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WHEN SINGULAR POINTS DETERMINE QUADRATIC SYSTEMS

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Abstract. When one considers a quadratic differential system, one realizes that it depends on 12 parameters of which one can be fixed by means of a time change. One also can notice that fixing 4 finite real singular points plus 3 infinite real ones (all its possible singular points) implies to fix 11 conditions, that is, 11 equations that the parameters must satisfy. Since these conditions are linear with respect to the parameters, it is obvious to think that the system will be determined, except that the fixed conditions are incompatible with a quadratic differential system having finitely many singular points.

In this paper we prove exactly this. That is, if we fix the position of the 7 singular points of a quadratic differential system in a distribution that does not force an infinite number of finite singular points, then the system is completely determined, and consequently its phase portrait is also determined. This determination includes the local behavior of all singular points, even if they are weak focus or centers, the global behavior of separatrices, and even the existence or not of limit cycles. This also implies that limit cycles are sensitive to small perturbations of the coordinates of singular points, even if they are far from the singular points.

The result of the paper goes far beyond this, since we state that this result is independent of the fact that the fixed singular points are real or complex, and it does not mind if the infinite singular points are simple or multiple due to the collision of several infinite singular points. Only when some data is lost due to the collision of finite singular points or to the collision of some finite singular points with infinite ones, this adds free parameters to the set of parameters at the same rate than the number of finite singular points are lost.

1. Introduction and statement of main results

We consider the real polynomial differential systems

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

where P and Q are polynomials in x and y with real coefficients; i.e. $P, Q \in \mathbb{R}[x,y]$. We say that systems (1.1) are *quadratic* if $\max\big(\deg(P), \deg(Q)\big) = 2$.

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Quadratic differential systems have been studied from many different points of view (the following lists clearly are not complete): studying their finite singular points [12, 13, 9, 10, 22, 23, 24, 51, 7], studying their infinite singular points [30, 33, 48], studying systems with limit cycles [18, 17], studying systems with invariant straight lines [37, 38, 39, 47, 46], studying systems with centers [25, 29, 26, 44, 15, 51, 45, 52], and systems with weak focus [5, 32, 6], studying systems with invariant algebraic curves or first integrals [4, 31], classifying phase portraits according to the number of finite singular points [27, 41, 42, 43], classifying phase portraits according to the structural stability of the portrait [3], and many others ways up to more than 1000 papers have been published on these systems [40].

But up to now no one seems to have noticed the relation between the number of parameters of a quadratic differential systems, and the number of conditions that are fixed by determining the situation of the 4 finite singular points and 3 infinite ones that a quadratic system can have. Intuitively one easily realizes that fixing 11 conditions forces a linear system of equations which if it is not incompatible, it depends just on one parameter and it determines uniquely a phase portrait since one parameter can be removed by means of a time change.

This case could be proved by means of simple algebra tools, and so we tried to go a bit further, and realized that it was not important whether the singular points were real or complex. The conditions would be the same. Moreover it did not matter whether the infinite singular points where simple or multiple, meanwhile their multiplicity was due only on the collision of infinite singular points.

It was also clear that when a finite singular point collided with another finite singular point or infinite, we would lose two data (of the coordinates of the singular point) but win one data from the non-hyperbolicity. Thus the system should now have one free parameter. However the conditions are no more linear and thus it is not so obvious that the result should be that. When one adds more collisions between singular points, and specially if several points collide all together, it is even less obvious how many degrees of freedom will be obtained.

In order to compute this we have used the theory of invariants developed by Sibirsky and his disciples (cf. [49], [50], [36], [8], [16]) and completed for the quadratic differential systems by Schlomiuk and Vulpe [48] when dealing with infinite singular points and by Artes, Llibre and Vulpe [7] when dealing with finite singular points.

The main result of this paper is the following.

Theorem 1.1. *We consider a family of quadratic systems* (1.1) *depending on* 12 *parameters. Assume that one parameter is removed via a time rescaling and that all the coordinates of the singularities (finite and infinite, real and/or complex, simple and/or multiple) are fixed and that this does not force an infinite number of finite singular points. Then we get a family of quadratic systems whose number of free parameters is four minus the number of distinct finite singular points (real or complex).*

In Section 2 we introduce some preliminary results and definitions needed for the rest of the work, and which deal mainly with the theory of invariants. The proof of the Main Theorem is split in five sections from the 3 to the 7 since we develop separately the cases with four finite singular points (real or complex) in Section 3, three finite singular points (real or complex) in Section 4, two finite singular points (real or complex) in Section 5, less than two finite singular points in Section 6, and systems with the infinite full of singular points which have been skipped into the previous sections are considered all together in Section 7.

It is worth to note that a quadratic system with 4 distinct finite singular points (real or complex) has its phase portrait completely determined once these four points and the infinite points (real or complex, simple or multiple) are fixed. That is, whether the singular points are saddles, nodes, foci or centers is imbedded by the position of the singular points. Even more, whether a focus is strong or weak, whether there is a separatrix connection or not, an invariant straight line, and even the existence of limit cycles comes determined in this case only by the position of the singular points.

This implies that the perturbation of one singular point (finite or infinite, real or complex) may affect the phase portrait. It may, for example, imply the born or death of limit cycles.

To illustrate this last case, we are going to provide an example. In [18] it is proved that the system:

$$\begin{aligned} \dot{x} &= 1 + xy, \\ \dot{y} &= 0.722 + 15.28x + 8.4y - 12x^2 - 1.398xy + 3y^2, \end{aligned} \tag{1.2}$$

has exactly three concentric limit cycles of visible size around the singular point $p_1 = (1, -1)$. The other finite singular point is $p_2 = (-0.7571298123634432\dots,$ $1.320777472595376\dots)$ and there are two finite complex singular points being $p_{3,4} = (0.515231572848387\dots \pm 0.2544224724528470\dots i,$ $-1.56038873629768\dots \pm 0.770523355317071\dots i)$. The infinite singular points are the $(0, 0)$ of the local chart U_2 and the points $z_1 = (-2.1247979307270, 0)$ and $z_2 = (2.8237979307270, 0)$ of the local chart U_1.

We must remark that the study of the limit cycles of polynomial differential systems (even for quadratic) is still far from be completed, and that numerical tools can only provide evidence of existence of large limit cycles but are useless to detect infinitesimal ones. So we will limit now to talk about these large limit cycles which can be observed in (1.2).

Now we will perturb a little one of the infinite singular points. Concretely we will take $z = z_2 + (\varepsilon, 0)$. With the help of the numerical program P4 [2], we can numerically show that when $\varepsilon = 0.00018$ the three large limit cycles persist, at $\varepsilon = 0.0002$ only one large limit cycle persists, and at $\varepsilon = 0.001$ there is no large limit cycles. Moreover the inner limit cycle of (1.2) has disappeared in a Hopf bifurcation changing the stability of the focus p_1 and the central and outer limit cycles have collapsed in a double semi–stable limit cycle.

It is also remarkable that all known situations of quadratic differential systems having the maximum number of known limit cycles correspond to cases with four finite singular points (two of them complex). That is they correspond to cases that are completely determined (up to a time change) by the location of the singular points. It is also known that some conditions (like having an invariant straight line) restrict the number of possible limit cycles (in the case of one invariant straight line the maximum is one [21, 19], and with two invariant straight lines none is possible [20]). It may happen that the maximum number of limit cycles is also related with the number of parameters which are not determined by the position of the singular points of the system.

2. Some preliminary results and definitions

2.1. **Zero-cycles associated to finite and infinite singularities.** Consider real quadratic systems of the form

$$\begin{aligned}\frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y),\\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y),\end{aligned} \tag{2.1}$$

with homogeneous polynomials p_i and q_i $(i = 0, 1, 2)$ of degree i in x and y, where

$$\begin{aligned} p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2,\\ q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

We associate with the two polynomials $P, Q \in \mathbb{R}[x, y]$ defining systems (2.1), the homogeneous polynomials P^*, Q^* in X, Y, Z of degree 2 with real coefficients defined as follows

$$P^*(X,Y,Z) = Z^2P(X/Z,Y/Z), \quad Q^*(X,Y,Z) = Z^2Q(X/Z,Y/Z),$$

and denote $C^*(X,Y,Z) = YP^*(X,Y,Z) - XQ^*(X,Y,Z)$.

We shall use the notions of *zero-cycle* and *divisor* in order to describe the number and multiplicity of singularities of a quadratic system (2.1) (for the definitions of these notions see [35]). The notions of zero-cycle and divisor were used for classification purposes of planar quadratic differential systems by Pal and Schlomiuk [35], Llibre and Schlomiuk [32], Schlomiuk and Vulpe [48] and by Artes and Llibre and Schlomiuk [6]. Following [35] (see also [48]) we define here the next zero-cycle and divisor.

For a system (S) belonging to family (2.1) we denote $\sigma(P,Q) = \{w \in \mathbb{C}_2 \mid P(w) = Q(w) = 0\}$ and we define the zero-cycle $\mathcal{D}_S^f(P,Q) = \sum_{w\in\sigma(P,Q)} I_w(P,Q)w$, where $I_w(P,Q)$ is the intersection number or multiplicity of intersection at w. It is clear that for a non–degenerate quadratic system $\deg(\mathcal{D}_S^f) = \sum I_w(P,Q) \le 4$. For a degenerate system (i.e. $\gcd(P,Q) \neq$ constant) the zero-cycle $\mathcal{D}_S^f(P,Q)$ is undefined.

Assume that a system (S) is such that $P(x,y)$ and $Q(x,y)$ are relatively prime over $\mathbb{C}$ and that $yp_2 - xq_2$ is not identically zero. The following divisor on the line at infinity $Z = 0$ is then well defined

$$\mathcal{D}_S^\infty = \sum_{w\in\{Z=0\}} \begin{pmatrix} I_w(P^*,Q^*) \\ I_w(C^*,Z) \end{pmatrix} w.$$

We note that the zero-cycle $\mathcal{D}_S^f$ describes the number of finite singularities which could arise from a perturbation of (S) from singularities in the phase plane of (S). On the other hand the divisor $\mathcal{D}_S^\infty$ describes the number of singularities which could arise from a perturbation of (S) from singularities at infinity of (S) in both the finite plane and at infinity.

2.2. **Affine invariant polynomials associated with infinite singularities.** It is known that on the set QS of all quadratic differential systems (2.1) acts the group $Aff(2,\mathbb{R})$ of the affine transformation on the plane (cf. [48]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on QS. We can identify the set QS

of systems (2.1) with a subset of $\mathbb{R}^{12}$ via the map $QS \to \mathbb{R}^{12}$ which associates to each system (2.1) the 12-tuple $(a_{00}, \dots, b_{02})$ of its coefficients.

For the definitions of a GL-comitant and invariant as well as for the definitions of a T-comitant and a CT-comitant we refer the reader to the paper [48]. Here we shall only construct the necessary T-comitants and CT-comitants associated to configurations of finite and infinite singularities (including multiplicities) of quadratic systems (2.1).

Using the so called *transvectant of order k* (see [28], [34]) of two polynomials $f, g \in \mathbb{R}[a, x, y]$,

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \ \frac{\partial^k g}{\partial x^h \partial y^{k-h}},$$

we shall construct the following invariant polynomials

$$\begin{aligned}
C_i(a,x,y) &= y p_i(x,y) - x q_i(x,y), \ \ i = 0,1,2;\\
K(a,x,y) &= \text{Jacob}\big(p_2(x,y), q_2(x,y)\big)/4;\\
\mu_0(a) &= \text{Res}_{\,x}(p_2,q_2)/y^4 = \text{Discrim}\big(K(a,x,y)\big)/16;\\
M(a,x,y) &= 2\ \text{Hess}\big(C_2(a,x,y)\big);\\
\eta(a) &= \text{Discrim}\big(C_2(a,x,y)\big);\\
H(a,x,y) &= -\,\text{Discrim}(\alpha p_2(x,y) + \beta q_2(x,y))\big|_{\{\alpha=y,\,\beta=-x\}};\\
\kappa(a) &= (M,K)^{(2)};\\
\kappa_1(a) &= (M,C_1)^{(2)};\\
L(a,x,y) &= 16K + 8H - M;\\
K_1(a,x,y) &= p_1(x,y) q_2(x,y) - p_2(x,y) q_1(x,y).
\end{aligned}$$

Consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ acting on $\mathbb{R}[a,x,y]$ constructed in [11], where

$$\begin{aligned}
\mathbf{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},\\
\mathbf{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.
\end{aligned}$$

Using this operator we construct the following polynomials

$$\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \ \ i = 1,\dots,4, \quad \text{where} \quad \mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)).$$

These polynomials are in fact comitants of systems (2.1) with respect to the group $GL(2,\mathbb{R})$ (see [11]). Their geometrical meaning is revealed in Lemmas 2.1 and 2.2 below.

2.3. Some useful assertions.

Lemma 2.1 ([11, 48])**.** *The system $P^*(X,Y,Z) = Q^*(X,Y,Z) = 0$ possesses m ($1 \le m \le 4$) solutions $[X_i : Y_i : Z_i]$ with $Z_i = 0$ ($i = 1,\dots,m$) (considered with multiplicities) if and only if for every $i \in \{0,1,\dots,m-1\}$ we have $\mu_i(a,x,y) = 0$ in $\mathbb{R}[a,x,y]$ and $\mu_m(a,x,y) \neq 0$.*

Lemma 2.2 ([11]). *The point $M_0(0,0)$ is a singular point of multiplicity k ($1 \leq k \leq 4$) for a quadratic system (2.1) if and only if for every $i \in \{0,1,\dots,k-1\}$ we have $\mu_{4-i}(a,x,y) = 0$ in $\mathbb{R}[a,x,y]$ and $\mu_{4-k}(a,x,y) \neq 0$.*

Remark 2.3. We note that according to Lemma 2.1 at least two finite singular points of a quadratic system have gone to infinity if and only if $\mu_0 = \mu_1 = 0$.

Remark 2.4. Assume that the polynomials $p_2(x,y)$ and $q_2(x,y)$ of systems (2.1) have a common linear factor $\alpha x + \beta y$. Then these systems have the infinite singular point $N(-\beta,\alpha,0)$ and via a rotation we can assume that this infinite singularity is located in the direction $x = 0$, i.e. x will be a factor of $p_2(x,y)$ and $q_2(x,y)$. This implies $a_{02} = b_{02} = 0$ for systems (2.1).

The next lemma follows directly from [48, Theorem 5.1].

