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NONLINEAR POTENTIAL FILTRATION EQUATION AND GLOBAL ACTIONS OF LIE SYMMETRIES

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Abstract. The Lie point symmetries of the nonlinear potential filtration equation break into five cases. Contact symmetries provide another two cases. By restricting to a natural class of functions, we show that these symmetries exponentiate to a global action of the corresponding Lie group in four of the cases of Lie point symmetries. Furthermore, the action is actually the composition of a linear action with a simple translation. In fact, as a crucial step in applying the machinery of representation theory, this is accomplished using induced representations. In the remaining case as well as the contact symmetries, we show that the infinitesimal action does not exponentiate to any global Lie group action on any reasonable space of functions.

1. Introduction

The theory of Lie groups began as a tool to study partial differential equations. Subsequently, the theory of Lie symmetries developed into a very powerful and systematic mechanism for the analysis of PDE's. For instance, the method of reduction of variables via Lie point symmetries is an extremely useful technique for simplifying or solving PDE's [5, 6, 14, 18, 3, 22, 24, 25, 29, 27, 34, 37, 7]. However, the main body of Lie theory and representation theory quickly diverged from the study of Lie point symmetries of PDE's. This separation occurred due to the fact that Lie point symmetry analysis of PDE's is based on the notion of local one-parameter actions of Lie groups. As a result, the various algorithms give rise to infinitesimal symmetries that only generate a Lie algebra. Typically, the corresponding local one-parameter actions do not exponentiate to a global action of the corresponding Lie group. As a result, the enormous body of literature devoted to the study of Lie groups and representation theory is frequently not applicable to the study of symmetries of PDE's.

However in [10, 11], M. Craddock made an important discovery. He found that, in certain cases, a global action of a Lie group is made possible by restricting to an appropriate subset of the solution space. This allows the full weight of representation theory to be brought to bear. For instance when this machinery is applied to the wave equation, representation theory naturally picks out a distinguished

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orthonormal basis that is extremely well behaved with respect to energy and momentum (actually consisting of smooth, rational, finite energy solutions when the space dimension is odd), [19]. Similarly nice results are achieved in the case of the heat and Schrödinger equations with links to the harmonic oscillator [35, 36].

Of course the cases mentioned above consist of linear PDE's and so it is not surprising that representation theory can be used on the problem. In this paper we examine a particular type of nonlinear diffusion equation. Diffusion-convection equations are used to model many types of problems from physics, chemistry, and biology [12, 8, 33]. They are well studied from the point of view of Lie symmetry analysis [31, 32, 30, 38, 20, 28, 21, 13, 26, 15, 39, 16, 9, 4, 1, 2], to name only a few). The equation we study here is the nonlinear potential filtration equation

$$w_t = K(w_{xx}) \tag{1.1}$$

where $\frac{dK}{dw_{xx}}$ is not a constant.

The classification of Lie point symmetries, contact symmetries, and equivalence transformations of Equation 1.1 are well known, [3, 17]. The Lie point symmetries of this equation fall into five categories and contact symmetries give two more categories, each of which will be examined. A priori, there is no reason to suppose that an algebra of Lie symmetries exponentiates to an action of the entire corresponding Lie group.

Nevertheless in four of the five cases of point symmetries, we show that the Lie algebra of symmetries extend to a *global* action of a corresponding Lie group by restricting to a natural subclass of functions. In each of these four cases, we explicitly write down this global action. Especially important, we accomplish this by means of the theory of induced representations which will allow much of the machinery of representation theory to be applied and which is a crucial step in [35] and [19]. Remarkably in each of these cases, we show that the action of the group is actually given by a *linear* action composed with a relatively simple (nonlinear) translation. Also interesting in its own right, in the remaining case and for both of the contact symmetries, we show that the Lie algebra of symmetries does not extend to a global action for any corresponding Lie group on any reasonable subclass of functions.

2. Symmetry Classification

The equivalence transformations for the nonlinear potential filtration equation

$$w_t = K(w_{xx}),$$

$\frac{dK}{dw_{xx}}$ not a constant, are given by

$$\begin{gathered}
\overline{t} = \alpha t + \gamma_1, \quad \overline{x} = \beta_1 x + \beta_2 w_x + \gamma_2, \\
\begin{aligned}
\overline{w} = \beta_1\Big(\beta_4 w + \frac{1}{2}\beta_3 x^2 + \gamma_3 x\Big) + \gamma_4 t + \gamma_5 \\
+ \beta_2\Big(\beta_3(x w_x - w) + \gamma_3 w_x + \frac{1}{2}\beta_4 w_x^2\Big),
\end{aligned} \\
\overline{K} = \frac{\beta_1\beta_4 - \beta_2\beta_3}{\alpha} K + \frac{\gamma_4}{\alpha}
\end{gathered}$$

where $\alpha \neq 0$ and $\beta_1\beta_4 - \beta_2\beta_3 \neq 0$ [3, 17]. Up to these transformations, the classification of the Lie point symmetries break into five cases: the generic case, $K = e^{w_{xx}}$, $K = \frac{1}{\sigma} w_{xx}^{\sigma}$ with $\sigma > 0$ and $\sigma \neq 1, \frac{1}{3}$, $K = 3 w_{xx}^{1/3}$, and $K = \ln(w_{xx})$.

Furthermore, additional Lie contact symmetries occur in two cases: $K = \arctan w_{xx}$ and $K = \frac{1}{\lambda} e^{\lambda \arctan w_{xx}}$ for $\lambda > 0$ [3, 17]. We will examine each case and determine when the symmetries exponentiate to a global action of a Lie group.

3. Generic Case

In the generic case, the symmetry Lie algebra is five dimensional and spanned by

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial w}, \\ X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w}, \quad X_5 = x\frac{\partial}{\partial w}. \end{gathered}$$

To find a globalization of the corresponding local one-parameter actions, consider the solvable group G_1 given by

$$G_1 = \Big\{ \begin{pmatrix} r^2 & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : r, s, v, u, z \in \mathbb{R},\ r > 0 \Big\}$$

along with the subgroup

$$D_1 = \Big\{ \begin{pmatrix} r^2 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : r, s, z \in \mathbb{R},\ r > 0 \Big\}.$$

Define the character $\chi_1 : D_1 \to \mathbb{C}^\times$ by

$$\chi_1\Big(\begin{pmatrix} r^2 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big) = r^2$$

and consider the representation of G_1,

$$\operatorname{Ind}_{D_1}^{G_1} \chi_1 = \big\{ \varphi \in C^\infty(G_1) : \varphi(gd) = \chi_1(d)^{-1}\varphi(g) \text{ for } g \in G_1, \quad d \in D_1 \big\}$$

with G_1-action given by

$$(g_1 \cdot f)(g_2) = f(g_1^{-1} g_2)$$

for $g_i \in G_1$. Using what would be called the noncompact picture if we were working in the semisimple case [23], let

$$\mathcal{I}_1 = \{ f \in C^\infty(\mathbb{R}^2) : f(x,t) = \varphi\Big(\begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big) \text{ for some } \varphi \in \operatorname{Ind}_{D_1}^{G_1} \chi_1 \}.$$

By requiring that the map $\varphi \to f$ be an intertwining operator, $\mathcal{I}_1$ inherits an action of G_1 so that $\mathcal{I}_1 \cong \operatorname{Ind}_{D_1}^{G_1} \chi_1$. Writing

$$\begin{pmatrix} r^2 & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & v \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r^2 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we see that φ can be reconstructed from f by

$$\varphi(\begin{pmatrix} r^2 & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) = r^{-2} f(w,v)$$

so that $\mathcal{I}_1 = C^\infty(\mathbb{R}^2)$. Of course it is easy to move out of the smooth category by studying, say, L^2-functions. All of our theorems can be easily extended to this case.

Theorem 3.1. *The (linear) action of G_1 on $\mathcal{I}_1$ is given by*

$$(\begin{pmatrix} r^2 & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot f)(x,t) = r^2 f(\frac{x-u}{r}, \frac{t-v}{r^2}).$$

Proof. Observe that

$$\begin{aligned} &\begin{pmatrix} r^2 & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & r^{-2}(t-v) \\ 0 & 1 & 0 & r^{-1}(x-u) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-2} & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & -r^{-3}s & r^{-2} & r^{-3}(su-sx) - r^{-2}z \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

It follows that

$$(\begin{pmatrix} r^2 & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot f)(x,t) = r^2 f(\frac{x-u}{r}, \frac{t-v}{r^2}).$$

where φ corresponds to f under the isomorphism $\mathcal{I}_1 \cong \operatorname{Ind}_{D_1}^{G_1} \chi_1$. □

To complete our picture, let $\tau_1 : G_1 \times \mathcal{I}_1 \to \mathcal{I}_1$ be given by

$$(\tau_1(\begin{pmatrix} r^2 & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) \cdot f)(x,t) = f(x,t) + r^{-1}s(x-u) + z.$$

Lemma 3.2. *The map τ_1 does not define an action of G_1 on $\mathcal{I}_1$. However, it is related to the the original action of G_1 on $\mathcal{I}_1$ (given in Theorem 3.1) by*

$$\tau_1(g_1 g_2) \cdot f = \tau_1(g_1) \cdot (g_1 \cdot (\tau_1(g_2) \cdot (g_1^{-1} \cdot f)))$$

for $g_i \in G_1$ and $f \in \mathcal{I}_1$.

