Electronic Journal of Differential Equations, Vol. 2009(2009), No. 100, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

POSITIVE ALMOST PERIODIC SOLUTIONS OF NON-AUTONOMOUS DELAY COMPETITIVE SYSTEMS WITH WEAK ALLEE EFFECT

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ABSTRACT. By using Mawhin's continuation theorem of coincidence degree theory, we obtain sufficient conditions for the existence of positive almost periodic solutions for a non-autonomous delay competitive system with weak Allee effect.

1. INTRODUCTION

The Lotka-Volterra type systems have been studied in various fields of epidemiology, chemistry, economics and biological science. In the past few years, there has been increasing interest in studying dynamical characteristics such as stability, persistence and periodicity of Lotka-Volterra type systems. There have been considerable works on the qualitative analysis of Lotka-Volterra type systems with delays; see [5, 6, 7, 8, 10, 18, 17]. Naturally, the study of almost periodic solutions for Lotka-Volterra type systems is important and of great interest.

There are two main approaches to obtain sufficient conditions for the existence and stability of the almost periodic solutions of biological models: One is using the fixed point theorem, Lyapunov functional method and differential inequality techniques [1, 9, 19]; the other is using functional hull theory and Lyapunov functional method [11, 12, 13]. However, to the best of our knowledge, there are very few published papers considering the almost periodic solutions for non-autonomous Lotka-Volterra type systems by applying the method of coincidence degree theory. Motivated by this, in this paper, we apply the coincidence theory to study the existence of positive almost periodic solutions for the following non-autonomous delay competitive systems with weak Allee effect

$$\begin{aligned} \dot{u}_i(t) = u_i(t)\Big[&F_i(t, u_i(t-\tau_{ii}(t))) - \sum_{j=1}^{n} b_{ij}(t)u_j(t) \\ &- \sum_{j=1, j\neq i}^{n} c_{ij}(t)u_j(t-\tau_{ij}(t)) - \sum_{j=1}^{n}\int_{-\sigma_{ij}}^{0} \mu_{ij}(t,s)u_j(t+s)\,\mathrm{d}s\Big], \end{aligned} \tag{1.1}$$

2000 *Mathematics Subject Classification.* 34K14, 92D25.
Key words and phrases. Positive almost periodic solution; coincidence degree; delay; non-autonomous competitive systems.

Submitted April 6, 2009. Published August 19, 2009.
Supported by grant 04Y239A from the Natural Sciences Foundation of Yunnan Province.

where $i = 1, 2, \dots, n$, $u_i(t)$ stands for the ith species population density at time $t \in \mathbb{R}, b_{ij}(t) \geq 0, c_{ij}(t) \geq 0, \tau_{ij}(t)$ are continuous almost periodic functions on R, $\mu_{ij}(t, s)$ are positive almost periodic functions on $\mathbb{R} \times [-\sigma_{ij}, 0]$, continuous with respect to $t \in \mathbb{R}$ and integrable with respect to $s \in [-\sigma_{ij}, 0]$, where σ_{ij} are nonnegative constants, moreover $\int_{-\sigma_{ij}}^{0} \mu_{ij}(t, s)\,\mathrm{d}s = 1, i, j = 1, 2, \dots, n$. The per capita growth rate $F_i \in C(\mathbb{R}^2, \mathbb{R})$ is defined by the form for each $i = 1, 2, \dots, n$,

$$F_i(t, x) = r_i(t) - f_i(t, x)x. \tag{1.2}$$

In this definition, r_i is the natural reproduction rate and f_i represents the inner-specific competition, c_{ij} in (1.1) represents the interspecific competition. In addition, f_i satisfies the following condition for each $i = 1, 2, \dots, n$,

$$\frac{\partial f_i(t, x)}{\partial x} > 0 \quad \text{and } f_i(t, x) \text{ are almost periodic in } t, \tag{1.3}$$

for each $t \in \mathbb{R}$, there exists a constant $\alpha_i > 0$ such that

$$f_i(t, \alpha_i) = 0. \tag{1.4}$$

The situations formulated by $\frac{\partial f_i}{\partial x} > 0$ and $\frac{\partial f_i}{\partial x} < 0$ are called the weak Allee effect and the strong Allee effect respectively. Details about the Allee effect can be found in [14, 15, 16]. The initial condition for (1.1) is

$$u_i(s) = \phi_i(s), \quad i = 1, 2, \dots, n, \tag{1.5}$$

where $\phi_i(s)$ are positive bounded continuous function on $[-\tau, 0]$, $i = 1, 2, \dots, n$ and $\tau = \max_{1 \leq i,j \leq n}\{\max_{t \in \mathbb{R}} |\tau_{ij}(t)|, \sigma_{ij}\}$.