Lemma 2.5. *Assume that at least two finite singular points of a quadratic system with finite number of infinite singularities have gone to infinity (i.e. $\mu_0 = \mu_1 = 0$). Then the configurations of the infinite singularities are given by the following conditions.*

(a) *3 real distinct points*

$$\binom{2}{1}p + \binom{0}{1}q + \binom{0}{1}r \Leftrightarrow \mu_2 \neq 0,\ \kappa \neq 0;$$
$$\binom{1}{1}p + \binom{1}{1}q + \binom{0}{1}r \Leftrightarrow \mu_2 \neq 0,\ \kappa = 0;$$
$$\binom{3}{1}p + \binom{0}{1}q + \binom{0}{1}r \Leftrightarrow \mu_2 = 0,\ \mu_3 \neq 0,\ \kappa \neq 0;$$
$$\binom{2}{1}p + \binom{1}{1}q + \binom{0}{1}r \Leftrightarrow \mu_2 = 0,\ \mu_3 \neq 0,\ \kappa = 0;$$
$$\binom{4}{1}p + \binom{0}{1}q + \binom{0}{1}r \Leftrightarrow \mu_2 = \mu_3 = 0,\ \mu_4 \neq 0,\ \kappa \neq 0;$$
$$\binom{3}{1}p + \binom{1}{1}q + \binom{0}{1}r \Leftrightarrow \mu_2 = \mu_3 = 0,\ \mu_4 \neq 0,\ \kappa = 0,\ K_1 \neq 0;$$
$$\binom{2}{1}p + \binom{2}{1}q + \binom{0}{1}r \Leftrightarrow \mu_2 = \mu_3 = 0,\ \mu_4 \neq 0,\ \kappa = 0,\ K_1 = 0;$$

(b) *one real and 2 complex singular points*

$$\binom{2}{1}p + \binom{0}{1}q^c + \binom{0}{1}r^c \Leftrightarrow \mu_2 \neq 0,\ \kappa \neq 0;$$
$$\binom{0}{1}p + \binom{1}{1}q^c + \binom{1}{1}r^c \Leftrightarrow \mu_2 \neq 0,\ \kappa = 0;$$
$$\binom{3}{1}p + \binom{0}{1}q^c + \binom{0}{1}r^c \Leftrightarrow \mu_2 = 0,\ \mu_3 \neq 0;$$
$$\binom{4}{1}p + \binom{0}{1}q^c + \binom{0}{1}r^c \Leftrightarrow \mu_2 = \mu_3 = 0,\ \mu_4 \neq 0,\ \kappa \neq 0;$$
$$\binom{0}{1}p + \binom{2}{1}q^c + \binom{2}{1}r^c \Leftrightarrow \mu_2 = \mu_3 = 0,\ \mu_4 \neq 0,\ \kappa = 0;$$

(c) *one double and one simple real singular points*

$\binom{2}{1}p+\binom{0}{2}q \Leftrightarrow \mu_2\neq 0,\ \kappa\neq 0;$

$\binom{0}{1}p+\binom{2}{2}q \Leftrightarrow \mu_2\neq 0,\ \kappa=0,\ L\neq 0;$

$\binom{1}{1}p+\binom{1}{2}q \Leftrightarrow \mu_2\neq 0,\ \kappa=0,\ L=0;$

$\binom{3}{1}p+\binom{0}{2}q \Leftrightarrow \mu_2=0,\ \mu_3\neq 0,\ \kappa\neq 0;$

$\binom{0}{1}p+\binom{3}{2}q \Leftrightarrow \mu_2=0,\ \mu_3\neq 0,\ \kappa=0,\ L\neq 0;$

$\binom{2}{1}p+\binom{1}{2}q \Leftrightarrow \mu_2=0,\ \mu_3\neq 0,\ \kappa=0,\ L=0,\ \kappa_1\neq 0;$

$\binom{1}{1}p+\binom{2}{2}q \Leftrightarrow \mu_2=0,\ \mu_3\neq 0,\ \kappa=0,\ L=0,\ \kappa_1=0;$

$\binom{4}{1}p+\binom{0}{2}q \Leftrightarrow \mu_2=\mu_3=0,\ \mu_4\neq 0,\ \kappa\neq 0;$

$\binom{0}{1}p+\binom{4}{2}q \Leftrightarrow \mu_2=\mu_3=0,\ \mu_4\neq 0,\ \kappa=0,\ L\neq 0;$

$\binom{3}{1}p+\binom{1}{2}q \Leftrightarrow \mu_2=\mu_3=0,\ \mu_4\neq 0,\ \kappa=0,\ L=0,\ \kappa_1\neq 0;$

$\binom{1}{1}p+\binom{3}{2}q \Leftrightarrow \mu_2=\mu_3=0,\ \mu_4\neq 0,\ \kappa=0,\ L=0,\ \kappa_1=0,\ K_1\neq 0;$

$\binom{2}{1}p+\binom{2}{2}q \Leftrightarrow \mu_2=\mu_3=0,\ \mu_4\neq 0,\ \kappa=0,\ L=0,\ \kappa_1=0,\ K_1=0;$

(d) *one real triple point:*

$$\binom{2}{3}p \Leftrightarrow \mu_2\neq 0; \quad \binom{3}{3}p \Leftrightarrow \mu_2=0,$$
$$\mu_3\neq 0; \quad \binom{4}{3}p \Leftrightarrow \mu_2=\mu_3=0,\ \mu_4\neq 0.$$

Lemma 2.6. *The polynomial $C_2(x,y)=yp_2(x,y)-xq_2(x,y)\not\equiv 0$ is completely determined up to a time rescaling by the coordinates of the three infinite singular points (real and/or complex, simple or multiple).*

Proof. It is known that the coordinates of infinite singular points of a quadratic system (2.1) are given by the linear factors over $\mathbb{C}$ of the polynomial

$$\begin{aligned} C_2(x,y) &= yp_2(x,y)-xq_2(x,y) \\ &= (u_1x+v_1y)(u_2x+v_2y)(u_3x+v_3y) \\ &= Ux^3+U_vx^2y+V_uxy^2+Vy^3, \end{aligned} \tag{2.2}$$

where $U=u_1u_2u_3$, $U_v=u_1u_2v_3+u_1v_2u_3+v_1u_2u_3$, $V=v_1v_2v_3$, $V_u=u_1v_2v_3+v_1u_2v_3+v_1v_2u_3$. Since $C_2(x,y)\not\equiv 0$ we have $U^2+U_v^2+V_u^2+V^2\neq 0$.

If $U \neq 0$ we may set $U = \theta$ and take $u_1 = \theta$, $u_2 = u_3 = 1$. Then the infinite singular points (real or complex) $N_i(k_i, 1, 0)$ $(i = 1, 2, 3)$ completely determine $v_1 = -\theta k_1$, $v_2 = -k_2$ and $v_3 = -k_3$. Thus $U = \theta$, $U_v = -\theta(k_1 + k_2 + k_3)$, $V_u = \theta(k_1 K_2 + k_1 k_3 + k_2 k_3)$, $V = -\theta k_1 k_2 k_3$.

If $U = 0 \neq U_v$ (i.e. $N_1(0, 1, 0)$ is an infinite singular point of systems (2.1)) we may set $U_v = \theta$ and take $u_1 = 0$, $v_1 = u_3 = 1$, $u_2 = \theta$. Then the remaining infinite singular points (real or complex) determine $v_2 = -\theta k_2$ and $v_3 = -k_3$. As in the previous case (and also the next two) the coefficients of C_2 become completely determined (up to a multiplication of the parameter θ) by the infinite singularities.

If $U = U_v = 0 \neq V_u$ (i.e. $N_1(0, 1, 0)$ is at least a double infinite singular point of systems (2.1)), we may set $V_u = \theta$ and take $u_1 = u_2 = 0$, $v_1 = v_2 = 1$, $u_3 = \theta$ and the remaining infinite singular point (which must be real) determines $v_3 = -\theta k_3$.

If $U = U_v = V_u = 0$ (then $V \neq 0$) we may set $V = \theta$ and the only infinite singularity is the point $N_1(0, 1, 0)$, which is at least of multiplicity three. □

Remark 2.7. We are trying to determine the number of free parameters of quadratic systems once the coordinates of the singular points and their configuration is fixed. In order to simplify calculations we shall use the group $Aff(2, \mathbb{R})$ of affine transformations. We say that an affine transformation is *admissible* if it is defined using only the coordinates of some singular points (finite or infinite). It is clear that an admissible affine transformation keeps the number of free parameters of the respective family of systems.

3. Quadratic systems with four distinct finite singularities

3.1. **Systems with four real simple singular points.** Evidently a quadratic system with four real simple singular points can be brought via an admissible (in the sense of Remark 2.7) affine transformation to the form

$$\begin{aligned} \dot{x} &= cx + dy - cx^2 + 2hxy - dy^2 \equiv P(x, y), \\ \dot{y} &= ex + fy - ex^2 + 2mxy - fy^2 \equiv Q(x, y). \end{aligned} \tag{3.1}$$

Each system from this family has the singular points $M_1(0, 0)$, $M_2(1, 0)$, $M_3(0, 1)$ and $M_4(\alpha, \beta)$. Now will find the dependence among the coefficients of systems (3.1) and the parameters α and β of the fourth singular point. Since $\alpha\beta \neq 0$ (we cannot have three distinct singular points placed on one line) from the identities $P(\alpha, \beta) = Q(\alpha, \beta) = 0$ we obtain

$$h = \frac{c\alpha(\alpha - 1) + d\beta(\beta - 1)}{2\alpha\beta}, \quad m = \frac{e\alpha(\alpha - 1) + f\beta(\beta - 1)}{2\alpha\beta}.$$

Therefore after a time rescaling ($t \to \alpha\beta t_1$) and some re–parametrization ($c\alpha \to c$, $e\alpha \to e$, $d\beta \to d$, $f\beta \to f$) we get the following family of systems

$$\begin{aligned} \dot{x} &= c\,\beta x(1 - x) + d\,\alpha y(1 - y) + \big[c(\alpha - 1) + d(\beta - 1)\big]xy, \\ \dot{y} &= e\,\beta x(1 - x) + f\alpha y(1 - y) + \big[e(\alpha - 1) + f(\beta - 1)\big]xy. \end{aligned} \tag{3.2}$$

Evidently each system of this family possesses the singular points $M_1(0, 0)$, $M_2(1, 0)$, $M_3(0, 1)$ and $M_4(\alpha, \beta)$ and for this family by Lemma 2.1 the following condition must be satisfied

$$\mu_0 = \alpha\beta(\alpha + \beta - 1)(cf - de)^2 \neq 0, \tag{3.3}$$

otherwise the systems become degenerate.

Since for systems (3.2) we have

$$C_2(x,y) = e\beta x^3 + [e(1-\alpha) + f(1-\beta) - c\beta]x^2y + [c(\alpha-1) + d(\beta-1) + f\alpha]xy^2 - d\alpha y^3,$$

considering the factorization (2.2) we get the following system of linear (with respect to the parameters c, d, e and f) equations

$$\begin{gathered}
e\,\beta = U, \quad e(1-\alpha) + f(1-\beta) - c\beta = U_v, \\
d\,\alpha = -V, \quad c(\alpha-1) + d(\beta-1) + f\alpha = V_u.
\end{gathered}$$

Solving this system we obtain

$$\begin{gathered}
c = -\frac{\alpha^2(\alpha-1)U + \alpha^2\beta U_v + \alpha\beta(\beta-1)V_u + \beta(\beta-1)^2V}{\alpha\beta(\alpha+\beta-1)}, \quad d = -\frac{V}{\alpha}, \\
f = \frac{\alpha(\alpha-1)^2U + \alpha\beta(\alpha-1)U_v + \alpha\beta^2)V_u + \beta^2(\beta-1)V}{\alpha\beta(\alpha+\beta-1)}, \quad e = \frac{U}{\beta},
\end{gathered}$$

where $\alpha\beta(\alpha+\beta-1) \neq 0$ by condition (3.3). Thus taking into account Lemma 2.6 we obtain that the coefficients of systems (3.2) depend exclusively on the coordinates of their singular points (finite and infinite).

3.2. **Systems with two real simple and two complex singular points.** By [7] such a quadratic system can be brought via an affine transformation to the form

$$\begin{aligned}
\dot{x} &= a - (a+g)x + gx^2 + 2hxy + ay^2, \\
\dot{y} &= b - (b+l)x + lx^2 + 2mxy + by^2.
\end{aligned} \tag{3.4}$$

These systems possess the singular points $M_{1,2}(0, \pm i)$, $M_3(1,0)$ and the fourth one, whose coordinates we will force to be (α, β).

First we observe that $\alpha \neq 0$, otherwise from the relation $P(0,\beta) = Q(0,\beta) = 0$ we obtain $a(1+\beta^2) = b(1+\beta^2) = 0$ and this leads to degenerate systems. So $\alpha \neq 0$ and we shall consider two subcases $\beta \neq 0$ and $\beta = 0$.

3.2.1. *The case* $\beta \neq 0$. From the identities $P(\alpha,\beta) = Q(\alpha,\beta) = 0$ for systems (3.4) we obtain

$$h = \frac{(\alpha-1)(a-g\alpha) - a\beta^2}{2\alpha\beta}, \quad m = \frac{(\alpha-1)(b-l\alpha) - b\beta^2}{2\alpha\beta}.$$

Therefore we get the following family of systems

$$\begin{aligned}
\dot{x} &= a(1-x+y^2) + gx(x-1) + \frac{1}{\alpha\beta}\big[(\alpha-1)(a-g\alpha) - a\beta^2\big]xy, \\
\dot{y} &= b(1-x+y^2) + \ lx(x-1) + \frac{1}{\alpha\beta}\big[(\alpha-1)(b-l\alpha) - b\beta^2\big]xy.
\end{aligned} \tag{3.5}$$

Evidently each system of this family possesses the singular points $M_{1,2}(0,\pm i)$, $M_3(1,0)$ and $M_4(\alpha,\beta)$ and for this family by Lemma 2.1 the following condition must be satisfied $\mu_0 = \big[(\alpha-1)^2 + \beta^2\big](al-bg)^2/(\alpha\beta^2) \neq 0$.

We shall consider now the infinite singular points. For systems (3.5) we have

$$\begin{aligned}
C_2(x,y) = -lx^3 &+ \frac{1}{\alpha\beta}\big[(\alpha-1)(l\alpha-b) + \beta(g\alpha+b\beta)\big]x^2y \\
&+ \frac{1}{\alpha\beta}\big[(\alpha-1)(a-g\alpha) - \beta(b\alpha+a\beta)\big]xy^2 + ay^3.
\end{aligned}$$

Using the factorization (2.2) and solving the respective linear system with respect to the parameters a, b, g and l we obtain

$$g = \frac{(1-\alpha+\beta^2)^2V + \alpha^2\beta(\alpha-1)U + \alpha\beta^2(U_v\alpha + V_u\beta) - \alpha\beta(\alpha-1)V_u}{\alpha\big[(\alpha-1)^2+\beta^2\big]}, \quad a = V,$$

$$b = -\frac{\beta(1-\alpha+\beta^2)V + \alpha(\alpha-1)^2U + \alpha\beta^2V_u + \alpha\beta(\alpha-1)U_v}{(\alpha-1)^2+\beta^2}, \qquad l = -U.$$

Therefore considering Lemma 2.6 we obtain the family of systems which depend exclusively on the coordinates of their singular points (finite and infinite).