Proof. To see that τ_1 is not an action, write

$$g_1 = \begin{pmatrix} r_1^2 & 0 & 0 & v_1 \\ 0 & r_1 & 0 & u_1 \\ 0 & s_1 & r_1^2 & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} r_2^2 & 0 & 0 & v_2 \\ 0 & r_2 & 0 & u_2 \\ 0 & s_2 & r_2^2 & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and note that

$$g_1g_2 = \begin{pmatrix} r_1^2r_2^2 & 0 & 0 & v_1 + r_1^2v_2 \\ 0 & r_1r_2 & 0 & u_1 + r_1u_2 \\ 0 & r_1^2s_2 + r_2s_1 & r_1^2r_2^2 & z_1 + r_1^2z_2 + s_1u_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned}
&(\tau_1(g_1g_2) \cdot f)(x,t) \\
&= f(x,t) + r_1^{-1}r_2^{-1}\left(r_1^2s_2 + r_2s_1\right)(x - u_1 - r_1u_2) + (z_1 + r_1^2z_2 + s_1u_2) \\
&= f(x,t) + z_1 + r_1^2z_2 + xr_1^{-1}s_1 - r_1^{-1}s_1u_1 - r_1^2r_2^{-1}s_2u_2 + xr_1r_2^{-1}s_2 - r_1r_2^{-1}s_2u_1
\end{aligned}$$

while

$$\begin{aligned}
&(\tau_1(g_1) \cdot (\tau_1(g_2) \cdot f))(x,t) \\
&= (\tau_1(g_2) \cdot f)(x,t) + r_1^{-1}s_1(x - u_1) + z_1 \\
&= f(x,t) + r_2^{-1}s_2(x - u_2) + z_2 + r_1^{-1}s_1(x - u_1) + z_1 \\
&= f(x,t) + z_1 + z_2 + xr_1^{-1}s_1 - r_1^{-1}s_1u_1 - r_2^{-1}s_2u_2 + xr_2^{-1}s_2.
\end{aligned}$$

Therefore, τ_1 is not an action. On the other hand, we can verify the relation $\tau_1(g_1g_2) \cdot f = \tau_1(g_1) \cdot (g_1 \cdot (\tau_1(g_2) \cdot (g_1^{-1} \cdot f)))$ by using Theorem 3.1 and calculating

$$\begin{aligned}
&(\tau_1(g_1) \cdot (g_1 \cdot (\tau_1(g_2) \cdot (g_1^{-1} \cdot f))))(x,t) \\
&= (g_1 \cdot (\tau_1(g_2) \cdot (g_1^{-1} \cdot f)))(x,t) + r_1^{-1}s_1(x - u_1) + z_1 \\
&= r_1^2((\tau_1(g_2) \cdot (g_1^{-1} \cdot f))(\frac{x - u_1}{r_1}, \frac{t - v_1}{r_1^2})) + r_1^{-1}s_1(x - u_1) + z_1 \\
&= r_1^2((g_1^{-1} \cdot f)(\frac{x - u_1}{r_1}, \frac{t - v_1}{r_1^2}) + r_2^{-1}s_2(\frac{x - u_1}{r_1} - u_2) + z_2) + r_1^{-1}s_1(x - u_1) + z_1 \\
&= r_1^2(r_1^{-2}f(x,t) + r_2^{-1}s_2(\frac{x - u_1}{r_1} - u_2) + z_2) + r_1^{-1}s_1(x - u_1) + z_1 \\
&= f(x,t) + z_1 + r_1^2z_2 + xr_1^{-1}s_1 - r_1^{-1}s_1u_1 - r_1^2r_2^{-1}s_2u_2 + xr_1r_2^{-1}s_2 - r_1r_2^{-1}s_2u_1.
\end{aligned}$$

□

As a result, consider $\delta_1 : G_1 \times \mathcal{I}_1 \to \mathcal{I}_1$ given by

$$\delta_1(g) \cdot f = \tau_1(g) \cdot (g \cdot f).$$

Explicitly, we see that

$$(\delta_1(\begin{pmatrix} r^2 & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) \cdot f)(x,t) = r^2 f(\frac{x - u}{r}, \frac{t - v}{r^2}) + r^{-1}s(x - u) + z. \tag{3.1}$$

Theorem 3.3. *The (nonlinear) action of G_1 on $\mathcal{I}_1$ given by δ_1 gives a globalization of the local one-parameter group action generated by the Lie point symmetries of the nonlinear potential filtration equation $w_t = K(w_{xx})$ in the generic case.*

Proof. First we check that δ_1 defines an action. For this use Lemma 3.2 to see that

$$\begin{aligned}
\delta_1(g_1g_2) \cdot f &= \tau_2(g_1g_2) \cdot (g_1g_2 \cdot f) = \tau_2(g_1g_2) \cdot (g_1 \cdot (g_2 \cdot f)) \\
&= \tau_2(g_1) \cdot (g_1 \cdot (\tau_2(g_2) \cdot (g_1^{-1} \cdot (g_1 \cdot (g_2 \cdot f)))))
\end{aligned}$$

$$\begin{aligned}
&= \tau_2(g_1)\cdot(g_1\cdot(\tau_2(g_2)\cdot(g_2\cdot f)))\\
&= \tau_2(g_1)\cdot(g_1\cdot(\delta_1(g_2)\cdot f))\\
&= \delta_1(g_1)\cdot(\delta_1(g_2)\cdot f)
\end{aligned}$$

as desired. Next, let

$$R=\begin{pmatrix}2&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&0\end{pmatrix},\quad S=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\quad V=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$U=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad Z=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

be a basis for the Lie algebra of G_1. Using Equation 3.1, it follows that

$$\begin{gathered}
(\delta_1(e^{sR})\cdot f)(x,t)=e^{2s}f(e^{-s}x,e^{-2s}t)\\
(\delta_1(e^{sS})\cdot f)(x,t)=f(x,t)+sx\\
(\delta_1(e^{sV})\cdot f)(x,t)=f(x,t-s)\\
(\delta_1(e^{sW})\cdot f)(x,t)=f(x-s,t)\\
(\delta_1(e^{sZ})\cdot f)(x,t)=f(x,t)+s.
\end{gathered}$$

Applying $\frac{\partial}{\partial s}|_{s=0}$ shows that

$$\begin{gathered}
\frac{\partial}{\partial s}(\delta_1(e^{sR})\cdot f)(x,t)|_{s=0}=-x\frac{\partial}{\partial x}f(x,t)-2t\frac{\partial}{\partial t}f(x,t)+2f(x,t)\\
\frac{\partial}{\partial s}(\delta_1(e^{sS})\cdot f)(x,t)|_{s=0}=x\\
\frac{\partial}{\partial s}(\delta_1(e^{sV})\cdot f)(x,t)|_{s=0}=-\frac{\partial}{\partial t}f(x,t)\\
\frac{\partial}{\partial s}(\delta_1(e^{sU})\cdot f)(x,t)|_{s=0}=-\frac{\partial}{\partial x}f(x,t)\\
\frac{\partial}{\partial s}(\delta_1(e^{sZ})\cdot f)(x,t)|_{s=0}=1.
\end{gathered}$$

Under the prolongation formalism [25], an easy application of the chain rule shows that the vector field Lie point symmetry

$$h_1(x,t)\frac{\partial}{\partial x}+h_2(x,t)\frac{\partial}{\partial t}+h_3(x,t,w)\frac{\partial}{\partial w}$$

on $\mathbb{R}^2\times\mathbb{R}$ gives rise to a local one-parameter group action on a function f whose partial with respect to s at $s=0$ is given by

$$-h_1(x,t)\frac{\partial}{\partial x}f(x,t)-h_2(x,t)\frac{\partial}{\partial t}f(x,t)+h_3(x,t,f(x,t)).$$

Therefore the one parameter groups corresponding to $\{R,S,V,U,Z\}$ give rise to the symmetry vector fields

$$X_4=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}+2w\frac{\partial}{\partial w},\quad X_5=x\frac{\partial}{\partial w}$$

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial w}.$$

Since the Lie point symmetries of $w_t = K(w_{xx})$ in the generic case are spanned by $\{X_1, X_2, X_3, X_4, X_5\}$, the proof is complete. $\square$

4. $K = e^{w_{xx}}$

Consider the case of $K = e^{w_{xx}}$. Then the symmetry Lie algebra is six dimensional and spanned by