The organization of the rest of this paper is as follows. In Section 2, we introduce some preliminary results which are needed in later sections. In Section 3, we establish our main results for the existence of almost periodic solutions of (1.1). Finally, we make the conclusion in Section 4.

2. Preliminaries

To obtain the existence of an almost periodic solution of (1.1), we firstly make the following preparations.

Definition 2.1 ([3])**.** Let $u(t) : \mathbb{R} \to \mathbb{R}$ be continuous in t. $u(t)$ is said to be almost periodic on $\mathbb{R}$, if, for any $\epsilon > 0$, the set $K(u, \epsilon) = \{\delta : |u(t + \delta) - u(t)| < \epsilon,\ \text{for any } t \in \mathbb{R}\}$ is relatively dense, that is for any $\epsilon > 0$, it is possible to find a real number $l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\delta = \delta(\epsilon)$ in this interval such that $|u(t + \delta) - u(t)| < \epsilon$, for any $t \in \mathbb{R}$.

Definition 2.2. A solution $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$ of (1.1) is called an almost periodic solution if and only if for each $i = 1, 2, \dots, n, u_i(t)$ is almost periodic.

For convenience, we denote $AP(\mathbb{R})$ is the set of all real valued, almost periodic functions on $\mathbb{R}$ and let

$$\wedge(f_j) = \Big\{\widetilde{\lambda} \in \mathbb{R} : \lim_{T \to \infty} \frac{1}{T} \int_0^T f_j(s) e^{-i\widetilde{\lambda}s}\,\mathrm{d}s \neq 0\Big\}, \quad j = 1, 2, \dots, n,$$

$$\mathrm{mod}(f_j) = \Big\{\sum_{i=1}^{N} n_i \widetilde{\lambda}_i : n_i \in Z, N \in N^+, \widetilde{\lambda}_i \in \wedge(f_j)\Big\}, \quad j = 1, 2, \dots, n$$

be the set of Fourier exponents and the module of f_j, respectively, where $f_j(\cdot)$ is almost periodic. Suppose $f_j(t,\phi_j)$ is almost periodic in t, uniformly with respect to $\phi_j \in C([-\tau,0],\mathbb{R})$. $K_j(f_j,\epsilon,\phi_j(s)) = \{\delta : |f_j(t+\delta,\phi_j(s)) - f_j(t,\phi_j(s))| < \epsilon, \forall t \in \mathbb{R}\}$ denote the set of ϵ-almost periods uniformly with respect to $\Phi_j(s) \in C([-\tau,0],\mathbb{R})$. $l_j(\epsilon)$ denote the length of inclusion interval. $m(f_j) = \frac{1}{T}\int_0^T f_j(s)\,\mathrm{d}s$ be the mean value of f_j on interval $[0,T]$, where $T > 0$ is a constant. Clearly, $m(f_j)$ depends on T. $m[f_j] = \lim_{T\to\infty} \frac{1}{T}\int_0^T f_j(s)\,\mathrm{d}s$.

Lemma 2.3 ([3])**.** *Suppose that f and g are almost periodic. Then the following statements are equivalent*

(i) $\mathrm{mod}(f) \supset \mathrm{mod}(g)$,
(ii) *for any sequence $\{t_n^*\}$, if $\lim_{n\to\infty} f(t+t_n^*) = f(t)$ for each $t \in \mathbb{R}$, then there exists a subsequence $\{t_n\} \subseteq \{t_n^*\}$ such that $\lim_{n\to\infty} g(t+t_n) = f(t)$ for each $t \in \mathbb{R}$.*

Lemma 2.4 ([2])**.** *Let $u \in AP(\mathbb{R})$. Then $\int_{t-\tau}^t u(s)\,\mathrm{d}s$ is almost periodic.*