3.2.2. *The case $\beta = 0$.* In this case from the identities $P(\alpha, 0) = Q(\alpha, 0) = 0$ for systems (3.4) we obtain

$$(1-\alpha)(a-g\alpha) = 0, \quad (1-\alpha)(b-l\alpha) = 0,$$

and since $1-\alpha \neq 0$ (otherwise the fourth point coincides with $M_3(1,0)$) we get the following family of systems

$$\dot{x} = g(x-1)(x-\alpha) + 2hxy + g\alpha y^2,$$
$$\dot{y} = l(x-1)(x-\alpha) + 2mxy + l\alpha y^2,$$

for which

$$\mu_0 = 4\alpha(gm-hl)^2 \neq 0, \quad C_2 = -lx^3 + (g-2m)x^2y + (2h-l\alpha)xy^2 + g\alpha y^3.$$

In the same way as above using the factorization (2.2) and solving the corresponding linear system with respect to the parameters g, h, l and m we obtain

$$g = \frac{V}{\alpha}, \quad h = \frac{V_u - \alpha U}{2}, \quad m = \frac{V-\alpha U_v}{2\alpha}, \quad l = -U.$$

Thus these parameters depend linearly on U, U_v, V_u and V and applying the same arguments and actions as above we again obtain the family of systems which depend exclusively on the coordinates of their singular points (finite and infinite).

In short the Main Theorem is proved for systems with two real simple and two complex singular points.

3.3. **Systems with four distinct complex singular points.** First we shall prove the following lemma concerning the complex singular points of quadratic systems.

Lemma 3.1. *If a quadratic system* (2.1) *with real coefficients possesses two complex (conjugated) singular points $M_{1,2}(\alpha \pm i\beta, \gamma \pm i\delta)$ then via an admissible affine transformation this system can be brought to the form*

$$\dot{x} = a + cx + gx^2 + 2hxy + ay^2,$$
$$\dot{y} = b + ex + lx^2 + 2mxy + by^2, \tag{3.6}$$

having the singular points $\widetilde{M}_{1,2}(0, \pm i)$.

Proof. Admit that a non-degenerate system (2.1) possesses the singular points $M_{1,2}(\alpha \pm i\beta, \gamma \pm i\delta)$ $(\beta^2 + \delta^2 \neq 0)$. Via the change $x \leftrightarrow y$ we can assume $\beta \neq 0$ and then we can consider $\beta = 1$ via the rescaling $x \to x/\beta$. Therefore the affine transformation

$$\tilde{x} = -\delta x + y + \alpha\delta - \gamma, \quad \tilde{y} = x - \alpha,$$

replace the singular points $M_{1,2}(\alpha \pm i, \gamma \pm i\delta)$ by the singular points $\widetilde{M}_{1,2}(0, \pm i)$. Then since $P(0,i) = Q(0,i) = 0$ yield $a_{00} + ia_{01} - a_{02} = 0$ and $b_{00} + ib_{01} - b_{02} = 0$. Thus $a_{01} = b_{01} = 0$, $a_{02} = a_{00}$ and $b_{02} = b_{00}$ and setting some new parameters we obtain the canonical system (3.6). □

We shall construct now the canonical form of the family of quadratic systems which possess four distinct complex singular points. By Lemma 3.1 doing an affine transformation we can locate two complex singularities at the points $M_{1,2}(0, \pm i)$. So we shall consider systems (3.6) which besides the singular points $(0, \pm i)$ have the singular points $M_{3,4}(x_{3,4}, y_{3,4})$ where

$$x_{3,4} = A \pm iB, \quad y_{3,4} = C \pm iD, \quad B^2 + D^2 \neq 0,$$

which are also complex. We claim that $x_{3,4} \neq 0$, i.e. $A^2 + B^2 \neq 0$. Indeed, if $A = B = 0$ then the point $(0, C + iD)$ is a singular point of system (3.6) and we have

$$P(x_3, y_3) = a + a(C + iD)^2 = a(1 + C^2 - D^2 + 2iCD) = 0,$$
$$Q(x_3, y_3) = b + b(C + iD)^2 = b(1 + C^2 - D^2 + 2iCD) = 0.$$

Since $C, D \in \mathbb{R}$ and $a^2 + b^2 \neq 0$ (otherwise systems (3.6) become degenerate) we obtain $C = 0$ and $D = \pm 1$. Hence $(x_{3,4}, y_{3,4}) = (0, \pm i)$ and the complex singular points have multiplicity 2. This proves our claim.

We note that the transformation $x_1 = \alpha x$, $y_1 = \beta x + y$ keeps the singular points $(0, \pm i)$ and transforms the singular points $(x_{3,4}, y_{3,4})$ to the points

$$(A\alpha \pm iB\alpha,\ A\beta + C \pm i(B\beta + D)) \tag{3.7}$$

Since $A^2 + B^2 \neq 0$ we shall consider two cases $B \neq 0$, and $B = 0, A \neq 0$.

3.3.1. *The case $B \neq 0$.* Then we may set $\alpha = 1/B$, $\beta = -D/B$ and the singular points (3.7) become $(p \pm i, q)$ $(p, q \in \mathbb{R})$. In this case the relations $P(p \pm i, q) = Q(p \pm i, q) = 0$ yield

$$a(1 + q^2) + g(p^2 - 1) + p(c + 2hq) \pm i(c + 2gp + 2hq) = 0,$$
$$b(1 + q^2) + l(p^2 - 1) + p(e + 2mq) \pm i(e + 2lp + 2mq) = 0.$$

Herein we obtain the relations $a = g(p^2 + 1)/(q^2 + 1)$, $c = -2(gp + hq)$, $b = l(p^2 + 1)/(q^2 + 1)$, $e = -2(lp + mq)$, and this leads to the following family of systems

$$\dot{x} = \frac{g(p^2 + 1)}{q^2 + 1}(1 + y^2) - 2(gp + hq)x + gx^2 + 2hxy,$$
$$\dot{y} = \frac{l(p^2 + 1)}{q^2 + 1}(1 + y^2) - 2(lp + mq)x + lx^2 + 2mxy,$$

with the singular points $M_{1,2}(0, \pm i)$ and $M_{3,4}(p \pm i, q)$. For these systems the condition $gm - lh \neq 0$ must be fulfilled (otherwise systems become degenerate) and calculations yield

$$C_2 = -lx^3 + (g - 2m)x^2y + \Big[2h - l\,\frac{p^2 + 1}{q^2 + 1}\Big]xy^2 + g\,\frac{p^2 + 1}{q^2 + 1}y^3.$$

Using the factorization (2.2) and solving the corresponding linear system with respect to the parameters g, h, l and m we obtain

$$g = \frac{q^2+1}{p^2+1}V, \quad m = \frac{(q^2+1)V - (p^2+1)U_v}{2(p^2+1)},$$
$$h = \frac{(q^2+1)V_u - (p^2+1)U}{2(q^2+1)}, \quad l = -U.$$

Thus taking into consideration Lemma 2.6 we again get a family of systems which depend exclusively on the coordinates of their singular points (finite and infinite).

3.3.2. *The case* $B = 0$. Then $A \neq 0$ and from (3.7) by setting $\alpha = 1/A$ and $\beta = -C/A$ we obtain the singular points $(1, \pm ip)$ with $p = D \neq 0$. In this case the identities $P(1, \pm ip) = Q(1, \pm ip) = 0$ yield

$$a(1-p^2) + c + g \pm 2ihp = 0, \quad b(1-p^2) + e + l \pm 2imp = 0.$$

Herein we obtain the relations

$$h = m = 0, \quad c = a(p^2-1) - g, \quad e = b(p^2-1) - l,$$

which lead to the systems

$$\dot{x} = a + [a(p^2-1) - g]x + gx^2 + ay^2,$$
$$\dot{y} = b + [b(p^2-1) - l]x + lx^2 + by^2$$

with $al - bg \neq 0$. These systems possess the singular points $M_{1,2}(0, \pm i)$, $M_{3,4}(1, \pm ip)$ and we calculate $C_2 = -lx^3 + gx^2y - bxy^2 + ay^3$. Considering the factorization (2.2) we obtain $a = V$, $b = -V_u$, $g = U_v$, $l = -U$. Thus considering Lemma 2.6 we obtain a family of systems which depends exclusively on the coordinates of their singular points (finite and infinite).

4. Quadratic systems with three distinct finite singularities

4.1. **Systems with one double and two simple real finite singular points.** Assume that a quadratic system (2.1) possesses one double and two simple real singular points. By [7] in this case it can be brought via an admissible (in the sense of Remark 2.7) affine transformation to the form

$$\begin{aligned} \dot{x} &= cx + cqy - c\,x^2 + 2hxy - cqy^2, \\ \dot{y} &= ex + eqy - e\,x^2 + 2mxy - eqy^2, \end{aligned} \tag{4.1}$$

with a double singular point $M_1(0,0)$ and two simple ones $M_2(1,0)$ and $M_3(0,1)$. For these systems we calculate

$$\mu_0 = 4q(cm - eh)^2 \neq 0, \quad C_2 = ex^3 - (c+2m)x^2y + (2h + eq)xy^2 - cqy^3.$$

Using the factorization (2.2) and solving the corresponding linear system with respect to the parameters c, e, h and m we obtain

$$c = -\frac{V}{q}, \quad e = U, \quad h = \frac{V_u - qU}{2}, \quad m = \frac{V - qU_v}{2q}. \tag{4.2}$$

Taking into consideration Lemma 2.6 we conclude that systems (4.1) with parameters c, e, h, m defined in (4.2), form a family of systems which depends on the coordinates of the infinite points (which can be complex and multiple) as well as of the parameter q.

4.2. **Systems with one double and two simple complex singular points.** Assume that a quadratic system (2.1) possesses one double and two complex singular points. By [7] in this case it can be brought via an admissible affine transformation to the form

$$\begin{aligned}\dot{x} &= cm\,x + cn\,y + g\,x^2 - cn\,xy + (g+cm)\,y^2,\\ \dot{y} &= em\,x + en\,y + l\,x^2 - en\,xy + (l+em)\,y^2.\end{aligned} \tag{4.3}$$

with a double singular point $M_1(0,0)$ and two complex $M_{2,3}(1,\pm i)$. For these systems we calculate

$$\mu_0 = (cl-eg)^2(m^2+n^2) \neq 0,$$
$$C_2 = -lx^3 + (g+en)x^2y - (l+em+cn)xy^2 + (g+cm)y^3.$$

Considering the factorization (2.2) we obtain

$$\begin{gathered} c = \frac{nU + mV - mU_v - nV_u}{m^2+n^2}, \quad e = \frac{mU - nV + nU_v - mV_u}{m^2+n^2},\\ g = \frac{n^2V - mnU + m^2U_v + mnV_u}{m^2+n^2}, \quad l = -U. \end{gathered} \tag{4.4}$$

Taking into account these relations we observe that all the coefficients of systems (4.3) are homogeneous functions on the parameters m and n. So, since $(m^2+n^2) \neq 0$ considering Lemma 2.6 we conclude that the family of systems (4.3) with parameters c, e, g and l defined in (4.4), is a family which depends on the coordinates of infinite points as well as of one parameter $\gamma = m/n$ or $\gamma = n/m$.

4.3. **Systems with three simple real finite singular points.** In this case only one finite singular point is gone to infinity (i.e. $\mu_0 = 0$) and hence the polynomials $p_2(x,y)$ and $q_2(x,y)$ have a common linear factor. By Remark 2.4 due to an admissible transformation we can assume $a_{02} = b_{02} = 0$ for systems (2.1). Then via a translation one of the finite singularities can be placed at the origin and we obtain the systems

$$\dot{x} = cx + dy + gx^2 + 2hxy \equiv P(x,y), \quad \dot{y} = ex + fy + lx^2 + 2mxy \equiv Q(x,y), \tag{4.5}$$

with $M_1(0,0)$.

We claim that the other real singular points $M_2(x_2,y_2)$ (and $M_3(x_3,y_3)$) of these systems has the coordinate $x \neq 0$. Indeed, if we suppose $x_2 = 0$ then we obtain $P(0,y_2) = d\,y_2 = 0$, $Q(0,y_2) = f\,y_2 = 0$, and since $y_2 \neq 0$ (M_1 and M_2 are distinct) we have $d = f = 0$. However the last relations yield degenerate systems. Thus $x_2 \neq 0$ and via the linear transformation $\bar{x} = x/x_2$ and either $\bar{y} = y$ if $y_2 = 0$, or $\bar{y} = x - x_2y/y_2$ if $y_2 \neq 0$ (which keeps the form (4.5)), we locate the singular point $M_2(x_2,y_2)$ at the point $M_2(1,0)$. In this way we obtain the systems

$$\dot{x} = cx + dy - cx^2 + 2hxy \equiv P(x,y), \quad \dot{y} = ex + fy - ex^2 + 2mxy \equiv Q(x,y) \tag{4.6}$$

with the three singular points $M_1(0,0)$, $M_2(1,0)$ and $M_3(\alpha,\beta)$. Now will find the dependence among the coefficients of system (4.6) and the parameters α and β.

Since $\beta \neq 0$ (we cannot have three distinct singular points placed on the line $y = 0$) and $P(\alpha, \beta) = Q(\alpha, \beta) = 0$, we obtain

$$d = \frac{c\alpha(\alpha - 1) - 2h\alpha\beta}{\beta}, \quad f = \frac{e\alpha(\alpha - 1) - 2m\alpha\beta}{\beta}.$$

Therefore after the time rescaling ($t \to \beta t_1$) and the re-parametrization ($h\beta \to h$, $m\beta \to m$) we get the following family of systems

$$\begin{aligned} \dot{x} &= c\,\beta x(1 - x) + c\,\alpha(\alpha - 1)y + 2h(x - \alpha)y, \\ \dot{y} &= e\,\beta x(1 - x) + e\,\alpha(\alpha - 1)y + 2m(x - \alpha)y. \end{aligned} \tag{4.7}$$

Evidently each system of this family possesses the singular points $M_1(0, 0)$, $M_2(1, 0)$ and $M_3(\alpha, \beta)$ and for this family according to Lemma 2.1 we have $\mu_0 = 0$ and the following condition is satisfied $\mu_1 = 4(cm - eh)^2\alpha\beta(1 - \alpha)x \neq 0$, otherwise the systems become degenerate. For these systems we calculate $C_2 = e\beta x^3 - (c\beta + 2m)x^2y + 2hxy^2$. Using the factorization (2.2) (in this case $V = 0$) and solving the corresponding linear system with respect to the parameters c, e, h and m we can determine only three parameters e, h and either c or m. So we get $e = U/\beta$, $h = V_u/2$, $c = -(2m + U_v)/\beta$. Therefore considering Lemma 2.6 we conclude that the family of systems (4.7) becomes a family which depends on the coordinates of singular points (finite and infinite) as well as on the parameter m.