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial w} \\ X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w} \\ X_5 = x\frac{\partial}{\partial w}, \quad X_6 = t\frac{\partial}{\partial t} - \frac{1}{2}x^2\frac{\partial}{\partial w}. \end{gathered}$$

To find a globalization of the corresponding local one-parameter actions, consider the solvable group G_2 given by

$$G_2 = \{ \begin{pmatrix} r^2 e^{-\frac{n}{r^2}} & 0 & 0 & 0 & v \\ 0 & r & 0 & 0 & u \\ 0 & s & r^2 & n & z \\ 0 & -ru & 0 & r^2 & -\frac{1}{2}u^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : r, s, n, v, u, z \in \mathbb{R}, \quad r > 0\}$$

along with the subgroup

$$D_2 = \{ \begin{pmatrix} r^2 e^{-\frac{n}{r^2}} & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 \\ 0 & s & r^2 & n & z \\ 0 & 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : r, s, n, z \in \mathbb{R}, \quad r > 0\}.$$

Define the character $\chi_2 : D_2 \to \mathbb{C}^\times$ by

$$\chi_2(\begin{pmatrix} r^2 e^{-\frac{n}{r^2}} & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 \\ 0 & s & r^2 & n & z \\ 0 & 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}) = r^2$$

and consider the representation of G_2

$$\operatorname{Ind}_{D_2}^{G_2} \chi_2 = \{\varphi \in C^\infty(G_2) : \varphi(gd) = \chi_2(d)^{-1}\varphi(g) \text{ for } g \in G_2, \quad d \in D_2\}$$

with G_2-action given by

$$(g_1 \cdot f)(g_2) = f(g_1^{-1} g_2)$$

for $g_i \in G_2$. Using what would be called the noncompact picture if we were working in the semisimple case [23], let $\mathcal{I}_2$ be the set

$$\{f \in C^\infty(\mathbb{R}^2) : f(x,t) = \varphi(\begin{pmatrix} 1 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -x & 0 & 1 & -\frac{1}{2}x^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}) \text{ for some } \varphi \in \operatorname{Ind}_{D_2}^{G_2} \chi_2\}.$$

By requiring that the map $\varphi \to f$ be an intertwining operator, $\mathcal{I}_2$ inherits an action of G_2 so that $\mathcal{I}_2 \cong \operatorname{Ind}_{D_2}^{G_2} \chi_2$. Writing

$$\begin{aligned}
&\begin{pmatrix} r^2e^{-\frac{n}{r^2}} & 0 & 0 & 0 & v \\ 0 & r & 0 & 0 & u \\ 0 & s & r^2 & n & z \\ 0 & -ru & 0 & r^2 & -\frac{1}{2}u^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & v \\ 0 & 1 & 0 & 0 & u \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -u & 0 & 1 & -\frac{1}{2}u^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r^2e^{-\frac{n}{r^2}} & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 \\ 0 & s & r^2 & n & z \\ 0 & 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

we see that φ can be reconstructed from f by

$$\varphi(\begin{pmatrix} r^2e^{-\frac{n}{r^2}} & 0 & 0 & 0 & v \\ 0 & r & 0 & 0 & u \\ 0 & s & r^2 & n & z \\ 0 & -ru & 0 & r^2 & -\frac{1}{2}u^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}) = r^{-2} f(u,v)$$

so that $\mathcal{I}_2 = C^\infty(\mathbb{R}^2)$.

Theorem 4.1. *The (linear) action of G_2 on $\mathcal{I}_2$ is given by*

$$(\begin{pmatrix} r^2e^{-\frac{n}{r^2}} & 0 & 0 & 0 & v \\ 0 & r & 0 & 0 & u \\ 0 & s & r^2 & n & z \\ 0 & -ru & 0 & r^2 & -\frac{1}{2}u^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot f)(x,t) = r^2 f(\frac{x-u}{r}, e^{\frac{n}{r^2}}(\frac{t-v}{r^2})).$$

Proof. Observe that

$$\begin{aligned}
&\begin{pmatrix} r^2e^{-\frac{n}{r^2}} & 0 & 0 & 0 & v \\ 0 & r & 0 & 0 & u \\ 0 & s & r^2 & n & z \\ 0 & -ru & 0 & r^2 & -\frac{1}{2}u^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -x & 0 & 1 & -\frac{1}{2}x^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & r^{-2}e^{\frac{n}{r^2}}(t-v) \\ 0 & 1 & 0 & 0 & r^{-1}(x-u) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -r^{-1}(x-u) & 0 & 1 & -\frac{1}{2}r^{-2}(x-u)^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&\times \begin{pmatrix} r^{-2}e^{\frac{n}{r^2}} & 0 & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 & 0 \\ 0 & r^{-4}(-nu-rs+nx) & r^{-2} & -r^{-4}n & \begin{matrix}\frac{1}{2}r^{-4}(nu^2+nx^2-2r^2z \\ +2rsu-2nux-2rsx)\end{matrix} \\ 0 & 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

It follows that

$$\Big(\begin{pmatrix} r^2 e^{\frac{n}{r^2}} & 0 & 0 & 0 & v \\ 0 & r & 0 & 0 & u \\ 0 & s & r^2 & n & z \\ 0 & ru & 0 & r^2 & \frac{1}{2}u^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot f\Big)(x,t) = r^2 f\Big(\frac{x-u}{r}, e^{\frac{n}{r^2}}\Big(\frac{t-v}{r^2}\Big)\Big).$$

where φ corresponds to f under the isomorphism $\mathcal{I}_2 \cong \operatorname{Ind}_{D_2}^{G_2} \chi_2$. □

To complete our picture, let $\tau_2 : G_2 \times \mathcal{I}_2 \to \mathcal{I}_2$ be given by

$$\begin{aligned} &\Big(\tau_2\Big(\begin{pmatrix} r^2 e^{-\frac{n}{r^2}} & 0 & 0 & 0 & v \\ 0 & r & 0 & 0 & u \\ 0 & s & r^2 & n & z \\ 0 & -ru & 0 & r^2 & -\frac{1}{2}u^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\Big) \cdot f\Big)(x,t) \\ &= f(x,t) + \frac{1}{2} r^{-2} n (x-u)^2 + r^{-1} s (x-u) + z. \end{aligned}$$

Lemma 4.2. *The map τ_2 does not define an action of G_2 on $\mathcal{I}_2$. However, it is related to the the original action of G_2 on $\mathcal{I}_2$ (given in Theorem 4.1) by*

$$\tau_2(g_1 g_2) \cdot f = \tau_2(g_1) \cdot (g_1 \cdot (\tau_2(g_2) \cdot (g_1^{-1} \cdot f)))$$

for $g_i \in G_2$ and $f \in \mathcal{I}_2$.

Proof. Since it is trivial, we leave the proof that τ_2 is not an action to the reader. For the relation of τ_2 to the original action, write

$$g_1 = \begin{pmatrix} r_1^2 e^{-\frac{n_1}{r_1^2}} & 0 & 0 & 0 & v_1 \\ 0 & r_1 & 0 & 0 & u_1 \\ 0 & s_1 & r_1^2 & n_1 & z_1 \\ 0 & -r_1 u_1 & 0 & r_1^2 & -\frac{1}{2}u_1^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} r_2^2 e^{-\frac{n_2}{r_2^2}} & 0 & 0 & 0 & v_2 \\ 0 & r_2 & 0 & 0 & u_2 \\ 0 & s_2 & r_2^2 & n_2 & z_2 \\ 0 & -r_2 u_2 & 0 & r_2^2 & -\frac{1}{2}u_2^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and note that $g_1 g_2$ is the matrix

$$\begin{pmatrix} r_1^2 r_2^2 e^{-\frac{n_1 r_2^2 + n_2 r_1^2}{r_1^2 r_2^2}} & 0 & 0 & 0 & v_1 + r_1^2 v_2 e^{-\frac{n_1}{r_1^2}} \\ 0 & r_1 r_2 & 0 & 0 & u_1 + r_1 u_2 \\ 0 & s_2 r_1^2 + r_2 s_1 + n_1 r_2 u_2 & r_1^2 r_2^2 & n_2 r_1^2 + n_1 r_2^2 & z_2 r_1^2 + \frac{1}{2} n_1 u_2^2 + s_1 u_2 + z_1 \\ 0 & -r_1 r_2 (u_1 + r_1 u_2) & 0 & r_1^2 r_2^2 & -\frac{1}{2}(u_1 + r_1 u_2)^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} &(\tau_2(g_1 g_2) \cdot f)(x,t) \\ &= f(x,t) + \frac{1}{2} r_1^{-2} r_2^{-2} (n_2 r_1^2 + n_1 r_2^2)\,(x - u_1 - r_1 u_2)^2 \\ &\quad + r_1^{-1} r_2^{-1} (s_2 r_1^2 + r_2 s_1 + n_1 r_2 u_2)(x - u_1 - r_1 u_2) + (z_2 r_1^2 + \frac{1}{2} n_1 u_2^2 + s_1 u_2 + z_1). \end{aligned}$$