Let X and Z be Banach spaces. A linear mapping $L : \mathrm{dom}(L) \subset X \to Z$ is called Fredholm mapping if its kernel, denoted by $\ker(L) = \{X \in \mathrm{dom}(L) : Lx = 0\}$, has finite dimension and its range, denoted by $\mathrm{Im}(L) = \{Lx : x \in \mathrm{dom}(L)\}$, is closed and has finite codimension. The index of L is defined by the integer $\dim K(L) - \mathrm{codim}\,\mathrm{dom}(L)$. If L is a Fredholm mapping with index 0, then there exists continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}(P) = \ker(L)$ and $\ker(Q) = \mathrm{Im}(L)$. Then $L|_{\mathrm{dom}(L)\cap\ker(P)} : \mathrm{Im}(L) \cap \ker(P) \to \mathrm{Im}(L)$ is bijective, and its inverse mapping is denoted by $K_P : \mathrm{Im}(L) \to \mathrm{dom}(L) \cap \ker(P)$. Since $\ker(L)$ is isomorphic to $\mathrm{Im}(Q)$, there exists a bijection $J : \ker(L) \to \mathrm{Im}(Q)$. Let Ω be a bounded open subset of X and let $N : X \to Z$ be a continuous mapping. If $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact, then N is called L-*compact* on Ω, where I is the identity.

Let L be a Fredholm linear mapping with index 0 and let N be a L-compact mapping on $\overline{\Omega}$. Define mapping $F : \mathrm{dom}(L) \cap \overline{\Omega} \to Z$ by $F = L - N$. If $Lx \neq Nx$ for all $x \in \mathrm{dom}(L) \cap \partial\Omega$, then by using P, Q, K_P, J defined above, the coincidence degree of F in Ω with respect to L is defined by

$$\deg_L(F,\Omega) = \deg(I - P - (J^{-1}Q + K_P(I-Q))N, \Omega, 0),$$

where $\deg(g,\Gamma,p)$ is the Leray-Schauder degree of g at p relative to Γ.

Then The Mawhin's continuous theorem is given as follows:

Lemma 2.5 ([4])**.** *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Z$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume*

(a) *for each $\lambda \in (0,1), x \in \partial\Omega \cap \mathrm{dom}(L), Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega \cap L, QNx \neq 0$;*
(c) $\deg(JNQ, \Omega \cap \ker(L), 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \mathrm{dom}(L)$.

3. Main Result

In this section, we state and prove the main results of this paper. By making the substitution $u_i(t) = \exp\{y_i(t)\}$, $i = 1, 2, \dots, n$, (1.1) can be reformulated as

$$\begin{aligned} \dot{y}_i(t) &= r_i(t) - f_i\big(t, \exp\{y_i(t-\tau_{ii}(t))\}\big) \exp\big\{y_i(t-\tau_{ii}(t))\big\} \\ &\quad - \sum_{j=1}^{n} b_{ij}(t) \exp\big\{y_j(t)\big\} - \sum_{j=1, i\neq j}^{n} c_{ij}(t) \exp\big\{y_j(t-\tau_{ij}(t))\big\} \\ &\quad - \sum_{j=1}^{n} \int_{-\sigma_{ij}}^{0} \exp\big\{y_j(t+s)\big\} ds, \quad i = 1, 2, \dots, n. \end{aligned} \tag{3.1}$$

The initial condition (1.5) can be rewritten as

$$y_i(s) = \ln \phi_i(s) =: \psi_i(s), \quad i = 1, 2, \dots, n \tag{3.2}$$

Set $X = Z = V_1 \oplus V_2$, where

$$\begin{gathered} V_1 = \Big\{ y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in C(\mathbb{R}, \mathbb{R}^n) : y_i(t) \in AP(\mathbb{R}), \\ \operatorname{mod}(y_i(t)) \subset \operatorname{mod}(H_i(t)), \forall \widetilde{\lambda}_i \in \wedge(y_i(t)) \text{ satisfies } |\widetilde{\lambda}_i| > \beta,\ i = 1, 2, \dots, n \Big\}, \\ V_2 = \{ y(t) \equiv (h_1, h_2, \dots, h_n)^T \in \mathbb{R}^n \}, \end{gathered}$$

$$\begin{aligned} H_i(t) &= r_i(t) - f_i\big(t, \exp\{\psi_i(-\tau_{ii}(t))\}\big) \exp\big\{\psi_i(-\tau_{ii}(t))\big\} \\ &\quad - \sum_{j=1}^{n} b_{ij}(t) \exp\big\{\psi_j(0)\big\} - \sum_{j=1, i\neq j}^{n} c_{ij}(t) \exp\big\{\psi_j(-\tau_{ij}(0))\big\} \\ &\quad - \sum_{j=1}^{n} \int_{-\sigma_{ij}}^{0} \mu_{ij}(t,s) \exp\big\{\psi_j(s)\big\} \,\mathrm{d}s \end{aligned}$$

and $\psi_i(\cdot)$ is defined as (3.2), $i = 1, 2, \dots, n$. β is a given constant. For $y = (y_1, y_2, \dots, y_n)^T \in Z$, define $\|y\| = \max_{1\le i\le n} \sup_{t\in\mathbb{R}} |y_i(t)|$.