4.4. **Systems with one real simple and two complex finite singular points.** In this case taking into account Remark 2.4 according to [7] via an admissible affine transformation the quadratic systems can be written into the form

$$\begin{aligned} \dot{x} &= -2(h + gq)x + g(q^2 + 1)y + gx^2 + 2hxy, \\ \dot{y} &= -2(m + lq)x + l(q^2 + 1)y + lx^2 + 2mxy, \end{aligned} \tag{4.8}$$

with three singular points $M_1(0, 0)$ and $M_{2,3}(q \pm i, 1)$. For these systems considering Lemma 2.1 we have $\mu_0 = 0$ and we calculate

$$\mu_1 = 4(gm - hl)^2(q^2 + 1)x \neq 0, \quad C_2 = -lx^3 + (g - 2m)x^2y + 2hxy^2.$$

Then using the factorization (2.2) we can determine only the three parameters l, h and either g or m. So we have $g = 2m + U_v$, $h = V_u/2$, $l = -U$. By Lemma 2.6 the family of systems (4.8) becomes a family which depends on the coordinates of singular points (finite and infinite) as well as on the parameter m.

5. Quadratic systems with two distinct finite singularities

5.1. **Systems with two double real finite singular points.** By [7] and doing an affine transformation (using only the coordinates of the two double real singularities) a such quadratic system can be written into the form

$$\begin{aligned} \dot{x} &= cx + cpy - cx^2 + 2cqxy + ky^2, \\ \dot{y} &= ex + epy - ex^2 + 2eqxy + ny^2, \end{aligned} \tag{5.1}$$

with two double singular points $M_1(0, 0)$ and $M_2(1, 0)$. For these systems we calculate

$$\mu_0 = (cn - ek)^2 \neq 0, \quad C_2 = ex^3 - (c + 2eq)x^2y + (2cq - n)xy^2 + ky^3.$$

Then using the factorization (2.2) and solving the corresponding system of equations with respect to the parameters c, e, n and k we obtain $c = -(2qU + U_v)$, $e = U$,

$k = V$, $n = -(4q^2U + 2qU_v + V_u)$. Considering Lemma 2.6 we get that the family of systems (5.1) becomes a family which depends on the coordinates of infinite singular points as well as on two independent parameters p and q.

5.2. **Systems with two double complex finite singular points.** First we shall construct the respective canonical form for this class of systems. Assume that a quadratic system possesses 2 complex singular points. Then according to [7] (see the proof of Lemma 4.3) these points can be replaced by the points $M_{1,2}(0, \pm i)$. Thus we consider the canonical system

$$\begin{aligned} \dot{x} &= a + cx + gx^2 + 2hxy + ay^2, \\ \dot{y} &= b + ex + lx^2 + 2mxy + by^2, \end{aligned} \tag{5.2}$$

which besides the singular points $(0, \pm i)$ has the singular points $M_{3,4}(x_{3,4}, y_{3,4})$ where

$$\begin{aligned} x_{3,4} &= \left(2d_{25}d_{56} - d_{26}d_{46} \pm 2d_{56}\sqrt{\tilde{D}}\right)/\mu_0, \\ y_{3,4} &= \left(d_{26}d_{45} - d_{24}d_{56} \mp d_{46}\sqrt{\tilde{D}}\right)/\mu_0, \end{aligned}$$

and $\tilde{D} = d_{25}^2 - d_{46}^2 - d_{24}d_{26} + 4d_{45}d_{56}$, $\mu_0 = d_{46}^2 - 4d_{45}d_{56} \neq 0$ (by Lemma 2.1) and

$$\begin{gathered} d_{24} = cl - eg, \quad d_{25} = cm - eh, \quad d_{26} = bc - ae, \\ d_{45} = gm - hl, \quad d_{46} = bg - al, \quad d_{56} = bh - am. \end{gathered}$$

In order to have two double complex singular points it is necessary that $\tilde{D} < 0$. Therefore we can have $x_3 = x_4 = 0$ if and only if $d_{56} = d_{26}d_{46} = 0$. Since the condition $\mu_0 \neq 0$ imply $d_{46} \neq 0$ we obtain $bc - ae = bh - am = 0$. Due to the condition $a^2 + b^2 \neq 0$ (otherwise systems (5.2) become degenerate) we may set $c = ap$, $e = bp$ and $h = aq$, $m = bq$ (where p and q are some new parameters). Then we obtain the needed family of quadratic systems

$$\begin{aligned} \dot{x} &= a + apx + gx^2 + 2aqxy + ay^2, \\ \dot{y} &= b + bpx + lx^2 + 2bqxy + by^2, \end{aligned} \tag{5.3}$$

possessing two double complex singular points $M_{1,2}(0, \pm i)$. For these systems we calculate

$$\mu_0 = (al - bg)^2 \neq 0, \quad C_2 = -lx^3 + (g - 2bq)x^2y + (2aq - b)xy^2 + ay^3.$$

Then using the factorization (2.2) and solving the corresponding system of linear equations with respect to the parameters c, e, n and k we obtain

$$a = V, \quad b = 2qV - V_u, \quad g = 4q^2V + U_v - 2qV_u, \quad l = -U.$$

By Lemma 2.6 we obtain that the family of systems (5.3) becomes a family which depends on the coordinates of the infinite singular points (finite points being fixed) as well as on two independent parameters p and q.

5.3. **Systems with one triple and one simple real finite singular points.** By [7] in this case via an affine transformation (depending only on the coordinates of the two finite singularities) a such quadratic system can be written into the form

$$\begin{aligned} \dot{x} &= cx + cpy - cx^2 + 2hxy + (2hp + cq)y^2, \\ \dot{y} &= ex + epy - ex^2 + 2mxy + (2mp + eq)y^2, \end{aligned} \tag{5.4}$$

with one triple $M_1(0,0)$ and one simple $M_2(1,0)$ real singular points. For these systems we calculate

$$\mu_0 = 4(p^2-q)(cm-eh)^2 \neq 0,$$
$$C_2 = ex^3-(c+2m)x^2y+(2h-2mp-eq)xy^2+(2hp+cq)y^3.$$

Using the factorization (2.2) and solving the corresponding linear system with respect to the parameters c, e, m and h (as $q-p^2\neq 0$) we obtain

$$c = \frac{-p\,qU+p^2U_v-pV_u+V}{q-p^2}, \quad h = \frac{q^2U-p\,qU_v+qV_u-pV}{2(q-p^2)},$$
$$m = \frac{p\,qU-qU_v+pV_u-V}{2(q-p^2)}, \quad e = U.$$

Taking into account Lemma 2.6 we obtain that the family of systems (5.4) becomes a family which depends on the coordinates of the infinite singular points (finite singular points being fixed) as well as on two independent parameters p and q.

5.4. **Systems with one double and one simple real finite singular points.** Since in this case only one finite singularity has gone to infinity we conclude that $p_2(x,y)$ and $q_2(x,y)$ have a linear common factor. Then taking into consideration Remark 2.4, according to [7] via an admissible affine transformation a such quadratic system can be written into the form

$$\begin{aligned} \dot{x} &= cx+cpy-cx^2+2hxy,\\ \dot{y} &= ex+epy-ex^2+2mxy, \end{aligned} \tag{5.5}$$

with one double $M_1(0,0)$ and one simple $M_2(1,0)$ real singular points. By Lemma 2.1 we have $\mu_0=0$ and the condition $\mu_1=-4p(cm-eh)^2x\neq 0$ holds. We calculate $C_2=ex^3-(c+2m)x^2y+2hxy^2$. Then using the factorization (2.2) and solving the corresponding linear system with respect to the parameters c, e and h we obtain $c=-(2m+U_v)$, $e=U$, $h=V_u/2$. By Lemma 2.6 we obtain that the family of systems (5.5) becomes a family which depends on the coordinates of the infinite singular points as well as on two independent parameters m and p.

5.5. **Systems with two simple real finite singular points.** In this case $\mu_0=\mu_1=0$ and two singular points of quadratic systems (2.1) have gone to infinity. Therefore the form of the respective canonical systems depends of the degree of $\gcd(p_2,q_2)$ and we shall investigate two cases $K\neq 0$ and $K=0$.

5.5.1. *The case $K\neq 0$.* Then $\deg(\gcd(p_2,q_2))=1$, i.e. $p_2(x,y)$ and $q_2(x,y)$ have a linear common factor. So taking into consideration Remark 2.4, according to [7] via an admissible affine transformation a quadratic system in this case can be written into the form

$$\begin{aligned} \dot{x} &= cx+dy-cx^2+2dqxy,\\ \dot{y} &= ex+fy-ex^2+2fqxy, \end{aligned} \tag{5.6}$$

which possess simple singular points $M_1(0,0)$ and $M_2(1,0)$. For these systems we calculate $\mu_0=\mu_1=0$, $\mu_2=(cf-de)^2(2q+1)x^2\neq 0$, $K=q(de-cf)x^2\neq 0$ and $C_2=ex^3-(c+2fq)x^2y+2dqxy^2$. Then using the factorization (2.2) and solving the corresponding system of linear equations with respect to the parameters c, e and d (since $q\neq 0$) we get the following relations $c=-2fq-U_v$, $e=U$, $d=V_u/(2q)$.

Thus the family of systems (5.6) becomes a family which depends on the coordinates of the infinite singular points as well as on two independent parameters f and q.

5.5.2. *The case* $K = 0$. In this case the polynomials $p_2(x, y)$ and $q_2(x, y)$ are proportional, i.e. the identity $\alpha p_2(x, y) - \beta q_2(x, y) = 0$ with $\alpha^2 + \beta^2 \neq 0$ holds in $\mathbb{R}[x, y]$. Moreover we can assume $\alpha \neq 0$ due to the change $x \leftrightarrow y$. Since $C_2(a, x, y) = yp_2(x, y) - xq_2(x, y)$ we conclude that one of the factors of C_2 in the factorization (2.2) coincides with $\alpha x - \beta y$. If $M_1(x_0, y_0)$ is a singular point of the quadratic systems, then via the admissible affine transformation $x_1 = \alpha x - \beta y - x_0$ and $y_1 = y - y_0$ we obtain the systems (keeping the old notation)

$$\begin{gathered}\dot{x} = cx + dy \equiv P(x, y),\\ \dot{y} = ex + fy + lx^2 + 2mxy + ny^2 \equiv Q(x, y).\end{gathered}$$

Besides the singular point $M_1(0, 0)$ these systems possess a real simple singular point $M_2(x_2, y_2)$. Since $x_2^2 + y_2^2 \neq 0$, using the transformation $x_1 = x$ and either $y_1 = y/y_2$ (if $y_2 \neq 0$), or $y_1 = x/x_2 + y$ (if $y_2 = 0$) the point M_2 could be placed at the coordinates $(q, 1)$, where q is some parameter. So, since $P(q, 1) = Q(q, 1) = 0$ we get the family of systems

$$\begin{gathered}\dot{x} = c(x - qy),\\ \dot{y} = e(x - qy) - (n + 2mq + lq^2)y + lx^2 + 2mxy + ny^2,\end{gathered} \tag{5.7}$$

for which $\mu_0 = \mu_1 = 0$, $\mu_2 = c^2(n + 2mq + lq^2)q_2(x, y) \neq 0$ and $C_2 = -lx^3 - 2mx^2y - nxy^2$. Then using the factorization (2.2) and solving the corresponding linear system with respect to the parameters l, m and n we get the following relations $l = -U$, $m = -U_v/2$, $n = -V_u$. Thus the family of systems (5.7) becomes a family which depends on the coordinates of the singular points (finite and infinite) as well as on two independent parameters e and c.

5.6. **Systems with two complex finite singular points.** We shall consider the canonical system (3.6) which has the singular points $(0, \pm i)$ and some two other singular points $(x_{3,4}, y_{3,4})$. To construct the needed canonical form we must find out the conditions on the parameters which locate both singular points $(x_{3,4}, y_{3,4})$ at infinity. For this according to Lemma 2.1 the conditions $\mu_0 = \mu_1 = 0$ have to be fulfilled.

For system (3.6) we calculate

$$\mu_0 = d_{46}^2 - 4d_{45}d_{56} = 0, \quad K = d_{45}x^2 + d_{46}xy + d_{56}y^2, \tag{5.8}$$

where $d_{45} = gm - hl$, $d_{46} = gb - al$, $d_{56} = bh - am$ and we shall consider two cases $K \neq 0$ and $K = 0$.

5.6.1. *The case* $K \neq 0$. Since μ_0 is the discriminant of the binary form $K(a, x, y) \neq 0$ then the condition $\mu_0 = 0$ implies $K(a, x, y) = \pm(\alpha x + \beta y)^2 \neq 0$.

On the other hand, computations yield that $\text{Resultant}[K(a, x, y), C_2(x, y), \gamma] = \mu_0 W(a)$, $\gamma = x/y$ or $\gamma = y/x$, where $W(a)$ is a polynomial of degree 4 in the coefficients of systems (2.1). Hence the condition $\mu_0 = 0$ also implies that K and C_2 have a common non–constant factor, which in this case is $\alpha x + \beta y$. Hence this factor indicates a real infinite singular point and we claim that this point cannot be in the direction $x = 0$, i.e. $\beta \neq 0$.

Indeed suppose $\beta = 0$. Then $K = \tilde{\alpha}x^2$ and from (5.8) we obtain $d_{56} = d_{46} = 0$, $d_{45} \neq 0$. Therefore we obtain the relations $d_{45} = gm - hl \neq 0$, $d_{46} = gb - al = 0$,

$d_{56} = bh - am = 0$, which imply $a = b = 0$. This leads to degenerate systems (3.6). So our claim is proved and $\beta \neq 0$.

Now via the admissible transformation $x_1 = x$, $y_1 = \alpha x/\beta + y$ (which keeps the singular points $(0, \pm i)$ and depends only on the coordinates of the infinite points) we obtain $K = d_{56}y^2 \neq 0$. So we have reached the relations $d_{45} = d_{46} = 0$ and hence we have $d_{45} = gm - hl = 0$, $d_{46} = gb - al = 0$, $d_{56} = bh - am \neq 0$. Herein we obtain $g = l = 0$. Then for system (3.6) we calculate $\mu_1 = -4(bh - am)(eh - cm)y$ and the condition $\mu_1 = 0$ implies $eh - cm = 0$. Since the condition $d_{56} = bh - am \neq 0$ yields $m^2 + h^2 \neq 0$ we may set $c = hq$ and $e = mq$. Thus we get the following systems

$$\begin{aligned} \dot{x} &= a + hqx + 2hxy + ay^2, \\ \dot{y} &= b + mqx + 2mxy + by^2, \end{aligned} \tag{5.9}$$

which have the singular points $M_{1,2}(0, \pm i)$ and the other two singular points have gone to infinity. For these systems we calculate $\mu_0 = \mu_1 = 0$, $\mu_2 = (am - bh)^2(q^2 + 4)y^2 \neq 0$ and $C_2 = -2mx^2y + (2h - b)xy^2 + ay^3$. Then using the factorization (2.2) and solving the corresponding linear system with respect to the parameters a, b and m we get $a = V$, $b = 2h - V_u$, $m = -U_v/2$. Thus the family of systems (5.9) becomes a family which depends on the coordinates of the singular points (finite and infinite) as well as on two independent parameters h and q.