We then verify the relation $\tau_2(g_1g_2)\cdot f=\tau_2(g_1)\cdot(g_1\cdot(\tau_2(g_2)\cdot(g_1^{-1}\cdot f)))$ by using Theorem 3.1 and calculating

$$\begin{aligned}
&(\tau_2(g_1)\cdot(g_1\cdot(\tau_2(g_2)\cdot(g_1^{-1}\cdot f))))(x,t)\\
&=(g_1\cdot(\tau_2(g_2)\cdot(g_1^{-1}\cdot f)))(x,t)+\frac{1}{2}r_1^{-2}n_1(x-u_1)^2\\
&\quad+r_1^{-1}s_1(x-u_1)+z_1\\
&=r_1^2((\tau_2(g_2)\cdot(g_1^{-1}\cdot f))(\frac{x-u_1}{r_1},e^{\frac{n_1}{r_1^2}}(\frac{t-v_1}{r_1^2})))+\frac{1}{2}r_1^{-2}n_1(x-u_1)^2\\
&\quad+r_1^{-1}s_1(x-u_1)+z_1\\
&=r_1^2((g_1^{-1}\cdot f)(\frac{x-u_1}{r_1},e^{\frac{n_1}{r_1^2}}(\frac{t-v_1}{r_1^2}))+\frac{1}{2}r_2^{-2}n_2\big(\frac{x-u_1}{r_1}-u_2\big)^2\\
&\quad+r_2^{-1}s_2(\frac{x-u_1}{r_1}-u_2)+z_2)+\frac{1}{2}r_1^{-2}n_1(x-u_1)^2+r_1^{-1}s_1(x-u_1)+z_1\\
&=r_1^2(r_1^{-2}f(x,t)+\frac{1}{2}r_2^{-2}n_2\big(\frac{x-u_1}{r_1}-u_2\big)^2+r_2^{-1}s_2(\frac{x-u_1}{r_1}-u_2)+z_2)\\
&+\frac{1}{2}r_1^{-2}n_1(x-u_1)^2+r_1^{-1}s_1(x-u_1)+z_1\\
&=(\tau_2(g_1g_2)\cdot f)(x,t).
\end{aligned}$$

□

As a result, consider $\delta_2:G_2\times\mathcal{I}_2\to\mathcal{I}_2$ given by $\delta_2(g)\cdot f=\tau_2(g)\cdot(g\cdot f)$. Explicitly, we see that

$$\begin{aligned}
&(\delta_2(\begin{pmatrix} r^2e^{-\frac{n}{r^2}} & 0 & 0 & 0 & v\\ 0 & r & 0 & 0 & u\\ 0 & s & r^2 & n & z\\ 0 & -ru & 0 & r^2 & -\frac{1}{2}u^2\\ 0 & 0 & 0 & 0 & 1\end{pmatrix})\cdot f)(x,t)\\
&=r^2f(\frac{x-u}{r},e^{\frac{n}{r^2}}(\frac{t-v}{r^2}))+\frac{1}{2}r^{-2}n(x-u)^2+r^{-1}s(x-u)+z.
\end{aligned}$$

Theorem 4.3. *The (nonlinear) action of G_2 on $\mathcal{I}_2$ given by δ_2 gives a globalization of the local one-parameter group action generated by the Lie point symmetries of the nonlinear potential filtration equation $w_t=e^{w_{xx}}$.*

Proof. The proof that δ_2 defines an action follows from Lemma 4.2 just as in the proof of Theorem 3.3. Next, let

$$R=\begin{pmatrix}2&0&0&0&0\\0&1&0&0&0\\0&0&2&0&0\\0&0&0&2&0\\0&0&0&0&0\end{pmatrix},\quad S=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad N=\begin{pmatrix}-1&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$V=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad U=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&0\end{pmatrix},\quad Z=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

be a basis for the Lie algebra of G_2. Using Equation 4.1, it follows that

$$\begin{gathered}
(\delta_2(e^{sR}) \cdot f)(x,t) = e^{2s} f(e^{-s}x, e^{-2s}t) \\
(\delta_2(e^{sS}) \cdot f)(x,t) = f(x,t) + sx \\
(\delta_2(e^{sN}) \cdot f)(x,t) = f(x, e^{s}t) + \frac{1}{2}sx^2 \\
(\delta_2(e^{sV}) \cdot f)(x,t) = f(x, t-s) \\
(\delta_2(e^{sU}) \cdot f)(x,t) = f(x-s, t) \\
(\delta_2(e^{sZ}) \cdot f)(x,t) = f(x,t) + s.
\end{gathered}$$

Applying $\frac{\partial}{\partial s}|_{s=0}$ shows that

$$\begin{gathered}
\frac{\partial}{\partial s}(\delta_2(e^{sR}) \cdot f)(x,t)|_{s=0} = -x\frac{\partial}{\partial x}f(x,t) - 2t\frac{\partial}{\partial t}f(x,t) + 2f(x,t) \\
\frac{\partial}{\partial s}(\delta_2(e^{sS}) \cdot f)(x,t)|_{s=0} = x \\
\frac{\partial}{\partial s}(\delta_2(e^{sN}) \cdot f)(x,t)|_{s=0} = t\frac{\partial}{\partial t}f(x,t) + \frac{1}{2}x^2 \\
\frac{\partial}{\partial s}(\delta_2(e^{sV}) \cdot f)(x,t)|_{s=0} = -\frac{\partial}{\partial t}f(x,t) \\
\frac{\partial}{\partial s}(\delta_2(e^{sU}) \cdot f)(x,t)|_{s=0} = -\frac{\partial}{\partial x}f(x,t) \\
\frac{\partial}{\partial s}(\delta_2(e^{sZ}) \cdot f)(x,t)|_{s=0} = 1.
\end{gathered}$$

Under the prolongation formalism [25], an easy application of the chain rule shows that the vector field Lie point symmetry

$$h_1(x,t)\frac{\partial}{\partial x} + h_2(x,t)\frac{\partial}{\partial t} + h_3(x,t,w)\frac{\partial}{\partial w}$$

on $\mathbb{R}^2 \times \mathbb{R}$ gives rise to a local one-parameter group action on a function f whose partial with respect to s at $s = 0$ is given by

$$-h_1(x,t)\frac{\partial}{\partial x}f(x,t) - h_2(x,t)\frac{\partial}{\partial t}f(x,t) + h_3(x,t,f(x,t)).$$

Therefore, the one parameter groups corresponding to $\{R, S, N, V, U, Z\}$ give rise to the symmetry vector fields

$$\begin{gathered}
X_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + 2w\frac{\partial}{\partial w}, \quad X_5 = x\frac{\partial}{\partial w}, \\
-X_6 = -t\frac{\partial}{\partial t} + \frac{1}{2}x^2\frac{\partial}{\partial w}, \quad X_1 = \frac{\partial}{\partial t}, \\
X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial w}.
\end{gathered}$$

Since $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ span the Lie point symmetries of $w_t = e^{w_{xx}}$, the proof is complete. $\square$

5. $K = \frac{1}{\sigma} w_{xx}^{\sigma}$

Consider the case of $K = \frac{1}{\sigma} w_{xx}^{\sigma}$ with $\sigma > 0$ and $\sigma \neq 1, \frac{1}{3}$. Then the symmetry Lie algebra is six dimensional and spanned by

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \\ X_3 = \frac{\partial}{\partial w}, \quad X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w}, \\ X_5 = x\frac{\partial}{\partial w}, \quad X_6 = (1-\sigma)\, t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}. \end{gathered}$$

To find a globalization of the corresponding local one-parameter actions, consider the solvable group G_3 given by

$$G_3 = \{ \begin{pmatrix} r^2 n^{1-\sigma} & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : r, s, n, v, u, z \in \mathbb{R}, \quad r, n > 0\}$$

along with the subgroup

$$D_3 = \{ \begin{pmatrix} r^2 n^{1-\sigma} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & s & r^2 n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : r, s, n, z \in \mathbb{R}, \quad r, n > 0\}.$$

Define the character $\chi_3 : D_3 \to \mathbb{C}^{\times}$ by

$$\chi_3(\begin{pmatrix} r^2 n^{1-\sigma} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & s & r^2 n & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) = r^2 n$$

and consider the representation of G_3

$$\operatorname{Ind}_{D_3}^{G_3} \chi_3 = \left\{ \varphi \in C^{\infty}(G_3) : \varphi(gd) = \chi_3(d)^{-1} \varphi(g) \text{ for } g \in G_3, \quad d \in D_3 \right\}$$

with G_3-action given by

$$(g_1 \cdot f)(g_2) = f(g_1^{-1} g_2)$$

for $g_i \in G_3$. Using what would be called the noncompact picture if we were working in the semisimple case [23], let

$$\mathcal{I}_3 = \{ f \in C^{\infty}(\mathbb{R}^2) : f(x,t) = \varphi(\begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) \text{ for some } \varphi \in \operatorname{Ind}_{D_3}^{G_3} \chi_3 \}.$$

By requiring that the map $\varphi \to f$ be an intertwining operator, $\mathcal{I}_3$ inherits an action of G_3 so that $\mathcal{I}_3 \cong \operatorname{Ind}_{D_3}^{G_3} \chi_3$. Writing

$$\begin{pmatrix} r^2 n^{1-\sigma} & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & v \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r^2 n^{1-\sigma} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & s & r^2 n & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we see that φ can be reconstructed from f by

$$\varphi(\begin{pmatrix} r^2n^{1-\sigma} & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2n & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) = r^{-2}n^{-1}f(u,v)$$

so that $\mathcal{I}_3 = C^\infty(\mathbb{R}^2)$.