Lemma 3.1. *Z is a Banach space equipped with the norm $\|\cdot\|$.*

Proof. If $y^{\{k\}} \subset V_1$ and $y^{\{k\}} = (y_1^{\{k\}}, y_2^{\{k\}}, \dots, y_n^{\{k\}})^T$ converges to $\overline{y} = (\overline{y}_1, \overline{y}_2, \dots, \overline{y}_n)^T$, that is $y_j^{\{k\}} \to \overline{y}_j$, as $k \to \infty$, $j = 1, 2, \dots, n$. Then it is easy to show that $\overline{y}_j \in AP(\mathbb{R})$ and $\operatorname{mod}(\overline{y}_j) \in \operatorname{mod}(H_j)$. For any $\widetilde{\lambda}_j \le \beta$, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T y_j^{\{k\}}(t) e^{-i\widetilde{\lambda}_j t} dt = 0, \quad j = 1, 2, \dots, n;$$

therefore,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{y}_j(t) e^{-i\widetilde{\lambda}_j t} dt = 0, \quad j = 1, 2, \dots, n,$$

which implies $\overline{y} \in V_1$. Then it is not difficult to see that V_1 is a Banach space equipped with the norm $\|\cdot\|$. Thus, we can easily verify that x and Z are Banach spaces equipped with the norm $\|\cdot\|$. The proof is complete. □

Lemma 3.2. *Let $L : X \to Z$, $Ly = \dot{y}$, then L is a Fredholm mapping of index 0.*

Proof. Clearly, L is a linear operator and $\ker(L) = V_2$. We claim that $\operatorname{Im}(L) = V_1$. Firstly, we suppose that $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T \in \operatorname{Im}(L) \subset Z$. Then there exist $z^{\{1\}}(t) = (z_1^{\{1\}}(t), z_2^{\{1\}}(t), \dots, z_n^{\{1\}}(t))^T \in V_1$ and constant vector $z^{\{2\}} = (z_1^{\{2\}}, z_2^{\{2\}}, \dots, z_n^{\{2\}})^T \in V_2$ such that

$$z(t) = z^{\{1\}}(t) + z^{\{2\}};$$

that is,

$$z_i(t) = z_i^{\{1\}}(t) + z_i^{\{2\}}, \quad i = 1, 2, \dots, n.$$

From the definition of $z_i(t)$ and $z_i^{\{1\}}(t)$, we can easily see that $\int_{t-\tau}^t z_i(s)\,\mathrm{d}s$ and $\int_{t-\tau}^t z_i^{\{1\}}(s)\,\mathrm{d}s$ are almost periodic function. So we have $z_i^{\{2\}} \equiv 0$, $i = 1, 2, \dots, n$, then $z^{\{2\}} \equiv (0, 0, \dots, 0)^T$, which implies $z(t) \in V_1$, that is $\operatorname{Im}(L) \subset V_1$.

On the other hand, if $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T \in V_1 \backslash \{0\}$, then we have $\int_0^t u_j(s)\,\mathrm{d}s \in AP(\mathbb{R}), j = 1, 2, \dots, n$. If $\widetilde{\lambda}_j \neq 0$, then we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big(\int_0^t u_j(s) ds \Big) e^{-i\widetilde{\lambda}_j t}\,\mathrm{d}t = \frac{1}{i\widetilde{\lambda}_j} \lim_{T\to\infty} \frac{1}{T} \int_0^T u_j(t) e^{-i\widetilde{\lambda}_j t}\,\mathrm{d}t,$$

$j = 1, 2, \dots, n$. It follows that

$$\wedge \Big[\int_0^t u_j(s)\,\mathrm{d}s - m\Big(\int_0^t u_j(s)\,\mathrm{d}s \Big) \Big] = \wedge(u_j(t)), \quad j = 1, 2, \dots, n,$$

hence

$$\int_0^t u(s)\,\mathrm{d}s - m\Big(\int_0^t u(s)\,\mathrm{d}s \Big) \in V_1 \subset X$$

Note that $\int_0^t u(s)\,\mathrm{d}s - m(\int_0^t u(s)\,\mathrm{d}s)$ is the primitive of $u(t)$ in X, we have $u(t) \in \operatorname{Im}(L)$, that is $V_1 \subset \operatorname{Im}(L)$. Therefore, $\operatorname{Im}(L) = V_1$.