5.6.2. *The case* $K = 0$. In this case the polynomials $p_2(x, y)$ and $q_2(x, y)$ are proportional and as above (see subsection 5.5.2) via an admissible linear transformation we can force $a_{20} = a_{11} = a_{02} = 0$. Therefore we obtain the systems

$$\begin{aligned} \dot{x} &= a + cx + dy \equiv P(x, y), \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 \equiv Q(x, y), \end{aligned}$$

which need to possess two complex singular points $M_{1,2}(A \pm iB, C \pm iD)$ with $B^2 + D^2 \neq 0$, and we shall consider two subcases $B \neq 0$ and $B = 0$.

Subcase $B \neq 0$. Using the transformation $x_1 = x/B$ and $y_1 = -Dx + By$ we place these points at $\tilde{M}_{1,2}(p \pm i, q)$ where p and q are some parameters. So, since $P(p + i, q) = Q(p + i, q) = 0$ we get the family of systems

$$\begin{aligned} \dot{x} &= d(y - q), \\ \dot{y} &= l(p^2 + 1) - q(f + nq) - 2(lp + mq)x + fy + lx^2 + 2mxy + ny^2, \end{aligned} \tag{5.10}$$

for which $\mu_0 = \mu_1 = 0$, $\mu_2 = d^2l(lx^2 + 2mxy + ny^2) \neq 0$. For these systems we calculate $C_2 = -lx^3 - 2mx^2y - nxy^2$. Then using the factorization (2.2) and solving the corresponding linear system with respect to the parameters l, m and n we get $l = -U$, $m = -U_v/2$ and $n = -V_u$. Thus the family of systems (5.10) becomes a family which depends on the coordinates of the singular points (finite and infinite) as well as on two independent parameters d and f.

Subcase $B = 0$. Then $D \neq 0$ and via the rescaling $y \to y/D$ we get the singular points $\tilde{M}_{1,2}(p, q \pm i)$ where $p = A$ and $q = C/D$. So, from $P(p, q + i) = Q(p, q + i) = 0$ we get the family of systems

$$\begin{aligned} \dot{x} &= c(x - p), \\ \dot{y} &= n(q^2 + 1) - p(e + lp) + ex - 2(mp + nq)y + lx^2 + 2mxy + ny^2, \end{aligned} \tag{5.11}$$

for which $\mu_0 = \mu_1 = 0$, $\mu_2 = c^2n(lx^2 + 2mxy + ny^2) \neq 0$. For these systems we obtain $C_2 = -lx^3 - 2mx^2y - nxy^2$, and taking into consideration the factorization

(2.2) and solving the corresponding linear system with respect to the parameters l, m and n we get $l = -U$, $m = -U_v/2$ and $n = -V_u$. Thus the family of systems (5.11) becomes a family which depends on the coordinates of the singular points (finite and infinite) as well as on two independent parameters c and e.

6. Systems with at most one finite singular point

In this section we shall use another point of view. Since this family of systems has at most one finite singularity, in order to use admissible (in the sense of Remark 2.7) affine transformations we shall use the possible configurations of infinite singular points.

6.1. **The case of three real infinite singular points.** In this case the polynomial $C_2 = yP(x,y) - xQ(x,y)$ has three real linear factors. Therefore via a linear transformation this binary cubic form can be written in the form $C_2 = xy(x-y)$. We note that the applied transformation is admissible (see Remark 2.7) because it depends only on the coordinates of infinite singularities of systems (2.1). Then using the factorization (2.2) and a time rescaling we get the systems

$$\dot{x} = a + cx + dy + gx^2 + (h-1)xy, \quad \dot{y} = b + ex + fy + (g-1)xy + hy^2, \quad (6.1)$$

with $\mu_0 = gh(g+h-1)$.

6.1.1. *The case* $\mu_0 \neq 0$. By Lemma 2.1 the unique finite singularity must be of multiplicity four. Translating this point to the origin of coordinates we get the family of systems

$$\dot{x} = cx + dy + gx^2 + (h-1)xy, \quad \dot{y} = ex + fy + (g-1)xy + hy^2. \quad (6.2)$$

Clearly if the singular point $(0,0)$ has at least multiplicity 2 the condition $cf - de = 0$ must hold.
Subcase $d \neq 0$. Then $e = cf/d$ and for systems (6.2) calculations yield

$$\mu_4 = \mu_3 = 0, \quad \mu_2 = \frac{\mathcal{F}_1}{d}\big[fgx^2 + (d - f - dg + fh)xy - dhy^2\big],$$

where $\mathcal{F}_1 = f(dg + c - ch) - c(d - dg + ch)$. By Lemma 2.2 in order to have a point of multiplicity 4 we must force the conditions $\mu_2 = \mu_1 = 0$ to be fulfilled. Since $\mu_0 \neq 0$ (i.e. $h \neq 0$) the condition $\mu_2 = 0$ is equivalent to $\mathcal{F}_1 = 0$.

We claim that for $dg + c - ch = 0$ we cannot have a point of multiplicity 4. Indeed, if $g = c(h-1)/d$ and we get the contradiction $\mu_0 = c(c+d)h(h-1)^2/d^2 \neq 0$ and $\mathcal{F}_1 = -c(c+d) = 0$. So $dg + c - ch \neq 0$ and the condition $\mathcal{F}_1 = 0$ gives $f = c(d - dg + ch)/(dg + c - ch)$. Then we calculate

$$\mu_1 = \frac{\mathcal{F}_2}{d(dg + c - ch)}\big[g(dg - d - ch)x + h(dg + c - ch)y\big],$$

where $\mathcal{F}_2 = (dg - ch)^2 - c^2h - d^2g$. Since $h(dg + c - ch) \neq 0$ the condition $\mu_1 = 0$ is equivalent to $\mathcal{F}_2 = 0$. We observe that $\text{Discrim}[\mathcal{F}_2, c] = 4d^2gh(g+h-1) = 4d^2\mu_0$. So for the existence of real parameters c, d, h and g in order to have a point of multiplicity 4 for a non–homogeneous quadratic system (2.1) it is necessary $\mu_0 > 0$. Then we have $c_{1,2} = (dgh \pm d\sqrt{gh(g+h-1)})/(h(h-1)) = d\tilde{c}_{1,2}$, and this leads to the families of systems

$$\dot{x} = dc_0x + dy + gx^2 + (h-1)xy,$$

$$\dot{y} = \frac{dc_0^2(1-g+c_0h)}{c_0+g-c_0h}x + \frac{dc_0(1-g+c_0h)}{c_0+g-c_0h}y + (g-1)xy + hy^2,$$

where c_0 is one of the values $\tilde{c}_{1,2}$. Therefore we get two families of systems, each of them depending on 3 parameters. However fixing the coordinates of three distinct real infinite singularities and fixing the multiplicity 4 for finite singularity we automatically get only one of these families, which depends on three parameters.
Subcase $d = 0$. Then we have $cf = 0$ and we claim that in order to have the singular point $M_0(0,0)$ of multiplicity 4 it is necessary that $c = 0$. Indeed if $c \neq 0$ then $f = 0$ and for systems (6.2) we get $\mu_4 = \mu_3 = 0$ and $\mu_2 = ch\big[-egx^2 + (e-c+cg-eh)xy + chy^2\big]$. Since $\mu_0 \neq 0$ (i.e. $h \neq 0$) we obtain $ch \neq 0$ and this yields $\mu_2 \neq 0$.

Now $c = 0$ and for systems (6.2) we calculate $\mu_4 = \mu_3 = 0$ and $\mu_2 = f(e+fg-eh)\big[gx+(h-1)y\big]x$. In this case the condition $\mu_2 = 0$ implies $f(e+fg-eh) = 0$. Then calculations yield either $\mu_1 = eh(1-h)\big[gx+(h-1)y\big]$ if $f = 0$, or $\mu_1 = e(h-1)(g+h-1)x$ if $f = e(h-1)/g$. Since $\mu_0 \neq 0$ in both cases we get $f = 0 = e(h-1)$. Therefore in the case $f = e = 0$ as well as in the case $f = h-1 = 0$ we obtain a family of quadratic systems depending on two parameters.

6.1.2. *Case* $\mu_0 = 0$, $\mu_1 \neq 0$. The condition $\mu_0 = 0$ yields $gh(g+h-1) = 0$ and without loss of generality we may assume that for systems (6.1) the condition $g = 0$ holds. Indeed, if $h = 0$ (respectively $g+h-1 = 0$) we can apply the admissible linear transformation which sends the straight line $y = 0$ to $x = 0$ (respectively $y = 0$ to $y = x$). So assuming $g = 0$ and translating the singular point to the origin of coordinates we get the family of systems

$$\dot{x} = cx + dy + (h-1)xy, \quad \dot{y} = ex + fy - xy + hy^2, \tag{6.3}$$

for which $\mu_1 = h(h-1)(e-eh-c)y \neq 0$ because $(0,0)$ must be of multiplicity 3. Clearly in order to have at least of multiplicity 2 the condition $cf - de = 0$ must hold.
Subcase $d \neq 0$. Then $e = cf/d$ and for systems (6.2) calculations yield $\mu_4 = \mu_3 = 0$ and $\mu_2 = c\big[f(1-h) - ch - d\big]\big[(fh-f+d)x - dhy\big]y/d$. By Lemma 2.2 to have a point of multiplicity 3 we must force the condition $\mu_2 = 0$ to be fulfilled. Since $\mu_1 = ch(1-h)(fh-f+d)y/d \neq 0$ (i.e. $ch \neq 0$) the condition $\mu_2 = 0$ is equivalent to $f(1-h) - ch - d = 0$ and this gives $c = (f - fh - d)/h$. Thus we obtain the family depending on three parameters d, f and h.
Subcase $d = 0$. Then we have $cf = 0$ and we claim that to in order to have the singular point $M_0(0,0)$ of multiplicity 3 it is necessary $c = 0$. Indeed, if $c \neq 0$ then $f = 0$ and for systems (6.3) we get $\mu_4 = \mu_3 = 0$ and $\mu_2 = ch\big[(e-eh-c)x + chy\big]y$. Since $\mu_1 \neq 0$ (i.e. $h \neq 0$) we obtain $ch \neq 0$ and this yields $\mu_2 \neq 0$. Thus $c = 0$ and in this case we calculate $\mu_1 = -eh(h-1)^2y$ and $\mu_2 = -ef(h-1)^2xy$, and by $\mu_1 \neq 0$ the condition $\mu_2 = 0$ gives $f = 0$. So we get the family of systems depending on two parameters e and h.

6.1.3. *Case* $\mu_0 = \mu_1 = 0$. In this case according to Lemma 2.1 at least two finite singularities have gone to infinity. As it was shown above from $\mu_0 = gh(g+h-1) = 0$ without loss of generality we may assume that for systems (6.1) the condition $g = 0$ holds. Then for these systems calculations yield

$$\mu_1 = (c-e+eh)h(1-h)y, \quad \kappa = 16h(1-h). \tag{6.4}$$

By Lemma 2.5 for $\mu_0 = \mu_1 = 0$ the configurations of infinite singularities are governed by the invariant polynomial $\kappa(a)$.

Subcase $\kappa \neq 0$. Then $h(h-1) \neq 0$ and he condition $\mu_1 = 0$ yields $c = e(1-h)$. Thus we get the 4–parameter family of systems

$$\dot{x} = a + e(1-h)x + dy + (h-1)xy, \quad \dot{y} = b + ex + fy - xy + hy^2, \tag{6.5}$$

for which

$$\mu_2 = h(h-1)\big[a + de + (h-1)(b + ef + e^2h)\big]y^2. \tag{6.6}$$

We separate the proof of this subcase in three pieces.
First $\mu_2 \neq 0$. In this case systems possess exactly one real singular point of multiplicity 2. Then translating this point to the origin of coordinate we obtain the systems

$$\dot{x} = e(1-h)x + dy + (h-1)xy, \quad \dot{y} = ex + fy - xy + hy^2, \tag{6.7}$$

for which $\mu_0 = \mu_1 = 0$, $\mu_2 = eh(h-1)\big[d + (h-1)(f + eh)\big]y^2$, $\mu_3 = e(d - f + fh)\big[(d + (h-1)(f + eh))x - dhy\big]y^2$ and $\mu_4 = 0$. By Lemma 2.2 in order to have a double point at the origin of coordinates we must force $\mu_3 = 0$. Since $\mu_2 \neq 0$ we obtain $d = f(1-h)$ and we get the family of systems

$$\dot{x} = (1-h)(ex + fy - xy), \quad \dot{y} = ex + fy - xy + hy^2,$$

which depends on three parameters e, f and h.
Second $\mu_2 = 0$ and $\mu_3 \neq 0$. In this case by Lemma 2.2 the singular point $M_0(0,0)$ of systems (6.7) is a simple real one. The conditions $\mu_2 = 0$ and $\mu_3 \neq 0$ imply $d = (1-h)(f + eh)$. Thus the family of systems (6.7) becomes again a family depending of three parameters e, f and h.
Third $\mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$. Since there are no finite singularities and $\mu_0 = \mu_1 = 0$, we shall consider systems (6.5) for which according to (6.6) the conditions $\mu_2 = 0$ and $\kappa \neq 0$ (i.e. $h(h-1) \neq 0$) yield $a = -de + (1-h)(b + ef + e^2h)$. Then we calculate

$$\mu_3 = h(1-h)(b + ef + e^2h)\big[d + (h-1)(f + 2eh)\big]y^3,$$
$$\mu_4 = (b + ef + e^2h)y^3\, W(b, e, f, h, x, y)$$

where $W(b, e, f, h, x, y)$ is a linear homogeneous polynomial in x and y. Taking into consideration the condition $\kappa\mu_4 \neq 0$ the relation $\mu_3 = 0$ yields $d = (1-h)(f + 2eh)$. In such a way we get a family of systems depending on four parameters b, e, f and h.