Theorem 5.1. *The (linear) action of G_3 on $\mathcal{I}_3$ is given by*

$$(\begin{pmatrix} r^2n^{1-\sigma} & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot f)(x,t) = r^2nf(\frac{x-u}{r}, \frac{t-v}{r^2n^{1-\sigma}}).$$

Proof. Observe that

$$\begin{aligned} &\begin{pmatrix} r^2n^{1-\sigma} & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2n & z \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & r^{-2}n^{\sigma-1}(t-v) \\ 0 & 1 & 0 & r^{-1}(x-u) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} r^{-2}n^{\sigma-1} & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & -r^{-3}n^{-1}s & r^{-2}n^{-1} & r^{-3}n^{-1}(su-sx-rz) \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

It follows that

$$(\begin{pmatrix} r^2n^{1-\sigma} & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot f)(x,t) = r^2nf(\frac{x-u}{r}, \frac{t-v}{r^2n^{1-\sigma}}).$$

where φ corresponds to f under the isomorphism $\mathcal{I}_3 \cong \operatorname{Ind}_{D_3}^{G_3} \chi_3$. □

To complete our picture, let $\tau_3 : G_3 \times \mathcal{I}_3 \to \mathcal{I}_3$ be given by

$$(\tau_3(\begin{pmatrix} r^2n^{1-\sigma} & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2n & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) \cdot f)(x,t) = f(x,t) + r^{-1}s(x-u) + z.$$

Lemma 5.2. *The map τ_3 does not define an action of G_3 on $\mathcal{I}_3$. However, it is related to the the original action of G_3 on $\mathcal{I}_3$ (given in Theorem 5.1) by*

$$\tau_3(g_1g_2) \cdot f = \tau_3(g_1) \cdot (g_1 \cdot (\tau_3(g_2) \cdot (g_1^{-1} \cdot f)))$$

for $g_i \in G_3$ and $f \in \mathcal{I}_3$.

Proof. Since it is trivial, we leave the proof that τ_3 is not an action to the reader. For the relation of τ_3 to the original action, write

$$g_1 = \begin{pmatrix} r_1^2 n_1^{1-\sigma} & 0 & 0 & v_1 \\ 0 & r_1 & 0 & u_1 \\ 0 & s_1 & r_1^2 n_1 & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} r_2^2 n_2^{1-\sigma} & 0 & 0 & v_2 \\ 0 & r_2 & 0 & u_2 \\ 0 & s_2 & r_2^2 n_2 & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and note that

$$g_1 g_2 = \begin{pmatrix} r_1^2 r_2^2 n_1^{1-\sigma} n_2^{1-\sigma} & 0 & 0 & v_1 + n_1^{1-\sigma} r_1^2 v_2 \\ 0 & r_1 r_2 & 0 & u_1 + r_1 u_2 \\ 0 & n_1 s_2 r_1^2 + r_2 s_1 & n_1 n_2 r_1^2 r_2^2 & n_1 z_2 r_1^2 + z_1 + s_1 u_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} &(\tau_3(g_1 g_2) \cdot f)(x,t) \\ &= f(x,t) + r_1^{-1} r_2^{-1} (n_1 s_2 r_1^2 + r_2 s_1)(x - u_1 - r_1 u_2) + (n_1 z_2 r_1^2 + z_1 + s_1 u_2). \end{aligned}$$

We then verify the relation $\tau_3(g_1 g_2) \cdot f = \tau_3(g_1) \cdot (g_1 \cdot (\tau_3(g_2) \cdot (g_1^{-1} \cdot f)))$ by using Theorem 3.1 and calculating

$$\begin{aligned} &(\tau_3(g_1) \cdot (g_1 \cdot (\tau_3(g_2) \cdot (g_1^{-1} \cdot f))))(x,t) \\ &= (g_1 \cdot (\tau_3(g_2) \cdot (g_1^{-1} \cdot f)))(x,t) + r_1^{-1} s_1 (x - u_1) + z_1 \\ &= r_1^2 n_1 ((\tau_3(g_2) \cdot (g_1^{-1} \cdot f))(\frac{x - u_1}{r_1}, \frac{t - v_1}{r_1^2 n_1^{1-\sigma}})) + r_1^{-1} s_1 (x - u_1) + z_1 \\ &= r_1^2 n_1 ((g_1^{-1} \cdot f)(\frac{x - u_1}{r_1}, \frac{t - v_1}{r_1^2 n_1^{1-\sigma}}) + r_2^{-1} s_2 (\frac{x - u_1}{r_1} - u_2) + z_2) \\ &\quad + r_1^{-1} s_1 (x - u_1) + z_1 \\ &= r_1^2 n_1 (r_2^2 n_2 f(x,t) + r_2^{-1} s_2 (\frac{x - u_1}{r_1} - u_2) + z_2) + r_1^{-1} s_1 (x - u_1) + z_1 \\ &= (\tau_3(g_1 g_2) \cdot f)(x,t). \end{aligned}$$

□

As a result, consider $\delta_3 : G_3 \times \mathcal{I}_3 \to \mathcal{I}_3$ given by

$$\delta_3(g) \cdot f = \tau_3(g) \cdot (g \cdot f).$$

Explicitly, we see that

$$(\delta_3(\begin{pmatrix} r^2 n^{1-\sigma} & 0 & 0 & v \\ 0 & r & 0 & u \\ 0 & s & r^2 n & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) \cdot f)(x,t) = r^2 n f(\frac{x-u}{r}, \frac{t-v}{r^2 n^{1-\sigma}}) + r^{-1} s (x-u) + z. \tag{5.1}$$

Theorem 5.3. *The (nonlinear) action of G_3 on $\mathcal{I}_3$ given by δ_3 gives a globalization of the local one-parameter group action generated by the Lie point symmetries of the nonlinear potential filtration equation $w_t = \frac{1}{\sigma} w_{xx}^{\sigma}$ with $\sigma > 0$ and $\sigma \neq 1, \frac{1}{3}$.*

Proof. The proof that δ_3 defines an action follows from Lemma 5.2 just as in the proof of Theorem 3.3. Next, let

$$R=\begin{pmatrix}2&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&0\end{pmatrix},\quad S=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\quad N=\begin{pmatrix}1-\sigma&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},$$
$$V=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad U=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad Z=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

be a basis for the Lie algebra of G_3. Using Equation 5.1, it follows that

$$\begin{gathered}
r^2nf(\frac{x-u}{r},\frac{t-v}{r^2n^{1-\sigma}})+r^{-1}s(x-u)+z,\\
(\delta_3(e^{sR})\cdot f)(x,t)=e^{2s}f(e^{-s}x,e^{-2s}t)\\
(\delta_3(e^{sS})\cdot f)(x,t)=f(x,t)+sx\\
(\delta_3(e^{sN})\cdot f)(x,t)=e^{s}f(x,e^{(\sigma-1)s}t)\\
(\delta_3(e^{sV})\cdot f)(x,t)=f(x,t-s)\\
(\delta_3(e^{sU})\cdot f)(x,t)=f(x-s,t)\\
(\delta_3(e^{sZ})\cdot f)(x,t)=f(x,t)+s.
\end{gathered}$$

Applying $\frac{\partial}{\partial s}|_{s=0}$ shows that

$$\begin{gathered}
\frac{\partial}{\partial s}(\delta_3(e^{sR})\cdot f)(x,t)|_{s=0}=-x\frac{\partial}{\partial x}f(x,t)-2t\frac{\partial}{\partial t}f(x,t)+2f(x,t)\\
\frac{\partial}{\partial s}(\delta_3(e^{sS})\cdot f)(x,t)|_{s=0}=x\\
\frac{\partial}{\partial s}(\delta_3(e^{sN})\cdot f)(x,t)|_{s=0}=(\sigma-1)t\frac{\partial}{\partial t}f(x,t)+f(x,t)\\
\frac{\partial}{\partial s}(\delta_3(e^{sV})\cdot f)(x,t)|_{s=0}=-\frac{\partial}{\partial t}f(x,t)\\
\frac{\partial}{\partial s}(\delta_3(e^{sU})\cdot f)(x,t)|_{s=0}=-\frac{\partial}{\partial x}f(x,t)\\
\frac{\partial}{\partial s}(\delta_3(e^{sZ})\cdot f)(x,t)|_{s=0}=1.
\end{gathered}$$