Furthermore, one can easily show that $\operatorname{Im}(L)$ is closed in Z and

$$\dim \ker(L) = n = \operatorname{codim} \operatorname{Im}(L);$$

therefore, L is a Fredholm mapping of index 0. The proof is complete. □

Lemma 3.3. *Let $N : X \to Z$, $Ny = (G_1^y, G_2^y, \dots, G_n^y)^T$, where*

$$\begin{aligned} G_i^y = {}& r_i(t) - f_i\big(t, \exp\{y_i(t - \tau_{ii}(t))\}\big) \exp\big\{y_i(t - \tau_{ii}(t))\big\} \\ & - \sum_{j=1}^n b_{ij}(t) \exp\big\{y_j(t)\big\} - \sum_{j=1, i\neq j}^n c_{ij}(t) \exp\big\{y_j(t - \tau_{ij}(t))\big\} \\ & - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 \exp\big\{y_j(t+s)\big\}\,\mathrm{d}s,\ i = 1, 2, \dots, n. \end{aligned}$$

Set

$$\begin{gathered} P : X \to Z, \quad Py = \big(m(y_1), m(y_2), \dots, m(y_n)\big)^T, \\ Q : Z \to Z, \quad Qz = \big(m[z_1], m[z_2], \dots, m[z_n]\big)^T. \end{gathered}$$

Then N is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X.

Proof. Obviously, P and Q are continuous projectors such that

$$\operatorname{Im} P = \ker(L),\ \operatorname{Im}(L) = \ker(Q).$$

It is clear that $(I-Q)V_2 = \{0\}$, $(I-Q)V_1 = V_1$. Hence

$$\operatorname{Im}(I-Q) = V_1 = \operatorname{Im}(L).$$

Then in view of

$$\operatorname{Im}(P) = \ker(L),\ \operatorname{Im}(L) = \ker(Q) = \operatorname{Im}(I-Q),$$

we obtain that the inverse $K_P : \operatorname{Im}(L) \to \ker(P) \cap \operatorname{dom}(L)$ of L_P exists and is given by

$$K_P(z) = \int_0^t z(s)\,\mathrm{d}s - m\Big[\int_0^t z(s)\,\mathrm{d}s\Big].$$

Thus,

$$\begin{gathered} QNy = \big(m[G_1^y], m[G_2^y], \dots, m[G_n^y]\big)^T, \\ K_P(I-Q)Ny = \big(f(y_1) - Q(f(y_1)), f(y_2) - Q(f(y_2)), \dots, f(y_n) - Q(f(y_n))\big)^T, \end{gathered}$$

where

$$f(y_i) = \int_0^t \big(G_i^y - m[G_i^y]\big)\,\mathrm{d}s, \quad i = 1, 2, \dots, n.$$

Clearly, QN and $(I-Q)N$ are continuous. Now we will show that K_P is also continuous. By assumptions, for any $0 < \epsilon < 1$ and any compact set $\phi_i \subset C\big([-\tau, 0], \mathbb{R}\big)$, let $l_i(\epsilon_i)$ be the length of the inclusion interval of $K_i(H_i, \epsilon_i, \phi_i)$, $i = 1, 2, \dots, n$. Suppose that $\{z^k(t)\} \subset \operatorname{Im}(L) = V_1$ and $z^k(t) = (z_1^k(t), z_2^k(t), \dots, z_n^k(t))^T$ uniformly converges to $\overline{z}(t) = (\overline{z}_1(t), \overline{z}_2(t), \dots, \overline{z}_n(t))^T$, that is $z_i^k \to \overline{z}_i$, as $k \to \infty$, $i = 1, 2, \dots, n$. Because of $\int_0^t z^k(s)\,\mathrm{d}s \in Z$, $k = 1, 2, \dots, n$, there exists $\sigma_i (0 < \sigma_i < \epsilon_i)$ such that $K_i(H_i, \sigma_i, \phi_i) \subset K_i(\int_0^t z_i^k(s)ds, \sigma_i, \phi_i)$, $i = 1, 2, \dots, n$. Let $l_i(\sigma_i)$ be the length of the inclusion interval of $K_i(H_i, \sigma_i, \phi_i)$ and

$$l_i = \max\big\{l_i(\epsilon_i), l_i(\sigma_i)\big\}, \quad i = 1, 2, \dots, n.$$

It is easy to see that l_i is the length of the inclusion interval of $K_i(H_i, \sigma_i, \phi_i)$ and $K_i(H_i, \epsilon_i, \phi_i)$, $i = 1, 2, \dots, n$. Hence, for any $t \notin [0, l_i]$, there exists $\xi_t \in K_i(H_i, \sigma_i, \phi_i) \subset K_i(\int_0^t z_i^k(s)\,\mathrm{d}s, \sigma_i, \phi_i)$ such that $t + \xi_t \in [0, l_i]$, $i = 1, 2, \dots, n$.