Subcase $\kappa = 0$. Then from (6.4) for systems (6.1) with $g = 0$ we have $h(h-1) = 0$. Without loss of generality we can assume $h = 0$ (if $h = 1$ we can apply the linear transformation which sends the straight line $x = 0$ to $y = x$). Thus we get the systems

$$\dot{x} = a + cx + dy - xy, \quad \dot{y} = b + ex + fy - xy, \tag{6.8}$$

for which $\mu_0 = \mu_1 = 0$ and $\mu_2 = (c-e)(f-d)xy$. We separate the proof of this subcase in three pieces.
First $\mu_2 \neq 0$. Then these systems possess exactly one real singular point of multiplicity 2 and translating it to the origin of coordinates we obtain the systems

$$\dot{x} = cx + dy - xy, \quad \dot{y} = ex + fy - xy, \tag{6.9}$$

for which the condition $cf - de = 0$ must be forced in order to have a double point. Since $c^2 + e^2 \neq 0$ (otherwise we get degenerate systems) without loss of generality we may assume $f = qe$ and $d = qc$. In such of way we get the family of systems

$$\dot{x} = cx + qcy - xy, \quad \dot{y} = ex + qey - xy, \tag{6.10}$$

which depends on three parameters c, e and $q \neq 0$.
Second $\mu_2 = 0$ and $\mu_3 \neq 0$. Then the singular point $M_0(0,0)$ of systems (6.9) is simple and we must force the condition $\mu_2 = (c-e)(f-d)xy = 0$. This yields either $c = e$ or $f = d$, and in each of these cases we get a family of systems depending on three free parameters.
Third $\mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$. Since there are no finite singularities, we shall consider the systems (6.8) for which the condition $\mu_2 = 0$ yields $(c-e)(f-d) = 0$ and without loss of generality we can consider $e = c$ via the replacing x with y, c with f, d with e, and a with b, which keeps the form of these systems. Then we get the systems

$$\dot{x} = a + cx + dy - xy, \quad \dot{y} = b + cx + fy - xy,$$

for which $\mu_0 = \mu_1 = \mu_2 = 0$ and $\mu_3 = (d-f)(a-b+cd-cf)xy^2$. Therefore the condition $\mu_3 = 0$ yields either $f = d$ or $a = b - cd + cf$ and in each of these cases we get a family of systems depending on four free parameters.

6.2. **The case of one real and two complex infinite singular points.** In this case the polynomial $C_2 = yP(x,y) - xQ(x,y)$ has one real and two complex linear factors. Therefore this cubic binary form can be written as $C_2 = x(x^2 + y^2)$ via an admissible linear transformation (depending only on the coordinates of the infinite singularities of the considered family of systems (2.1); see Remark 2.7). Using the factorization (2.2) and a time rescaling we get the family of systems

$$\dot{x} = a + cx + dy + gx^2 + (h+1)xy, \quad \dot{y} = b + ex + fy - x^2 + gxy + hy^2, \tag{6.11}$$

with $\mu_0 = -h\big[g^2 + (h+1)^2\big]$.

6.2.1. *The case* $\mu_0 \neq 0$. By Lemma 2.1 the unique finite singularity must be of the multiplicity four and translating it to the origin of coordinates we get the family of systems

$$\dot{x} = cx + dy + gx^2 + (h+1)xy, \quad \dot{y} = ex + fy - x^2 + gxy + hy^2. \tag{6.12}$$

Clearly in order that the singular point $(0,0)$ has at least of multiplicity 2 the condition $cf - de = 0$ must hold.
Subcase $d \neq 0$. Then $e = cf/d$ and for systems (6.12) we obtain $\mu_4 = \mu_3 = 0$ and $\mu_2 = -\mathcal{F}_3\big[(fg+d)x^2 + (f - dg + fh)xy - dhy^2\big]/d$, where $\mathcal{F}_3 = f(c - dg + ch) - (d^2 + cdg - c^2h)$. By Lemma 2.2 in order to have a point of multiplicity 4 we must force the conditions $\mu_2 = \mu_1 = 0$ to be fulfilled. Since $d \neq 0$ and $\mu_0 \neq 0$ (i.e. $h \neq 0$) the condition $\mu_2 = 0$ is equivalent to $\mathcal{F}_3 = 0$.

We observe that $c - dg + ch \neq 0$, otherwise $g = c(h+1)/d$ and then $\mathcal{F}_3 = -(c^2 + d^2) \neq 0$. So $c - dg + ch \neq 0$ and from $\mathcal{F}_3 = 0$ we get $f = (d^2 + cdg - c^2h)/(c - dg + ch)$. Then we get $\mu_1 = \mathcal{F}_4\big[(d + dg^2 + dh - cgh)x + h(dg - c - ch)y\big]/(d(dg - c - ch))$, where $\mathcal{F}_4 = (dg - ch)^2 + h(c^2 + d^2) + d^2$. Since $h(dg - c - ch) \neq 0$ the condition $\mu_1 = 0$ is equivalent to $\mathcal{F}_4 = 0$. We observe that $\mathrm{Discrim}[\mathcal{F}_4, c] = -4d^2h\big[g^2 + (h+1)^2\big] = 4d^2\mu_0$. So for the existence of real parameters c, d, h and g in order to have a point of multiplicity 4 for a non–homogeneous quadratic system (2.1) it is necessary $\mu_0 > 0$.

Then we have either $c_{1,2} = (dgh \pm d\sqrt{-h\big[g^2+(h+1)^2\big]})/(h(h+1)) = d\tilde{c}_{1,2}$ if $h \neq -1$, or $c_3 = -dg/2 = d\tilde{c}_3$ if $h = -1$, and this leads to the families of systems

$$\begin{gathered}\dot{x} = dc_0x + dy + gx^2 + (h+1)xy,\\ \dot{y} = \frac{dc_0(1+c_0g-c_0^2h)}{c_0-g+c_0h}x + \frac{d(1+c_0g-c_0^2h)}{c_0-g+c_0h}y - x^2 + gxy + hy^2,\end{gathered}$$

where c_0 is one of the values $\tilde{c}_i$ for $i = 1,2,3$. Thus in the generic case fixing the coordinates of singularities and fixing the multiplicity 4 for the finite singularity we obtain a family of systems depending on three parameters d, g and h.

Subcase $d = 0$. Then we have $cf = 0$ and we claim that in order to have the singular point $M_0(0,0)$ of multiplicity 4 it is necessary $c = 0$. Indeed if $c \neq 0$ then $f = 0$ and for systems (6.12) with $d = f = 0$ we get $\mu_4 = \mu_3 = 0$ and $\mu_2 = ch\big[-(c+eg)x^2 + (-e+cg-eh)xy + chy^2\big]$. Since $\mu_0 \neq 0$ (i.e. $h \neq 0$) we obtain $ch \neq 0$ and this yields $\mu_2 \neq 0$.

Since $c = 0$ for systems (6.12) we calculate $\mu_4 = \mu_3 = 0$ and $\mu_2 = f(-e+fg-eh)\big[gx+(h+1)y\big]x$. In this case the condition $\mu_2 = 0$ implies $f(-e+fg-eh) = 0$.

If $h \neq -1$ then calculations yield either $\mu_1 = -eh(1+h)\big[gx+(h+1)y\big]$ if $f = 0$, or $\mu_1 = -f[g^2+(h+1)^2]x$ if $e = fg/(h+1)$. Since $\mu_0 \neq 0$ in both cases we get $e = f = 0$ and this leads to a family of homogeneous quadratic systems depending on two parameters.

Assume now $h = -1$. Then the condition $\mu_0 \neq 0$ implies $g \neq 0$. Hence the conditions $\mu_2 = f^2g^2x^2 = 0$ and $\mu_1 = -2fg^2x = 0$ yield $f = 0$. In this case we obtain the family of systems

$$\dot{x} = gx^2, \quad \dot{y} = ex - x^2 + gxy - y^2,$$

which also depends on two parameters.

6.2.2. *Case* $\mu_0 = 0$, $\mu_1 \neq 0$. In this case by Lemma 2.1 a single singularity of systems (6.11) must be of multiplicity 3. We observe that for systems (6.11) the polynomial μ_1 can be represented in the form $\mu_1 = gW_1(x,y) + (h+1)W_2(x,y)$, where W_1 and W_2 are polynomials in the coefficients of systems (6.11) as well as homogeneous of degree one in x and y. Then we conclude that the conditions $\mu_0 = 0$ and $\mu_1 \neq 0$ yield $h = 0$ and after a translation we get the systems

$$\dot{x} = cx + dy + gx^2 + xy, \quad \dot{y} = ex + fy - x^2 + gxy, \tag{6.13}$$

for which $\mu_1 = (dg-f)(g^2+1)x \neq 0$. Since $(0,0)$ is of multiplicity 3 the condition $cf - de = 0$ must hold.

Subcase $d \neq 0$. Then $e = cf/d$ and for systems (6.13) we obtain $\mu_4 = \mu_3 = 0$ and $\mu_2 = \big[c(dg-f) + d(d+fg)\big]\big[(fg+d)x + (f-dg)y\big]x/d$. By Lemma 2.2 in order to have a point of multiplicity 3 we must force the condition $\mu_2 = 0$ to be fulfilled. Since $\mu_1 \neq 0$ (i.e. $dg - f \neq 0$) the condition $\mu_2 = 0$ is equivalent to $c = d(d+fg)/(f-dg)$ and then systems (6.13) become a family of systems which depends on three parameters d, f and g.

Subcase $d = 0$. Then we have $cf = 0$ and $\mu_1 = -f(g^2+1)x \neq 0$. Therefore $c = 0$ and calculations yield $\mu_4 = \mu_3 = 0$, $\mu_2 = f(fg-e)(gx+y)x$ and $\mu_1 = -f(g^2+1)x$. Since $\mu_1 \neq 0$ the condition $\mu_2 = 0$ gives $e = fg$. So in this case we get a family of systems which depends on two parameters f and g.

6.2.3. *Case* $\mu_0 = \mu_1 = 0$. In this case according with Lemma 2.1 at least two finite singularities have gone to infinity. For systems (6.11) we calculate

$$\mu_0 = -h\big[(h+1)^2 + g^2\big], \quad \kappa = -16\big[g^2 + (1+h)(1-3h)\big] \tag{6.14}$$

and for forcing $\mu_0 = 0$ we have to distinguish two possibilities: $\kappa \neq 0$ and $\kappa = 0$.

Subcase $\kappa \neq 0$. In this case the condition $\mu_0 = 0$ yields $h = 0$ and then the condition $\mu_1 = (dg - f)(g^2+1)x = 0$ yields $f = dg$. So we get the family of systems

$$\dot{x} = a + cx + dy + gx^2 + xy, \quad \dot{y} = b + ex + dgy - x^2 + gxy, \tag{6.15}$$

with the configuration $\binom{2}{1}\nu_1 + \binom{0}{1}\nu_2^c + \binom{0}{1}\nu_3^c$ at infinity (see Lemma 2.5) and

$$\mu_2 = (ag - b + d^2 + de - cdg + d^2g^2)(g^2+1)x^2. \tag{6.16}$$

We divide the proof of this subcase in three steps.

First $\mu_2 \neq 0$. Then according with Lemma 2.1 the unique finite singularity must be of multiplicity 2. So translating this singularity to the origin we obtain the family (6.15) with $a = b = 0$ and

$$\mu_2 = d(d+e-cg+dg^2)(g^2+1)x^2, \quad \mu_3 = d(e-cg)\big[(c+eg)x + (d+e-cg+dg^2)y\big]x^2. \tag{6.17}$$

Since $\mu_2 \neq 0$ the condition $\mu_3 = 0$ yields $e = cg$ and we obtain the family of systems

$$\dot{x} = cx + dy + gx^2 + xy, \quad \dot{y} = cgx + dgy - x^2 + gxy,$$

which has all singularities fixed (its configuration at infinity corresponds to $\binom{2}{1}\nu_1 + \binom{0}{1}\nu_2^c + \binom{0}{1}\nu_3^c$ and depends on three parameters c, d and g.

Second $\mu_2 = 0$ and $\mu_3 \neq 0$. Then systems (6.11) possess exactly one real singular point. Hence without loss of generality we can consider $a = b = 0$ translating this point to the origin and we obtain systems (6.15) with $a = b = 0$. For these systems we have $\mu_0 = \mu_1 = 0$ and the values of μ_2 and μ_3 are given in (6.17). But in this case the conditions $\mu_2 = 0$ and $\mu_3 \neq 0$ must hold. Evidently this implies $e = cg - d - dg^2$ and we get the family of systems

$$\dot{x} = cx + dy + gx^2 + xy, \quad \dot{y} = (cg - d - dg^2)x + dgy - x^2 + gxy,$$

depending on three parameters c, d and g.

Third $\mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$. Since there are no finite singularities we consider systems (6.15) with $\mu_0 = \mu_1 = 0$ and the value of the polynomial μ_2 is given by (6.16). Hence the condition $\mu_2 = 0$ yields $b = ag + d^2 + de - cdg + d^2g^2$ and then for these systems we obtain $\mu_3 = (g^2+1)(a - cd + d^2g)(2d + e - cg + 2dg^2)x^3$ and $\mu_4 = (a - cd + d^2g)x^3W_3(a, c, d, e, g, x, y)$. Thus the conditions $\mu_3 = 0$ and $\mu_4 \neq 0$ imply $e = cg - 2d(g^2+1)$. In such a way we get a family of systems which depends on four parameters a, c, d and g.

Subcase $\kappa = 0$. Then considering (6.14) for systems (6.11) the conditions $\mu_0 = \kappa = 0$ yield $g = 0 = h + 1$. Thus we get the systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b + ex + fy - x^2 - y^2, \tag{6.18}$$

for which $\mu_0 = \mu_1 = 0$ and $\mu_2 = (c^2 + d^2)(x^2 + y^2)$. We separate the proof in two parts.

First $\mu_2 \neq 0$. Now systems (6.18) have exactly one (double) real singular point. Then without loss of generality we can consider $a = b = 0$ translating it to the origin of coordinates. Then we have $\mu_4 = 0$ and $\mu_3 = (cf - de)(cx + dy)(x^2 + y^2)$, and by Lemma 2.2 for having a double singular point $M_0(0, 0)$ of systems (6.18) with $a = b = 0$ we have to force the condition $\mu_3 = 0$. Hence since $\mu_2 \neq 0$ we get $cf - de = 0$. Due to the fact taht $c^2 + e^2 \neq 0$ without loss of generality we may assume $f = qe$ and $d = qc$. In such of way we get the family of systems

$$\dot{x} = cx + qcy, \quad \dot{y} = ex + qey - x^2 - y^2,$$

which depends on three parameters c, q and e.
Second $\mu_2 = 0$. Then we have $\mu_2 = (c^2 + d^2)(x^2 + y^2) = 0$ which implies $c = d = 0$. So we obtain the family of systems

$$\dot{x} = a, \quad \dot{y} = b + ex + fy - x^2 - y^2,$$

for which $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \quad \mu_4 = a^2(x^2 + y^2)^2 \neq 0$. We observe that this family depends on four parameters a, b, e and f.

6.3. **The case of one double and one simple real infinite singular points.** In this case the polynomial $C_2 = yP(x, y) - xQ(x, y)$ due to a linear transformation can be written in the form $C_2 = x^2y$. Then using the factorization (2.2) and a time rescaling we get the family of systems

$$\dot{x} = a + cx + dy + gx^2 + hxy, \quad \dot{y} = b + ex + fy + (g - 1)xy + hy^2,$$

for which $\mu_0 = gh^2$ and $\kappa = -16h^2$.