Under the prolongation formalism [25], an easy application of the chain rule shows that the vector field Lie point symmetry

$$h_1(x,t)\frac{\partial}{\partial x}+h_2(x,t)\frac{\partial}{\partial t}+h_3(x,t,w)\frac{\partial}{\partial w}$$

on $\mathbb{R}^2\times\mathbb{R}$ gives rise to a local one-parameter group action on a function f whose partial with respect to s at $s=0$ is given by

$$-h_1(x,t)\frac{\partial}{\partial x}f(x,t)-h_2(x,t)\frac{\partial}{\partial t}f(x,t)+h_3(x,t,f(x,t)).$$

Therefore, the one parameter groups corresponding to $\{R, S, N, V, U, Z\}$ give rise to the symmetry vector fields

$$\begin{gathered} X_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + 2w\frac{\partial}{\partial w}, \quad X_5 = x\frac{\partial}{\partial w} \\ X_6 = (1-\sigma)t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}, \quad X_1 = \frac{\partial}{\partial t} \\ X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial w}. \end{gathered}$$

Since the Lie point symmetries of $w_t = \frac{1}{\sigma} w_{xx}^{\sigma}$ with $\sigma > 0$ and $\sigma \neq 1, \frac{1}{3}$ are spanned by $\{X_1, X_2, X_3, X_4, X_5, X_6\}$, the proof is complete. □

6. $K = 3(w_{xx})^{1/3}$

Consider the case of $K = 3(w_{xx})^{1/3}$. Then the symmetry Lie algebra is seven dimensional and spanned by

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial w} \\ X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w}, \quad X_5 = x\frac{\partial}{\partial w} \\ X_6 = \frac{2}{3}t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}, \quad X_7 = w\frac{\partial}{\partial x}. \end{gathered}$$

It turns out that local action of X_7 does not globalize except in trivial cases.

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}^2$ contain a nonempty open set and let $\mathcal{I} \subseteq \{f : \Omega \to \mathbb{C}\}$ contain a nonzero function. Then the local infinitesimal action of X_7 on $\mathcal{I}$ extends to a global action of a one dimensional Lie group if and only if $\Omega = \mathbb{R} \times T$ for some $T \subseteq \mathbb{R}$ and $\mathcal{I}$ consists of functions constant in x.*

Proof. Consider the local action (see [25]) of $\mathbb{R}$ on $\mathbb{R}^2 \times \mathbb{R}$ generated by the Lie symmetry vector field $X_7 = w\frac{\partial}{\partial x}$, written

$$s \cdot (x, t, w) = (\Xi_{1,s}(x, t, w), \Xi_{2,s}(x, t, w), \Phi_s(x, t, w))$$

for $s \in \mathbb{R}$ in a neighborhood of 0. This action must satisfy the vector field equation

$$\frac{d}{ds}(\Xi_{1,s}(x, t, w), \Xi_{2,s}(x, t, w), \Phi_s(x, t, w)) = (\Phi_s(x, t, w), 0, 0)$$

with initial condition

$$(\Xi_{1,0}(x, t, w), \Xi_{2,0}(x, t, w), \Phi_0(x, t, w)) = (x, t, w).$$

Clearly

$$\Xi_{1,s}(x, t, w), \Xi_{2,s}(x, t, w), \Phi_s(x, t, w)) = (x + sw, t, w).$$

In particular, this action is defined for all $s \in \mathbb{R}$ and requires Ω to be of the form $\mathbb{R} \times T$.

Now suppose G is a Lie group with Lie algebra identified with $\mathbb{R}X_7$ and that G acts on $\mathcal{I}$ in such a way to give a globalization of the local one-parameter group action generated by X_7. Write this action as $g \cdot f$ for $g \in G$ and $f \in \mathcal{I}$. For $f \in \mathcal{I}$, this action is related to the above action by the requirement that $s \cdot \Gamma_f = \Gamma_{\exp_G(sX_7) \cdot f}$ for all $s \in \mathbb{R}$ where $\Gamma_f = \{(x, t, f(x, t)) : (x, t) \in \Omega\} \subseteq \mathbb{R}^2 \times \mathbb{R}$ is the graph of f. More precisely, this means that when $s\cdot(x,t,f(x,t)) = (\widetilde{x}_s, \widetilde{t}_s, \widetilde{v}_s)$, then $(\exp(sX_7)\cdot f)(\widetilde{x}_s,\widetilde{t}_s) = \widetilde{v}_s$ for $s\in\mathbb{R}$ and $(x,t)\in\Omega$. In other words,

$$(\exp_G(sX_7)\cdot f)(x,t) = [\Phi_s\circ(1\times f)]\circ[\Xi_s\circ(1\times f)]^{-1}(x,t). \tag{6.1}$$

In particular, this means that $s\cdot\Gamma_f$ is the graph of some function in $\mathcal{I}$. However, we finish the proof by showing that this is possible if and only if $\mathcal{I}$ consists of functions constant in x.

Arguing via contradiction, suppose there exists $f\in\mathcal{I}$ and $(x_1,t),(x_2,t)\in\Omega$ with $f(x_1,t)\neq f(x_2,t)$. Then consider $s=\frac{x_1-x_2}{f(x_2,t)-f(x_1,t)}$ and compute that

$$\begin{gathered}
\frac{x_1-x_2}{f(x_2,t)-f(x_1,t)}\cdot(x_1,t,f(x_1,t)) = \Big(\frac{x_1 f\,(x_2,t)-x_2 f\,(x_1,t)}{f\,(x_2,t)-f\,(x_1,t)}, t, f(x_1,t)\Big)\\
\frac{x_1-x_2}{f(x_2,t)-f(x_1,t)}\cdot(x_2,t,f(x_2,t)) = \Big(\frac{x_1 f\,(x_2,t)-x_2 f\,(x_1,t)}{f\,(x_2,t)-f\,(x_1,t)}, t, f(x_2,t)\Big).
\end{gathered}$$

Since $s\cdot\Gamma_f$ is the graph of a function, it follows that $f(x_1,t)=f(x_2,t)$ and the proof is complete. $\square$

Since the Lie symmetries of $\{X_1,X_2,X_3,X_4,X_5,X_6\}$ globalize in the same way as Theorem 5.3 (with $\sigma=\frac{1}{3}$) and do not preserve the set of functions constant in x, it follows that $\{X_1,X_2,X_3,X_4,X_5,X_6,X_7\}$ does not exponentiate to give a global action of a Lie group.

Remark. Theorem 6.1 shows there is no *global* action generated by X_7 on nonconstant functions. Of course this result says nothing about the *local* action generated by X_7.

As an example, consider the nonconstant function

$$f(x) = ax+b,$$

$a\neq 0$, which is a solution to $w_t = 3(w_{xx})^{1/3}$. Recalling Equation 6.1, $[\Xi_s\circ(1\times f)](x,t) = (x+s(ax+b),t)$ so that $[\Xi_s\circ(1\times f)]^{-1}(x,t) = (\frac{x-sb}{1+sa},t)$. It follows that X_7 generates a perfectly fine local action on f given by

$$(s\cdot f)(x,t) = a\frac{x-sb}{1+sa}+b = \frac{ax+b}{1+as}$$

which is defined only when $s\neq-\frac{1}{a}$ (and so does not give rise to a global action).

7. $K=\ln(w_{xx})$

Consider the case of $K=3(w_{xx})^{1/3}$. Then the symmetry Lie algebra is six dimensional and spanned by

$$\begin{gathered}
X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x}\\
X_3=\frac{\partial}{\partial w},\quad X_4=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+2w\frac{\partial}{\partial w}\\
X_5=x\frac{\partial}{\partial w},\quad X_6=t\frac{\partial}{\partial t}+(t+w)\frac{\partial}{\partial w}.
\end{gathered}$$

To find a globalization of the corresponding local one-parameter actions, consider the solvable group G_4 given by

$$G_4 = \{\begin{pmatrix} r^2e^n & 0 & 0 & v \\ 0 & r & 0 & u \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : r, s, n, v, u, z \in \mathbb{R}, \quad r > 0\}$$

along with the subgroup

$$D_4 = \{\begin{pmatrix} r^2e^n & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : r, s, n, z \in \mathbb{R}, \quad r > 0\}.$$

Define the character $\chi_4 : D_4 \to \mathbb{C}^\times$ by

$$\chi_4(\begin{pmatrix} r^2e^n & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) = r^2e^n$$

and consider the representation of G_4

$$\operatorname{Ind}_{D_4}^{G_4} \chi_4 = \big\{\varphi \in C^\infty(G_4) : \varphi(gd) = \chi_4(d)^{-1}\varphi(g) \text{ for } g \in G_4, \quad d \in D_4\big\}$$

with G_4-action given by

$$(g_1 \cdot f)(g_2) = f(g_1^{-1} g_2)$$

for $g_i \in G_4$. Using what would be called the noncompact picture if we were working in the semisimple case [23], let

$$\mathcal{I}_4 = \{f \in C^\infty(\mathbb{R}^2) : f(x,t) = \varphi(\begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) \text{ for some } \varphi \in \operatorname{Ind}_{D_4}^{G_4} \chi_4\}.$$

By requiring that the map $\varphi \to f$ be an intertwining operator, $\mathcal{I}_4$ inherits an action of G_4 so that $\mathcal{I}_4 \cong \operatorname{Ind}_{D_4}^{G_4} \chi_4$. Writing

$$\begin{pmatrix} r^2e^n & 0 & 0 & v \\ 0 & r & 0 & u \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & v \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r^2e^n & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we see that φ can be reconstructed from f by

$$\varphi(\begin{pmatrix} r^2e^n & 0 & 0 & v \\ 0 & r & 0 & u \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix}) = r^{-2}e^{-n} f(u,v)$$

so that $\mathcal{I}_4 = C^\infty(\mathbb{R}^2)$.