Hence, by the definition of almost periodic function we have

$$\begin{aligned}
&\Big\| \int_0^t z^k(s)\,\mathrm{d}s \Big\| \\
&= \max_{1\le i\le n} \sup_{t\in\mathbb{R}} \Big| \int_0^t z_i^k(s)\,\mathrm{d}s \Big| \\
&\le \max_{1\le i\le n} \sup_{t\in[0,l_i]} \Big| \int_0^t z_i^k(s)\,\mathrm{d}s \Big| + \max_{1\le i\le n} \sup_{t\notin[0,l_i]} \Big| \int_0^t z_i^k(s)\,\mathrm{d}s - \int_0^{t+\xi_t} z_i^k(s)\,\mathrm{d}s \\
&\quad + \int_0^{t+\xi_t} z_i^k(s)\,\mathrm{d}s \Big| \\
&\le 2\max_{1\le i\le n} \sup_{t\in[0,l_i]} \Big| \int_0^t z_i^k(s)\,\mathrm{d}s \Big| + \max_{1\le i\le n} \sup_{t\notin[0,l_i]} \Big| \int_0^t z_i^k(s)\,\mathrm{d}s - \int_0^{t+\xi_t} z_i^k(s)\,\mathrm{d}s \Big| \\
&\le 2\max_{1\le i\le n} \Big| \int_0^{l_i} z_i^k(s)\,\mathrm{d}s \Big| + \max_{1\le i\le n} \epsilon_i.
\end{aligned} \tag{3.3}$$

From this inequality, we can conclude that $\int_0^t z(s)ds$ is continuous, where $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T \in \mathrm{Im}(L)$. Consequently, K_P and $K_P(I-Q)Ny$ are continuous.

From (3.3), we also have $\int_0^t z(s)\,\mathrm{d}s$ and $K_P(I-Q)Ny$ also are uniformly bounded in $\overline{\Omega}$. Further, it is not difficult to verify that $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)Ny$ is equicontinuous in $\overline{\Omega}$. By the Arzela-Ascoli theorm, we have immediately conclude that $K_P(I-Q)N(\overline{\Omega})$ is compact. Thus N is L-compact on $\overline{\Omega}$. The proof is complete. $\square$

By (1.3), $f_i(t,x)$ can be represented as a power-series at α_i of x, in form of

$$f_i(t,x) = f_i(t,\alpha_i) + \frac{\partial f_i}{\partial x}\Big|_{(t,\alpha_i)} x + o(x), \quad i = 1,2,\dots,n,$$

where $o(x)$ is a higher-order infinitely small quantity of x. By (1.4), we conclude that $f_i(t,\alpha_i) = 0$, $i = 1,2,\dots,n$. For convenience, we denote $\frac{\partial f_i}{\partial x}\big|_{(t,\alpha_i)} := c_{ii}(t)$, $i = 1,2,\dots,n$. By (1.3), $c_{ii}(t) > 0$.

Theorem 3.4. *Assume that*

$$m[r_i(t)] = \lim_{T\to\infty} \frac{1}{T}\int_0^T r_i(t)\,\mathrm{d}t > 0,$$
$$m\Big[\sum_{j=1}^n (b_{ij}(t) + c_{ij}(t))\Big] = \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{j=1}^n (b_{ij}(t) + c_{ij}(t))\,\mathrm{d}t > 0\,.$$

Then (1.1) *has at least one positive almost periodic solution.*

Proof. To use the continuation theorem of coincidence degree theorem to establish the existence of a solution of (3.1), we set Banach space X and Z the same as those in Lemma 3.1 and set mappings L, N, P, Q the same as those in Lemma 3.2 and Lemma 3.3, respectively. Then we can obtain that L is a Fredholm mapping of index 0 and N is a continuous operator which is L-compact on $\overline{\Omega}$.