6.3.1. *The case* $\mu_0 \neq 0$. According with Lemma 2.1 the unique finite singularity must be of multiplicity 4. Translating this point to the origin of coordinates we get the family of systems

$$\dot{x} = cx + dy + gx^2 + hxy, \quad \dot{y} = ex + fy + (g - 1)xy + hy^2. \tag{6.19}$$

Clearly in order that the singular point $(0, 0)$ has at least of multiplicity 2 the condition $cf - de = 0$ must hold.
Subcase $d \neq 0$. Then $e = cf/d$ and for systems (6.19) calculations yield $\mu_4 = \mu_3 = 0$ and $\mu_2 = \mathcal{F}_5\big[fgx^2 + (d - dg + fh)xy - dhy^2\big]/d$, where $\mathcal{F}_5 = f(dg - ch) - c(d - dg + ch)$. According with Lemma 2.2 for having a point of multiplicity 4 we must force the conditions $\mu_2 = \mu_1 = 0$ to be fulfilled. Since $\mu_0 \neq 0$ (i.e. $h \neq 0$) the condition $\mu_2 = 0$ is equivalent to $\mathcal{F}_5 = 0$.

We claim that for $dg - ch = 0$ we cannot have a point of multiplicity 4. Indeed, supposing $g = ch/d$ we get the contradiction $\mu_0 = ch^3/d \neq 0$ and $\mathcal{F}_5 = cd = 0$. So $dg - ch \neq 0$ and the condition $\mathcal{F}_5 = 0$ gives $f = c(d - dg + ch)/(dg - ch)$. Then we get $\mu_1 = \mathcal{F}_6\big[g(dg - d - ch)x + h(dg - ch)y\big]/(d(dg - ch))$, where $\mathcal{F}_6 = (dg - ch)^2 - d^2g$. Since $h(dg - ch) \neq 0$ the condition $\mu_1 = 0$ is equivalent to $\mathcal{F}_6 = 0$. We observe that $\mathrm{Discrim}[\mathcal{F}_6, c] = 4d^2gh^2 = 4d^2\mu_0$. So for the existence of real parameters c, h, d and g such that a non–homogeneous quadratic system (6.19) has a point of multiplicity 4 it is necessary that $\mu_0 > 0$. Then we have $c_{1,2} = (dg \pm d\sqrt{g})/h = d\tilde{c}_{1,2}$, and this leads to the two families of systems

$$\dot{x} = dc_0x + dy + gx^2 + hxy,$$
$$\dot{y} = \frac{dc_0^2(1 - g + c_0h)}{g - c_0h}x + \frac{dc_0(1 - g + c_0h)}{g - c_0h}y + (g - 1)xy + hy^2,$$

where c_0 is one of the values $\tilde{c}_{1,2}$. Thus we get two families of systems each of them depending on three parameters d, g and h. However fixing the coordinates of singularities and fixing the multiplicity 4 for the finite singularity we automatically get only one of these families depending on three parameters.

Subcase $d = 0$. Then we have $cf = 0$ and we claim that to have the singular point $M_0(0,0)$ of multiplicity 4 it is necessary that $c = 0$. Indeed, if $c \neq 0$ then $f = 0$ and for systems (6.19) we get $\mu_4 = \mu_3 = 0$ and $\mu_2 = ch\big[-egx^2+(cg-c-eh)xy+chy^2\big]$. Since $\mu_0 \neq 0$ (i.e. $h \neq 0$) we obtain $ch \neq 0$ and this yields $\mu_2 \neq 0$. Thus $c = 0$ and for systems (6.19) obtain $\mu_4 = \mu_3 = 0$ and $\mu_2 = f(fg-eh)(gx+hy)x$. In this case the condition $\mu_2 = 0$ implies $f(fg-eh)) = 0$. Then calculations yield either $\mu_1 = -eh^2(gx+hy)$ if $f = 0$, or $\mu_1 = fghx$ if $e = fg/h$. Since $\mu_0 \neq 0$ in both cases we get $e = f = 0$ and this leads to the family of homogeneous quadratic systems depending on two parameters.

6.3.2. *The case* $\mu_0 = 0$. Since $\kappa = -16h^2$ we shall consider two subcases $\kappa \neq 0$ and $\kappa = 0$.

Subcase $\kappa \neq 0$. Then $h \neq 0$ and the condition $\mu_0 = 0$ yields $g = 0$. So we get the family of systems

$$\dot{x} = a + cx + dy + hxy, \quad \dot{y} = b + ex + fy - xy + hy^2, \tag{6.20}$$

for which $\mu_1 = -h^2(c+eh)y$. We separate the proof of this subcase in four steps.

First $\mu_1 \neq 0$. In this case by Lemma 2.1 a single singularity of systems (6.11) must be of multiplicity 3. After the respective translation we get the systems

$$\dot{x} = cx + dy + hxy, \quad \dot{y} = ex + fy - xy + hy^2, \tag{6.21}$$

for which $cf - de = 0$ because $(0,0)$ is of multiplicity 3.

Subcase $d \neq 0$. Then $e = cf/d$ and for systems (6.21) we obtain $\mu_4 = \mu_3 = 0$, $\mu_2 = -c(d+ch+fh)\big[(fh+d)x-dhy\big]y/d$ and $\mu_1 = -ch^2(fh+d)y/d$. By Lemma 2.2 for having a singular point of multiplicity 3 the condition $\mu_2 = 0$ must hold. Since $\mu_1 \neq 0$ the condition $\mu_2 = 0$ is equivalent to $c = -(fh+d)/h$. Thus the family of systems (6.21) becomes a family which depends on three parameters d, f and h.

Subcase $d = 0$. Then we have $cf = 0$ and we claim that in order to have the singular point $M_0(0,0)$ of multiplicity 3 it is necessary that $c = 0$. Indeed, if $c \neq 0$ then $f = 0$ and for systems (6.21) we get $\mu_4 = \mu_3 = 0$ and $\mu_2 = ch\big[-(eh+c)x+chy\big]y$. Since $\kappa \neq 0$ (i.e. $h \neq 0$) we obtain $ch \neq 0$ and this yields $\mu_2 \neq 0$, but on the other hand we must have $\mu_2 = 0$. This contradiction proves our claim. Thus $c = 0$ and in this case we obtain $\mu_1 = -eh^3y$ and $\mu_2 = -efh^2xy$, and since $\mu_1 \neq 0$ the condition $\mu_2 = 0$ gives $f = 0$. So we get the family of systems depending on two parameters e and h.

Second $\mu_1 = 0$ and $\mu_2 \neq 0$. In this case by Lemma 2.1 the finite singularity must be double. Since $\kappa \neq 0$ (i.e. $h \neq 0$) the condition $\mu_1 = -h^2(c+eh)y = 0$ yields $c = -eh$. Thus after the respective translation systems (6.20) become the family of systems

$$\dot{x} = -ehx + dy + hxy, \quad \dot{y} = ex + fy - xy + hy^2, \tag{6.22}$$

for which we have

$$\mu_2 = eh^2(d+fh+eh^2)y^2, \quad \mu_3 = e(d+fh)\big[(d+fh+eh^2)x-dhy\big]y^2 \tag{6.23}$$

and according with Lemma 2.2 for having a double point at the origin we must force $\mu_3 = 0$. Since $\mu_2 \neq 0$ the condition $\mu_3 = 0$ yields $d = -fh$. Thus the family of systems (6.22) becomes a family which depends on three parameters e, f and h.
Third $\mu_1 = \mu_2 = 0$ and $\mu_3 \neq 0$. Then systems (6.20) possess exactly one real singular point and due to a translation, by the same reasons as above we get the family of systems (6.22) and we must force $\mu_2 = 0$. Considering (6.23) and $\mu_3 \neq 0$ we have $d = -h(f + eh)$ and we again arrive to the family depending on three parameters e, f and h.
Fourth $\mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$. Since there are no finite singularities, we consider systems (6.20) for which $\mu_0 = 0$ and due to the fact that $\kappa \neq 0$ the condition $\mu_1 = -h^2(c + eh)y = 0$ yields $c = -eh$. Thus we obtain the family of systems

$$\dot{x} = a - ehx + dy + hxy, \quad \dot{y} = b + ex + fy - xy + hy^2, \tag{6.24}$$

for which the condition $\mu_2 = h^2(a+de+bh+efh+e^2h^2)y^2 = 0$ implies $a = -(de+bh+efh+e^2h^2)$. Then calculations yield $\mu_3 = -h^2(b+ef+e^2h)(d+fh+2eh^2)y^3$ and $\mu_4 = (b+ef+e^2h)y^3\, W_4(b, d, e, f, h, x, y)$, where $W_4(b, d, e, f, h, x, y)$ is a polynomial in the indicated parameters and is linear in x and y. Therefore the conditions $\mu_3 = 0$ and $\mu_4 \neq 0$ give $d = -(fh + 2eh^2)$ and in such a way the family of systems (6.24) becomes a family depending on four parameters b, e, f, h.

Subcase $\kappa = 0$. Then $h = 0$ and we get systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + ex + fy + (g - 1)xy, \tag{6.25}$$

for which $\mu_0 = 0$ and $\mu_1 = dg(g - 1)^2x$. We divide the proof of this subcase in two steps.
First $\mu_1 \neq 0$. In this case by Lemma 2.1 a single singularity of systems (6.25) must be of multiplicity 3. After the respective translation we get the systems

$$\dot{x} = cx + dy + gx^2, \quad \dot{y} = ex + fy + (g - 1)xy, \tag{6.26}$$

for which $cf - de = 0$ because $(0, 0)$ is of multiplicity 3. Since $\mu_1 \neq 0$ yields $d \neq 0$. We obtain $e = cf/d$ and for systems (6.26) we get $\mu_4 = \mu_3 = 0$, $\mu_2 = (cg - c + fg)\big[fgx + d(1 - g)y\big]x$ and $\mu_1 = dg(g - 1)^2x$, and since $\mu_1 \neq 0$ the condition $\mu_2 = 0$ yields $f = c(1 - g)/g$. Thus the family (6.26) becomes a family depending on three parameters c, d and g.
Second $\mu_1 = 0$. According with Lemma 2.5 for $\kappa = 0$ and $\mu_0 = \mu_1 = 0$ the configurations of infinite singularities are governed by the invariant polynomial $L(a, x, y)$. For systems (6.25) we have $\mu_1 = dg(g - 1)^2x$ and $L = 8gx^2$.

1) The case $L \neq 0$. Then $g \neq 0$ and the condition $\mu_1 = 0$ implies $d(g - 1) = 0$.

α) Assume that $\mu_2 \neq 0$, i.e. the finite singularity is of multiplicity 2. Then without loss of generality translating this singular point at the origin we can consider $a = b = 0$ for systems (6.25). Thus we obtain the systems

$$\dot{x} = cx + dy + gx^2, \quad \dot{y} = ex + fy + (g - 1)xy, \tag{6.27}$$

for which $d(g - 1) = 0$. Therefore we get

$$\mu_2 = fg(fg + c - cg)x^2, \quad \mu_3 = (de - cf)\big[egx + (fg + c - cg)y\big]x. \tag{6.28}$$

Since $\mu_2 \neq 0$ the condition $\mu_3 = 0$ yields $de - cf = 0$. As $f \neq 0$ without loss of generality we introduce a new parameter q through $e = qf$, and then we get $c = dq$. Thus we obtain the family of systems

$$\dot{x} = dqx + dy + gx^2, \quad \dot{y} = fqx + fy + (g - 1)xy,$$

for which $d(g-1)=0$. Therefore for $d=0$ (respectively $g=1$) we get a family of systems depending on three parameters f, q and g (respectively f, q and d).

β) For $\mu_2=0$ and $\mu_3\neq 0$ systems (6.25) possess exactly one simple real singular point and doing a translation, by the same reasons as above, we get the family of systems (6.27) and we must force $\mu_2=0$, taking into account the relation $d(g-1)=0$.

If $d=0$ from (6.28) and since $\mu_3\neq 0$ we have $f\neq 0$ and then the conditions $\mu_2=0$ implies $f=c(g-1)/g$. Thus we get the family of systems

$$\dot{x}=cx+gx^2,\quad \dot{y}=ex+c(g-1)y/g+(g-1)xy,$$

depending on three parameters c, e and g.

Assume $d\neq 0$. Then $g=1$ and from (6.28) the condition $\mu_2=f^2x^2=0$ yields $f=0$. So we again get the family of systems

$$\dot{x}=cx+dy+x^2,\quad \dot{y}=ex,$$

depending on three parameters c, d and e.

γ) Assume finally that $\mu_2=\mu_3=0$ and $\mu_4\neq 0$, i.e. systems (6.25) have no finite singularities. Then considering the condition $L\neq 0$ (i.e. $g\neq 0$) we obtain as above that the condition $\mu_1=dg(g-1)^2x=0$ yields $d(g-1)=0$.

Suppose $d=0$. Then we calculate $\mu_2=g\big[a(g-1)^2-cf(g-1)+f^2g\big]x^2$.

If $g\neq 1$ then since $g\neq 0$ the condition $\mu_2=0$ gives $a=[cf(g-1)-f^2g]/(g-1)^2$ and we obtain $\mu_3=g(b+ef-bg)(c-cg+2fg)x^3/(1-g)$ and $\mu_4=(b+ef-bg)x^3W_5(b,c,e,f,g,x,y)/(1-g)^2$, where $W_5(b,c,e,f,g,x,y)$ is a polynomial in the indicated parameters and linear in x and y. Thus the conditions $\mu_3=0$ and $\mu_4\neq 0$ imply $c=2fg/(g-1)$. So we obtain the family of systems

$$\dot{x}=g(f-x+gx)^2/(g-1)^2,\quad \dot{y}=b+ex+fy+(g-1)xy$$

depending on four parameters b, e, f and g.

If $g=1$ then $\mu_2=f^2x^2=0$ gives $f=0$ and then $\mu_3=0$. This leads to the family

$$\dot{x}=a+cx+x^2,\quad \dot{y}=b+ex,$$

which also depends on four parameters a, b, c and e.

Assume now $d\neq 0$. Hence the condition $\mu_1=0$ gives $g=1$ and from $\mu_2=f^2x^2=0$ we get $f=0$. Then $\mu_3=de^2x^3=0$ and since $d\neq 0$ we obtain $e=0$. Thus we get the family of systems

$$\dot{x}=a+cx+dy+x^2,\quad \dot{y}=b$$

depending on four parameters a, b, c and d.

2) The case $L=0$. Then $g=0$ and systems (6.25) become

$$\dot{x}=a+cx+dy,\quad \dot{y}=b+ex+fxy,$$

for which we have $\mu_1=0$ and $\mu_2=-cd\,xy$.

