y_0'(\tau))d\tau\Big)ds\\
&\ge \int_0^\gamma \phi^{-1}\Big(\int_\gamma^{\delta} m(\tau)f(\tau,y_0(\tau),y_0'(\tau))d\tau\Big)ds\\
&\ge \int_0^\gamma \phi^{-1}\Big(\int_\gamma^{\delta} m(\tau)g(\tau,e^{-\theta\tau}y_0(\tau))d\tau\Big)ds\\
&> \gamma\,\phi^{-1}\Big(\int_\gamma^{\delta} m(\tau)\ell\phi(y_0(\tau))\,d\tau\Big)\\
&\ge \gamma\,\phi^{-1}\Big(\phi(\min_{t\in[\gamma,\delta]} y_0(t))\Big)\phi^{-1}\Big(\ell\int_\gamma^{\delta} m(\tau)\,d\tau\Big)\\
&\ge \gamma\,\phi^{-1}\Big(\ell\int_\gamma^{\delta} m(\tau)\,d\tau\Big)\min_{t\in[\gamma,\delta]} y_0(t)\\
&\ge \min_{t\in[\gamma,\delta]} y_0(t).
\end{aligned}$$

Hence for any $t \in [\gamma,\delta]$, $y_0(t) > \min_{t\in[\gamma,\delta]} y_0(t)$, contradicting the continuity of the function y_0 on the compact interval $[\gamma,\delta]$. This implies that (3.4) holds. As a consequence, Lemma 3.2 yields

$$i\,(F,\Omega_r \cap P, P) = 0. \tag{3.10}$$

To sum up, from (3.8), (3.10) and the fact that $\overline{\Omega}_{R_1} \subset \Omega_r, \overline{\Omega}_r \subset \Omega_{R_2}$, we deduce that $i(F,(\Omega_r \setminus \overline{\Omega}_{R_1}) \cap P, P) = -1$ and $i(F,(\Omega_{R_2} \setminus \overline{\Omega}_r) \cap P, P) = 1$. Therefore, there exist three fixed points $y_1, y_2, y_3 \in P$ satisfying $0 \le \|y_1\|_\theta < R_1 < \|y_2\|_\theta < r < \|y_3\|_\theta < R_2$. In addition, if

$$\min_{I\times[0,R_1]\times[0,R_1]} f(t,e^{\theta t}y,z) \ge 1,$$

then we can check as in Theorem 3.1 that $y_1(t) \ge w(t)$ for all $t \in I$. $\square$

Remark 3.2. Notice that at least the two solutions y_2 and y_3 are positive, whence nontrivial and that, due to the range of values the constant r may take, we can obtain as much pairs of solutions y_2, y_3 as we need.

4. Example

Consider the increasing homeomorphism defined by

$$\phi(x)=\begin{cases}(\frac{1}{8}\times 10^{-2}x^2)+(\frac{1}{4}\times 10^{-2}), & x\geq 1;\\ \frac{3}{8}\times 10^{-2}x, & x\leq 1.\end{cases}$$

Let $a(t)=b(t)=e^{-kt}$ $(k>0)$, $c(t)=2$, and

$$m(t)=\begin{cases}\frac{1}{t^{10^{-2}}}, & 0<t\leq 1;\\ \frac{1}{t^{10^{2}}}, & t\geq 1.\end{cases}$$

To check the inequality (3.1) in Assumption (H3), we take $\alpha=1/2$, $\eta=1$, $\gamma=1/5$ and $\delta=1/3$; thus we can choose $\theta=1$, $k=30$, and $R=50$. Moreover

$$\phi^{-1}(x)=\begin{cases}\sqrt{8\times 10^2x-2}, & x\geq \frac{3}{8}\times 10^{-2}\\ \frac{8}{3}\times 10^2x, & x\leq \frac{3}{8}\times 10^{-2},\end{cases}$$

$\max\{1,K\}=\max\{1,1/\sqrt{e}\}=1$, and

$$\phi^{-1}\left((|ma|_{L_1}+|mb|_{L_1})\phi(R)+|mc|_{L_1}\right)=42.4724<50.$$

Therefore, Assumptions (H1)-(H3) are satisfied. As a consequence, the singular boundary value problem

$$\begin{gathered}-(\phi(y'))'(t)=m(t)f(t,y(t),y'(t)),\quad t\in I\\ y(0)=\frac{1}{2}y'(1),\quad \lim_{t\to+\infty}y'(t)=0,\end{gathered}\tag{4.1}$$

where $f(t,y,z)=a(t)\phi(e^{-\theta t}y)+b(t)\phi(|z|)+c(t)$, $(t,y,z)\in I\times\mathbb{R}^+\times\mathbb{R}$ has at least one nonnegative, concave, and nondecreasing solution y. Moreover

$$f(t,e^{\theta t}y,z)\geq 1,\quad \forall\,(t,y,z)\in I\times\mathbb{R}^+\times\mathbb{R}.$$

Hence $y(t)\geq w(t)$, for $t\in I$ where $w(t):=1.2330+\int_0^t\phi^{-1}(\int_s^{+\infty}m(\tau)\,d\tau)ds$.

To check the inequality (3.6) in Theorem 3.2, we take $R_1=42$ and $R_2=50$ to get the numerical values

$$\begin{gathered}\phi^{-1}\left((|ma|_{L_1}+|mb|_{L_1})\phi(R_1)+|mc|_{L_1}\right)=41.8670<42.\\ \phi^{-1}\left((|ma|_{L_1}+|mb|_{L_1})\phi(R_2)+|mc|_{L_1}\right)=42.4724<50.\end{gathered}$$

Moreover,

$$f(t,y(t),y'(t))\geq g(t,e^{-\theta t}y),\quad \forall\,(t,y,z)\in I\times\mathbb{R}^+\times\mathbb{R}$$

with $g(t,e^{-\theta t}y)=a(t)\phi(e^{-\theta t}y(t))+c(t)$. If we take $\eta=45>R_1$, then for any $y\leq\eta$ and $t\in[\gamma,\delta]$, we have the estimates

$$\frac{g(t,e^{-\theta t}y)}{\phi(y)}\geq\frac{c(t)}{\phi(y)}\geq\frac{2}{\phi(\eta)}=0.7893>\frac{\phi(\frac{1}{\gamma})}{\int_\gamma^\delta m(\tau)\,d\tau}=0.2498.$$

Therefore, (H4) in Theorem 3.2 is satisfied. All the computations have been done using Matlab 7. As a consequence, for any $r\in(42,45)$, the singular boundary-value problem (4.1) has at least three nonnegative, concave, and nondecreasing solutions $y_1,y_2,y_3\in P$ satisfying

$$0\leq\|y_1\|_\theta<42<\|y_2\|_\theta<r<\|y_3\|_\theta<50.$$

Acknowledgments. The authors are thankful to the anonymous referees for their careful reading of the original manuscript, which led to substantial improvements.

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