Now, we are in the position of searching for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation

$$Ly = \lambda Ny, \lambda \in (0,1),$$

we obtain

$$\begin{aligned} \dot{y}_i(t) = \lambda\Big[& r_i(t) - c_{ii}(t)\exp\{y_i(t-\tau_{ii}(t))\} - o(\exp\{2y_i(t-\tau_{ii}(t))\}) \\ & - \sum_{j=1}^n b_{ij}(t)\exp\{y_j(t)\} - \sum_{j=1, i\neq j}^n c_{ij}(t)\exp\{y_j(t-\tau_{ij}(t))\} \\ & - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 \mu_{ij}(t,s)\exp\{y_j(t+s)\}\,\mathrm{d}s\Big], \quad i = 1,2,\dots,n. \end{aligned} \tag{3.4}$$

Assume that $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in X$ is a solution of (3.4) for some $\lambda \in (0,1)$. Denote

$$M_1 = \max_{1\le i\le n} \sup_{t\in\mathbb{R}}\{y_i(t)\}, \quad M_2 = \min_{1\le i\le n} \inf_{t\in\mathbb{R}}\{y_i(t)\},$$

by (3.4), we derive

$$\begin{aligned} m[r_i(t)] = m\Big[& c_{ii}(t)\exp\{y_i(t-\tau_{ii}(t))\} + o(\exp\{2y_i(t-\tau_{ii}(t))\}) \\ & + \sum_{j=1}^n b_{ij}(t)\exp\{y_j(t)\} + \sum_{j=1, i\neq j}^n c_{ij}(t)\exp\{y_j(t-\tau_{ij}(t))\} \\ & + \sum_{j=1}^n \int_{-\sigma_{ij}}^0 \mu_{ij}(t,s)\exp\{y_j(t+s)\}\,\mathrm{d}s\Big], \quad i = 1,2,\dots,n \end{aligned}$$

and consequently

$$m[r_i(t)] \le \exp\{M_1\}\Big\{m\Big[\sum_{j=1}^n (b_{ij}(t) + c_{ij}(t))\Big] + n + 1\Big\}, \quad i = 1,2,\dots,n;$$

that is,

$$M_1 \ge \ln \frac{m[r_i(t)]}{m[\sum_{j=1}^n (b_{ij}(t) + c_{ij}(t))] + n + 1}, \quad i = 1,2,\dots,n. \tag{3.5}$$

Similarly, we can get

$$M_2 \le \ln \frac{m[r_i(t)]}{m[\sum_{j=1}^n (b_{ij}(t) + c_{ij}(t))] + n - 1}, \quad i = 1,2,\dots,n. \tag{3.6}$$

By (3.5) and (3.6), we find that there exist $t_1^i \in \mathbb{R}$, $i = 1,2,\dots,n$ such that $y_i(t_1^i) \le R_1$, where

$$R_1 = \max_{1\le i\le n}\Big| \ln \frac{m[r_i(t)]}{m[\sum_{j=1}^n (b_{ij}(t) + c_{ij}(t))] + n - 1}\Big| + 1.$$

Furthermore, we have

$$
\begin{aligned}
\|y\| &\le \max_{1\le i\le n} |y_i(t_1^i)| + \max_{1\le i\le n} \sup_{t\in\mathbb{R}} \Big| \int_{t_1^i}^{t} \dot{y}_i(s)\,\mathrm{d}s \Big| \\
&\le R_1 + 2 \max_{1\le i\le n} \sup_{t\in[t_1^i, t_1^i + l_i]} \Big| \int_{t_1^i}^{t} \dot{y}_i(s)\,\mathrm{d}s \Big| + \max_{1\le i\le n} \epsilon_i \\
&\le R_1 + 2 \max_{1\le i\le n} \Big| \int_{t_1^i}^{t_1^i + l_i} \dot{y}_i(s)\,\mathrm{d}s \Big| + 1.
\end{aligned} \tag{3.7}
$$

Choose a point $\widetilde{\tau}_i$ such that $\widetilde{\tau}_i - t_1^i \in [l_i, 2l_i] \cap K_i(H_i, \sigma_i, \phi_i)$, where $\sigma_i (0 < \sigma_i < \epsilon_i)$ satisfies $K_i(H_i, \sigma_i, \phi_i) \subset K_i(y_i, \epsilon_i, \phi_i)$, $i = 1, 2, \dots, n$. Integrating (3.4) from t_1^i to $\widetilde{\tau}_i$, we get

$$
\begin{aligned}
&\lambda \int_{t_1^i}^{\widetilde{\tau}_i} \Big[c_{ii}(t) \exp\{y_i(t - \tau_{ii}(t))\} + o(\exp\{2y_i(t - \tau_{ii}(t))\}) + \sum_{j=1}^{n} b_{ij}(t) \exp\{y_j(t)\} \\
&+ \sum_{j=1, i\ne j}^{n} c_{ij}(t) \exp\{y_j(t - \tau_{ij}(t))\} + \sum_{j=1}^{n} \int_{-\sigma_{ij}}^{0} \mu_{ij}(t,s) \exp\{y_j(t+s)\}\,\mathrm{d}s \Big]\,\mathrm{d}t \\
&= \lambda \int_{t_1^i}^{\widetilde{\tau}_i} r_i(t)\,\mathrm{d}t - \int_{t_1^i}^{\widetilde{\tau}_i} \dot{y}_i(t)\,\mathrm{d}t \\
&\le \lambda \int_{t_1^i}^{\widetilde{\tau}_i} |r_i(t)|\,\mathrm{d}t + \epsilon_i, \quad i = 1, 2, \dots, n.
\end{aligned}
$$