Theorem 7.1. *The (linear) action of G_4 on $\mathcal{I}_4$ is given by*

$$(\begin{pmatrix} r^2e^n & 0 & 0 & v \\ 0 & r & 0 & u \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot f)(x,t) = r^2e^n f(\frac{x-u}{r}, \frac{t-v}{r^2e^n}).$$

Proof. Observe that

$$
\begin{aligned}
&\begin{pmatrix} r^2e^n & 0 & 0 & v \\ 0 & r & 0 & u \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & r^{-2}e^{-n}(t-v) \\ 0 & 1 & 0 & r^{-1}(x-u) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\times \begin{pmatrix} r^{-2}e^{-n} & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ -nr^{-2}e^{-n} & -r^{-3}e^{-n}s & r^{-2}e^{-n} & r^{-3}e^{-n}s(u-x)+r^{-2}e^{-n}(nv-z-nt) \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

It follows that

$$
\Big(\begin{pmatrix} r^2e^n & 0 & 0 & v \\ 0 & r & 0 & u \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot f\Big)(x,t) = r^2e^n f\Big(\frac{x-u}{r}, \frac{t-v}{r^2e^n}\Big).
$$

where φ corresponds to f under the isomorphism $\mathcal{I}_4 \cong \operatorname{Ind}_{D_4}^{G_4} \chi_4$. $\square$

To complete our picture, let $\tau_4 : G_4 \times \mathcal{I}_4 \to \mathcal{I}_4$ be given by

$$
\Big(\tau_4\Big(\begin{pmatrix} r^2e^n & 0 & 0 & v \\ 0 & r & 0 & u \\ nr^2e^n & s & r^2e^n & z \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big) \cdot f\Big)(x,t) = f(x,t) + r^{-1}s(x-u) + n(t-v) + z.
$$

Lemma 7.2. *The map τ_4 does not define an action of G_4 on $\mathcal{I}_4$. However, it is related to the the original action of G_4 on $\mathcal{I}_4$ (given in Theorem 7.1) by*

$$
\tau_4(g_1g_2) \cdot f = \tau_4(g_1) \cdot (g_1 \cdot (\tau_4(g_2) \cdot (g_1^{-1} \cdot f)))
$$

for $g_i \in G_4$ and $f \in \mathcal{I}_4$.

Proof. Since it is trivial, we leave the proof that τ_4 is not an action to the reader. For the relation of τ_4 to the original action, write

$$
g_1 = \begin{pmatrix} r_1^2e^{n_1} & 0 & 0 & v_1 \\ 0 & r_1 & 0 & u_1 \\ n_1r_1^2e^{n_1} & s_1 & r_1^2e^{n_1} & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} r_2^2e^{n_2} & 0 & 0 & v_2 \\ 0 & r_2 & 0 & u_2 \\ n_2r_2^2e^{n_2} & s_2 & r_2^2e^{n_2} & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

and note that g_1g_2 is the matrix

$$
\begin{pmatrix} r_1^2r_2^2e^{n_1}e^{n_2} & 0 & 0 & v_1 + r_1^2v_2e^{n_1} \\ 0 & r_1r_2 & 0 & u_1 + r_1u_2 \\ e^{n_2}e^{n_1}r_1^2r_2^2\,(n_1+n_2) & r_2s_1 + r_1^2s_2e^{n_1} & r_1^2r_2^2e^{n_1}e^{n_2} & z_1 + s_1u_2 + r_1^2z_2e^{n_1} + n_1r_1^2v_2e^{n_1} \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Thus

$$
\begin{aligned}
&(\tau_1(g_1g_2) \cdot f)(x,t) \\
&= f(x,t) + r_1^{-1}r_2^{-1}(r_2s_1 + r_1^2s_2e^{n_1})(x - u_1 - r_1u_2) \\
&\quad + (n_1+n_2)(t - v_1 - r_1^2v_2e^{n_1}) + (z_1 + s_1u_2 + r_1^2z_2e^{n_1} + n_1r_1^2v_2e^{n_1}).
\end{aligned}
$$

We can verify the relation $\tau_4(g_1g_2)\cdot f=\tau_4(g_1)\cdot(g_1\cdot(\tau_4(g_2)\cdot(g_1^{-1}\cdot f)))$ by using Theorem 7.1 and calculating

$$
\begin{aligned}
&(\tau_4(g_1)\cdot(g_1\cdot(\tau_4(g_2)\cdot(g_1^{-1}\cdot f))))(x,t)\\
&=(g_1\cdot(\tau_4(g_2)\cdot(g_1^{-1}\cdot f)))(x,t)+r_1^{-1}s_1(x-u_1)+n_1(t-v_1)+z_1\\
&=r_1^2e^{n_1}((\tau_4(g_2)\cdot(g_1^{-1}\cdot f))(\frac{x-u_1}{r_1},\frac{t-v_1}{r_1^2e^{n_1}}))+r_1^{-1}s_1(x-u_1)+n_1(t-v_1)+z_1\\
&=r_1^2e^{n_1}\Big((g_1^{-1}\cdot f)(\frac{x-u_1}{r_1},\frac{t-v_1}{r_1^2e^{n_1}})+r_2^{-1}s_2(\frac{x-u_1}{r_1}-u_2)+n_2(\frac{t-v_1}{r_1^2e^{n_1}}-v_2)+z_2\Big)\\
&\quad+r_1^{-1}s_1(x-u_1)+n_1(t-v_1)+z_1\\
&=r_1^2e^{n_1}(r_1^{-2}e^{-n_1}f(x,t)+r_2^{-1}s_2(\frac{x-u_1}{r_1}-u_2)+n_2\Big(\frac{t-v_1}{r_1^2e^{n_1}}-v_2\Big)\\
&\quad+z_2)+r_1^{-1}s_1(x-u_1)+n_1(t-v_1)+z_1\\
&=(\tau_1(g_1g_2)\cdot f)(x,t).
\end{aligned}
$$

□

As a result, consider $\delta_4:G_4\times\mathcal{I}_4\to\mathcal{I}_4$ given by

$$\delta_4(g)\cdot f=\tau_4(g)\cdot(g\cdot f).$$

Explicitly, we see that

$$
\begin{aligned}
&(\delta_4(\begin{pmatrix} r^2e^n & 0 & 0 & v\\ 0 & r & 0 & u\\ nr^2e^n & s & r^2e^n & z\\ 0 & 0 & 0 & 1\end{pmatrix})\cdot f)(x,t)\\
&=r^2e^nf(\frac{x-u}{r},\frac{t-v}{r^2e^n})+r^{-1}s(x-u)+n(t-v)+z.
\end{aligned}
\tag{7.1}
$$

Theorem 7.3. *The (nonlinear) action of G_4 on $\mathcal{I}_4$ given by δ_4 gives a globalization of the local one-parameter group action generated by the Lie point symmetries of the nonlinear potential filtration equation $w_t=\ln(w_{xx})$.*

Proof. The proof that δ_3 defines an action follows from Lemma 7.2 just as in the proof of Theorem 3.3. First we check that δ_1 defines an action. For this use Lemma 3.2 to see that

$$
R=\begin{pmatrix}2&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&0\end{pmatrix},\quad
S=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\quad
N=\begin{pmatrix}1&0&0&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix},
$$

$$
V=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad
U=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad
Z=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}
$$

be a basis for the Lie algebra of G_4. Using Equation 7.1, it follows that

$$
\begin{gathered}
(\delta_4(e^{sR})\cdot f)(x,t)=e^{2s}f(e^{-s}x,e^{-2s}t)\\
(\delta_4(e^{sS})\cdot f)(x,t)=f(x,t)+sx\\
(\delta_4(e^{sN})\cdot f)(x,t)=e^sf(x,e^{-s}t)+e^st
\end{gathered}
$$

$$
\begin{aligned}
(\delta_4(e^{sV})\cdot f)(x,t) &= f(x,t-s)\\
(\delta_4(e^{sW})\cdot f)(x,t) &= f(x-s,t)\\
(\delta_4(e^{sZ})\cdot f)(x,t) &= f(x,t)+s.
\end{aligned}
$$