From the above inequality and (3.4), we obtain

$$
\begin{aligned}
&\int_{t_1^i}^{\widetilde{\tau}_i} |\dot{y}_i(t)|\,\mathrm{d}t \\
&\le \lambda \int_{t_1^i}^{\widetilde{\tau}_i} |r_i(t)|\,\mathrm{d}t + \lambda \int_{t_1^i}^{\widetilde{\tau}_i} \Big[c_{ii}(t) \exp\{y_i(t - \tau_{ii}(t))\} + o(\exp\{2y_i(t - \tau_{ii}(t))\}) \\
&\quad + \sum_{j=1}^{n} b_{ij}(t) \exp\{y_j(t)\} + \sum_{j=1, i\ne j}^{n} c_{ij}(t) \exp\{y_j(t - \tau_{ij}(t))\} \\
&\quad + \sum_{j=1}^{n} \int_{-\sigma_{ij}}^{0} \mu_{ij}(t,s) \exp\{y_j(t+s)\}\,\mathrm{d}s \Big]\,\mathrm{d}t \\
&\le 2 \int_{t_1^i}^{\widetilde{\tau}_i} |r_i(t)| dt + \epsilon_i \\
&\le 2 \int_{t_1^i}^{\widetilde{\tau}_i} |r_i(t)|\,\mathrm{d}t + 1, \quad i = 1, 2, \dots, n,
\end{aligned}
$$

which together with (3.7) and $\widetilde{\tau} \ge t_1^i + l_i$, $i = 1, 2, \dots, n$, we have $\|y\| \le \overline{R}$, where

$$
\overline{R} = R_1 + 4 \max_{1\le i\le n} \int_0^{\widetilde{\tau}} |r_i(t)| dt + 3.
$$

Clearly, $\overline{R}$ is independent of λ. Take

$$
\Omega = \{ y = (y_1, y_2, \dots, y_n)^T \in X : \|y\| < \overline{R} + 1 \}.
$$

It is clear that Ω satisfies the requirement (a) in Lemma 2.5. when $y \in \partial\Omega \cap \ker(L)$, $y = (y_1, y_2, \dots, y_n)^T$ is a constant vector in $\mathbb{R}^n$ with $\|y\| = \overline{R} + 1$. Then

$$QNy = \big(m[G_1], m[G_2], \dots, m[G_n]\big)^T, \quad y \in X$$

where

$$\begin{aligned} G_i &= r_i(t) - f_i(t, \exp\{y_i\}) \exp\{y_i\} \\ &\quad - \sum_{j=1}^{n} b_{ij}(t) \exp\{y_j\} - \sum_{j=1, i\neq j}^{n} c_{ij}(t) \exp\{y_j\} \\ &\quad - \sum_{j=1}^{n} \int_{-\sigma_{ij}}^{0} \mu_{ij}(t,s) \exp\{y_j\}\, \mathrm{d}s, \quad i = 1, 2, \dots, n, \end{aligned}$$

thus $QNy \neq 0$, which implies that the requirement (b) in Lemma 2.5 is satisfied. Furthermore, take the isomorphism $J : \mathrm{Im}(Q) \to \ker(L), Jz \equiv z$ and let $\Phi(\gamma; y) = -\gamma y + (1-\gamma) JQNy$, then for any $y \in \partial\Omega \cap \ker(L), y^T \Phi(\gamma; y) < 0$, we have

$$\deg\{JQN, \Omega \cap \ker(L), 0\} = \deg\{-y, \Omega \cap \ker(L), 0\} \neq 0.$$

So, the requirement (c) in Lemma 2.5 is satisfied. Hence, (3.1) has at least one almost periodic solution in $\overline{\Omega}$, that is (1.1) has at least one positive almost periodic solution. The proof is complete. □

We remark that when $n = 1$ in (1.1), if $F_1(t, x)$ is a linear function and $\mu_{11}(t) \equiv 0$, then Theorem 3.4 is the same as [13, Theorem 3.1].

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