Applying $\frac{\partial}{\partial s}|_{s=0}$ shows that

$$
\begin{gathered}
\frac{\partial}{\partial s}(\delta_4(e^{sR})\cdot f)(x,t)|_{s=0} = -x\frac{\partial}{\partial t}f(x,t) - 2t\frac{\partial}{\partial t}f(x,t) + 2f(x,t)\\
\frac{\partial}{\partial s}(\delta_4(e^{sS})\cdot f)(x,t)|_{s=0} = x\\
\frac{\partial}{\partial s}(\delta_4(e^{sN})\cdot f)(x,t)|_{s=0} = -t\frac{\partial}{\partial t}f(x,t) + f(x,t) + t\\
\frac{\partial}{\partial s}(\delta_4(e^{sV})\cdot f)(x,t)|_{s=0} = -\frac{\partial}{\partial t}f(x,t)\\
\frac{\partial}{\partial s}(\delta_4(e^{sU})\cdot f)(x,t)|_{s=0} = -\frac{\partial}{\partial x}f(x,t)\\
\frac{\partial}{\partial s}(\delta_4(e^{sZ})\cdot f)(x,t)|_{s=0} = 1.
\end{gathered}
$$

Under the prolongation formalism [25], an easy application of the chain rule shows that the vector field Lie point symmetry

$$
h_1(x,t)\frac{\partial}{\partial x} + h_2(x,t)\frac{\partial}{\partial t} + h_3(x,t,w)\frac{\partial}{\partial w}
$$

on $\mathbb{R}^2\times\mathbb{R}$ gives rise to a local one-parameter group action on a function f whose partial with respect to s at $s=0$ is given by

$$
-h_1(x,t)\frac{\partial}{\partial x}f(x,t) - h_2(x,t)\frac{\partial}{\partial t}f(x,t) + h_3(x,t,f(x,t)).
$$

Therefore the one parameter groups corresponding to $\{R,S,N,V,U,Z\}$ give rise to the symmetry vector fields

$$
\begin{gathered}
X_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + 2w\frac{\partial}{\partial w},\quad X_5 = x\frac{\partial}{\partial w},\\
X_6 = t\frac{\partial}{\partial t} + (t+w)\frac{\partial}{\partial w},\quad X_1 = \frac{\partial}{\partial t},\\
X_2 = \frac{\partial}{\partial x},\quad X_3 = \frac{\partial}{\partial w}.
\end{gathered}
$$

Since the Lie point symmetries of $w_t = \ln(w_{xx})$ in the generic case are spanned by $\{X_1,X_2,X_3,X_4,X_5,X_6\}$, the proof is complete. $\square$

8. Non Lie Point Symmetries

The potential filtration equation also admits a further contact symmetries in two cases. When $K=\arctan(w_{xx})$, then

$$
X_6 = -w_x\frac{\partial}{\partial x} + (t + \frac{1}{2}(x^2 - w_x^2))\frac{\partial}{\partial w} + x\frac{\partial}{\partial w_x}
$$

is the first prolongation of a contact symmetry and when $K=\frac{1}{\lambda}e^{\lambda\arctan(w_{xx})}$, $\lambda>0$, then

$$
X_6' = -\lambda t\frac{\partial}{\partial t} - w_x\frac{\partial}{\partial x} + \frac{1}{2}(x^2 - w_x^2)\frac{\partial}{\partial w} + x\frac{\partial}{\partial w_x}
$$

is the first prolongation a contact symmetry. Neither of these lead to any interesting additional global actions of a Lie group.

Theorem 8.1. *Let $\Omega \subseteq \mathbb{R}^2$ contain a nonempty open set and let $\mathcal{I} \subseteq \{f : \Omega \to \mathbb{C}\}$ be nonempty. Then the local infinitesimal actions of X_6 and X_6' on $\mathcal{I}$ never extends to a global action of a one dimensional Lie group on $\mathcal{I}$.*

Proof. Consider X_6 first and its local action of $\mathbb{R}$ on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ generated by the prolonged Lie symmetry vector field X_6, written

$$s \cdot (x,t,w) = (\Xi_{1,s}(x,t,w,w_x), \Xi_{2,s}(x,t,w,w_x), \Phi_s(x,t,w,w_x), \Psi_s(x,t,w,w_x))$$

for $s \in \mathbb{R}$ in a neighborhood of 0. This action must satisfy the vector field equation

$$\begin{aligned} &\frac{d}{ds}(\Xi_{1,s}(x,t,w,w_x), \Xi_{2,s}(x,t,w,w_x), \Phi_s(x,t,w,w_x), \Psi_s(x,t,w,w_x)) \\ &= (-\Psi_s(x,t,w,w_x), 0, \Xi_{2,s}(x,t,w,w_x) + \frac{1}{2}(\Xi_{1,s}(x,t,w)^2 \\ &\quad - \Psi_s(x,t,w,w_x)^2), \Xi_{1,s}(x,t,w,w_x)) \end{aligned}$$

with initial condition

$$(\Xi_{1,0}(x,t,w,w_x), \Xi_{2,0}(x,t,w,w_x), \Phi_0(x,t,w,w_x), \Psi_0(x,t,w,w_x)) = (x,t,w,w_x).$$

It is easy to verify that

$$\begin{aligned} s \cdot (x,t,w,w_x) = \Big(&x\cos s - w_x \sin s, t, w + st \\ &+ \frac{1}{2}(\sin s)(x^2\cos s - 2vx\sin s - v^2\cos s), x\sin s + w_x\cos s\Big). \end{aligned}$$

With an argument similar to Theorem 6.1 (but looking at $\frac{\partial}{\partial x} f$ instead of f), a necessary requirement for X_6 to exponentiate to a global action on $f \in \mathcal{I}$ is that

$$\{(x\cos s - \frac{\partial}{\partial x} f(x,t)\sin s, t, x\sin s + \frac{\partial}{\partial x} f(x,t)\cos s) : (x,t) \in \Omega\} \tag{8.1}$$

be the graph of a function for each $s \in \mathbb{R}$. To see this cannot happen, fix any $(x,t),(x',t) \in \Omega$ with $x \neq x'$ and pick $s \in (0,\pi)$ so that $\cot s = \frac{\frac{\partial}{\partial x} f(x',t) - \frac{\partial}{\partial x} f(x,t)}{x'-x}$. Then

$$\begin{aligned} x\cos s - \sin s\, f(x,t) &= (x\frac{\frac{\partial}{\partial x} f(x',t) - \frac{\partial}{\partial x} f(x,t)}{x'-x} - \frac{\partial}{\partial x} f(x,t))\sin s \\ &= (\frac{x'\frac{\partial}{\partial x} f(x,t) - x\frac{\partial}{\partial x} f(x',t)}{x-x'})\sin s \end{aligned}$$

and

$$\begin{aligned} x'\cos s - \sin s\, f(x',t) &= (x'\frac{\frac{\partial}{\partial x} f(x',t) - \frac{\partial}{\partial x} f(x,t)}{x'-x} - \frac{\partial}{\partial x} f(x',t))\sin s \\ &= (\frac{x'\frac{\partial}{\partial x} f(x,t) - x\frac{\partial}{\partial x} f(x',t)}{x-x'})\sin s \end{aligned}$$

so that

$$(x\cos s - \sin s\frac{\partial}{\partial x} f(x,t), t) = (x'\cos s - \sin s\frac{\partial}{\partial x} f(x',t), t).$$

However (noting that $\sin s \neq 0$),

$$\big[x\sin s + \cos s\frac{\partial}{\partial x} f(x,t)\big] - \big[x'\sin s + \cos s\frac{\partial}{\partial x} f(x',t)\big]$$

$$
\begin{aligned}
&= \Big([x + \frac{\frac{\partial}{\partial x} f(x',t) - \frac{\partial}{\partial x} f(x,t)}{x'-x} \frac{\partial}{\partial x} f(x,t)] \\
&\quad - [x' + \frac{\frac{\partial}{\partial x} f(x',t) - \frac{\partial}{\partial x} f(x,t)}{x'-x} \frac{\partial}{\partial x} f(x',t)] \Big) \sin s \\
&= \frac{\sin s}{x-x'} \Big((x-x')^2 + \big(\frac{\partial}{\partial x} f(x,t) - \frac{\partial}{\partial x} f(x',t) \big)^2 \Big) \neq 0.
\end{aligned}
$$

It follows that (8.1) cannot be the graph of a function and the proof is complete.

The argument for X_6' is similar. The main difference is that the action on the t-coordinate is now given by $e^{-\lambda s} t$ instead of just t. With this change, it is easy to see that the above argument for X_6 pushes through for X_6'. □

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