α) Assume that $\mu_2\neq 0$, i.e. the finite singularity is of multiplicity 2. Then translating this singular point at the origin we can consider $a=b=0$. Thus we get the systems

$$\dot{x}=cx+dy,\quad \dot{y}=ex+fy-xy, \tag{6.29}$$

for which calculations yield $\mu_2=-cd\,xy$ and $\mu_3=(de-cf)(cx+dy)xy$. Since $\mu_2\neq 0$ (i.e. $cd\neq 0$) the condition $\mu_3=0$ gives $e=cf/d$ and hence the family of systems (6.29) becomes a family depending on three parameters c, d and f.

β) Suppose that $\mu_2 = 0$ and $\mu_3 \neq 0$. By Lemma 2.1 there exists exactly one simple real singular point on the phase plane of systems (6.25) and doing a translation we get the family of systems (6.29) for which we must force $\mu_2 = -cd\,xy = 0$. So $cd = 0$ and since $\mu_3 \neq 0$ the condition $c^2 + d^2 \neq 0$ holds. We conclude that in the case $c = 0$ as well as in the case $d = 0$ the family of systems (6.29) becomes a family depending respectively on the remaining three parameters.

γ) Assume finally $\mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$, i.e. systems (6.25) have no finite singularities. For these systems we have $\mu_2 = -cd\,xy$ and the condition $\mu_2 = 0$ yields $cd = 0$. By Lemma 2.5 in this case the configurations of infinite singularities are governed by the invariant polynomial $\kappa_1 = -32d$.

If $\kappa_1 \neq 0$ (i.e. $d \neq 0$) then $c = 0$ and we calculate $\mu_3 = d(a + de)x^2y$. Hence $\mu_3 = 0$ gives $a = -de$ and we get the family of systems

$$\dot{x} = d(y - e), \quad \dot{y} = b + ex + fy - xy,$$

depending on four parameters b, d, e and f. The configuration of infinite singular points corresponds to $\binom{3}{1}p + \binom{1}{2}q$.

Assume $\kappa_1 = 0$. Then $d = 0$ and for systems (6.25) we obtain $\mu_3 = -c(a+cf)x^2y$ and $K_1 = -cx^2y$. Thus for $K_1 \neq 0$ the condition $\mu_3 = 0$ gives $a = -cf$ and we get the family of systems

$$\dot{x} = c(x - f), \quad \dot{y} = b + ex + fy - xy,$$

depending on four parameters b, c, e, f and with the configuration $\binom{1}{1}p + \binom{3}{2}q$ at infinity.

If $K_1 = 0$ we have $c = 0$ and this leads to the family of systems

$$\dot{x} = a, \quad \dot{y} = b + ex + fy - xy,$$

which also depends on four parameters a, b, e and f, but with the configuration $\binom{2}{1}p + \binom{2}{2}q$ at infinity.

6.4. **Systems with one triple real infinite singular point.** In this case the polynomial $C_2 = yP(x, y) - xQ(x, y)$ due to an admissible linear transformation can be written into the form $C_2 = x^3$. Then using the factorization (2.2) as well as a time rescaling we get the family of systems

$$\dot{x} = a + cx + dy + gx^2 + hxy, \quad \dot{y} = b + ex + fy - x^2 + gxy + hy^2, \tag{6.30}$$

for which $\mu_0 = -h^3$.

6.4.1. *The case* $\mu_0 \neq 0$. By Lemma 2.1 the unique finite singularity must be of multiplicity four and translating this point to the origin of coordinates we get the family of systems

$$\dot{x} = cx + dy + gx^2 + hxy, \quad \dot{y} = ex + fy - x^2 + gxy + hy^2. \tag{6.31}$$

Clearly the singular point $(0, 0)$ will be at least of multiplicity 2 if the condition $cf - de = 0$ holds.

If $d \neq 0$ then $e = cf/d$ and for systems (6.31) we get $\mu_4 = \mu_3 = 0$ and $\mu_2 = \mathcal{F}_7\big[-(d + fg)x^2 + (dg - fh)xy + dhy^2\big]/d$, where $\mathcal{F}_7 = (c + f)(ch - dg) - d^2$. So since $\mu_0 \neq 0$ (i.e. $h \neq 0$) the condition $\mu_2 = 0$ implies $\mathcal{F}_7 = 0$. We observe that $ch - dg \neq 0$, otherwise we get the contradiction $\mathcal{F}_7 = -d^2 = 0$. So the condition $\mathcal{F}_7 = 0$ gives $f = d^2/(ch - dg) - c$ and we obtain $\mu_1 = \mathcal{F}_8\big[(dg^2 + dh - cgh)x + h(dg - ch)y\big]/(d(dg - ch))$, where $\mathcal{F}_8 = (dg - ch)^2 + d^2h$. Since $h(dg - ch) \neq 0$ the condition $\mu_1 = 0$ is equivalent to $\mathcal{F}_8 = 0$. We observe that

$\text{Discrim}[\mathcal{F}_8, c] = -4d^2h^3 = 4d^2\mu_0$. So it is necessary that $\mu_0 > 0$ in order that a non–homogeneous quadratic system (6.31) has a singular point of multiplicity 4. Then we have $c_{1,2} = (dg \pm d\sqrt{-h})d = d\tilde{c}_{1,2}$, and this leads to the two families of systems

$$\dot{x} = dc_0x + dy + gx^2 + hxy,$$
$$\dot{y} = -\frac{dc_0(1 + c_0g - c_0^2h)}{g - c_0h}x - \frac{d(1 + c_0g - c_0^2h)}{g - c_0h}y - x^2 + gxy + hy^2,$$

where c_0 is one of the values $\tilde{c}_{1,2}$ above. Thus we get two families of systems each of them depending on 3 parameters. However fixing the coordinates of singularities and fixing the multiplicity 4 for the finite singularity we automatically get only one of these families depending on three parameters.

Assume now $d = 0$. Then we have $cf = 0$ and we claim that for having the singular point $M_0(0,0)$ of multiplicity 4 it is necessary that $c = 0$. Indeed, if $c \neq 0$ then $f = 0$ and for systems (6.31) we get $\mu_4 = \mu_3 = 0$ and $\mu_2 = ch\big[-(c + eg)x^2 + (cg - eh)xy + chy^2\big]$. Since $\mu_0 \neq 0$ (i.e. $h \neq 0$) we obtain $ch \neq 0$ and this yields $\mu_2 \neq 0$. Thus $c = 0$ and for systems (6.31) we get $\mu_4 = \mu_3 = 0$, $\mu_2 = f(fg - eh)(gx + hy)x$ and $\mu_1 = h\big[(fg^2 - fg - egh)x + h(fg - eh)y)\big]$. In this case the conditions $\mu_2 = \mu_1 = 0$ imply $e = f = 0$ and this leads to the family of homogeneous quadratic systems

$$\dot{x} = gx^2 + hxy, \quad \dot{y} = -x^2 + gxy + hy^2,$$

which depends on two parameters.

6.4.2. *Case* $\mu_0 = 0$ *and* $\mu_1 \neq 0$. In this case by Lemma 2.1 the single singularity of systems (6.30) must be of multiplicity 3. The condition $\mu_0 = 0$ yields $h = 0$ and doing a translation we obtain the systems

$$\dot{x} = cx + dy + gx^2, \quad \dot{y} = ex + fy - x^2 + gxy, \tag{6.32}$$

for which $\mu_1 = dg^3x \neq 0$. Clearly the singular point $(0,0)$ will be at least of multiplicity 2 if the condition $cf - de = 0$ holds. Since $d \neq 0$ we can write $e = cf/d$, and then $\mu_4 = \mu_3 = 0$ and $\mu_2 = (d + cg + fg)\big[(fg + d)x - dgy\big]x$. Since $\mu_1 \neq 0$ the condition $\mu_2 = 0$ yields $f = -(d + cg)/g$. Thus we obtain the family

$$\dot{x} = cx + dy + gx^2, \quad \dot{y} = -c(cg + d)x/(dg) - (cg + d)y/g - x^2 + gxy,$$

depending on three parameters c, d and g.

6.4.3. *Case* $\mu_0 = \mu_1 = 0$ *and* $\mu_2 \neq 0$. By 2.1 a single singularity of systems (6.30) must be of multiplicity 2. Therefore after a translation we obtain systems (6.32) for which $\mu_1 = dg^3x = 0$ and the singular point $(0,0)$ is double. So $dg = 0$ and then $\mu_2 = (d^2 - cfg^2 + f^2g^2)x^2 \neq 0$. Moreover we must force the condition $cf - de = 0$ to have a double singular point at the origin.

If $d = 0$ we have $cf = 0$ and since $f \neq 0$ (otherwise systems (6.32) are degenerate) we obtain $c = 0$. This leads to the systems

$$\dot{x} = gx^2, \quad \dot{y} = ex + fy - x^2 + gxy$$

depending on three parameters e, f and g.

If $g = 0$ then $\mu_2 = d^2x^2 \neq 0$ and then we have $e = cf/d$. This leads to the family of systems

$$\dot{x} = cx + dy, \quad \dot{y} = cfx/d + fy - x^2,$$

depending also on three parameters c, d and f.

6.4.4. *Case* $\mu_0 = \mu_1 = \mu_2 = 0$, $\mu_3 \neq 0$. Then systems (6.30) possess exactly one simple real singular point and due to a translation we shall consider systems (6.32) with the conditions $\mu_1 = dg^3x = 0$ and $\mu_2 = (d^2 - cfg^2 + f^2g^2)x^2 = 0$.
Subcase $g = 0$. Then the condition $\mu_2 = d^2x^2 = 0$ gives $d = 0$ and we get the family

$$\dot{x} = cx, \quad \dot{y} = ex + fy - x^2,$$

which depends on three parameters c, e and f.
Subcase $g \neq 0$. Then $d = 0$ and for systems (6.32) we obtain $\mu_2 = fg^2(f - c)x^2$ and $\mu_3 = -cf\big[(c + eg)x + g(f - c)y\big]x^2$, and since $\mu_3 \neq 0$ and $g \neq 0$ the condition $\mu_2 = 0$ implies $f = c$. This leads to the family of systems

$$\dot{x} = cx + gx^2, \quad \dot{y} = ex + cy - x^2 + gxy$$

depending also on three parameters c, e and g.

6.4.5. *Case* $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ *and* $\mu_4 \neq 0$. In this case systems (6.30) have no finite singularities and the condition $\mu_0 = 0$ yields $h = 0$. So we obtain the systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + ex + fy - x^2 + gxy \tag{6.33}$$

for which the condition $\mu_1 = 0$ yields $dg = 0$.
Subcase $g = 0$. In this case for systems (6.33) the condition $\mu_2 = d^2x^2 = 0$ gives $d = 0$ and then $\mu_3 = -c^2fx^3 = 0$ and $\mu_4 = \big[(a^2 + ace - bc^2)x + acfy\big]x^3 \neq 0$. Thus we get the 5–parameter family of systems

$$\dot{x} = a + cx, \quad \dot{y} = b + ex + fy - x^2$$

for which the condition $cf = 0$ holds. Clearly this condition, in each of two cases, leads to the family depending respectively on the remaining four parameters.
Subcase $g \neq 0$. Then $d = 0$ and for systems (6.33) the condition $\mu_2 = g^2(ag - cf + f^2)x^2 = 0$ implies $a = f(c - f)/g$. This leads to the systems

$$\dot{x} = (f + gx)(c - f + gx)/g, \quad \dot{y} = b + ex + fy - x^2 + gxy$$

for which $\mu_4 = (f^2 + efg - bg^2)x^3W_6(b, c, e, f, g, x, y)/g^2$ and $\mu_3 = (c - 2f)(f^2 + efg - bg^2)x^3$, where $W_6(b, c, e, f, g, x, y)$ is a polynomial in the given parameters and is linear in x and y. Thus, since $\mu_4 \neq 0$ the condition $\mu_3 = 0$ gives $c = 2f$ and this leads to the family of systems depending on four parameters b, e, f and g.

7. Systems with the infinite line full of singularities

Assume that the polynomial $C_2 = yP(x, y) - xQ(x, y) = 0$ in $\mathbb{R}[x, y]$. Clearly this class of quadratic systems have the form

$$\dot{x} = a + cx + dy + gx^2 + hxy, \quad \dot{y} = b + ex + fy + gxy + hy^2, \tag{7.1}$$

for which $\mu_0 = 0$. So by Lemma 2.1 these systems can possess finite singularities of total multiplicity at most three.

7.1. **Systems with finite singularities of total multiplicity 3.**

7.1.1. *Systems with three finite real simple singularities.* Assuming that $M_i(x_i, y_i)$ (i=1,2,3) are real distinct singular points of systems (7.1), due to an admissible affine transformation we can move them to the points $(0,0)$, $(0,1)$ and $(1,0)$. Therefore we get the family of systems

$$\dot{x} = cx - cx^2 - fxy, \quad \dot{y} = fy - cxy - fy^2,$$

which up to a time rescaling depends on a single parameter.

7.1.2. *Systems with one real and two complex finite singularities.* As it was mentioned early (see Subsection 3.2) due to an admissible affine transformation which moves the respective singularities to the points $M_{1,2}(0, \pm i)$, $M_3(1,0)$ we get the family of quadratic systems

$$\begin{aligned} \dot{x} &= a - (a+g)x + gx^2 + 2hxy + ay^2, \\ \dot{y} &= b - (b+l)x + lx^2 + 2mxy + by^2. \end{aligned}$$

Now taking into consideration the identity $C_2(a, x, y) = 0$ in $\mathbb{R}[x, y]$ we get $a = l = 0$, $m = g/2$ and $h = b/2$. Hence we get the family of systems

$$\dot{x} = -gx + gx^2 + bxy, \quad \dot{y} = b - bx + gxy + by^2,$$

which up to a time rescaling depends on one parameter.

7.1.3. *Systems with one simple and one double real finite singularities.* Via an admissible affine transformation we can assume that the two distinct singularities of systems (7.1) are placed at the points $(0,0)$ and $(1,0)$. Then evidently we get the relations $a = b = e = 0$, $g = -c$ and these systems become

$$\dot{x} = cx + dy - cx^2 + hxy, \quad \dot{y} = fy - cxy + hy^2, \tag{7.2}$$

and one of the singular points, say $(0,0)$, is double. Then the relation $cf = 0$ must hold. However the relation $c = 0$ yields degenerate systems and, hence we obtain $f = 0$. Clearly this leads to a family of systems which up to a time rescaling depends on two parameters.

7.1.4. *Systems with one finite real triple singularity.* In this case due to a translation we have the systems (7.1) with $a = b = 0$ (then $\mu_4 = 0$) and we must force $\mu_3 = \mu_2 = 0$ and $\mu_1 \neq 0$ in order to have a point of multiplicity 3 at the origin. Thus for systems (7.1) with $a = b = 0$ we have $cf - de = 0$.
The case $d \neq 0$. Then we have $e = cf/d$ and calculations yield $\mu_4 = \mu_3 = 0$,

$$\mu_2 = \frac{c+f}{d}(dg - ch)(fx - dy)(gx + hy), \quad \mu_1 = \frac{dg - ch}{d}(dg + fh)(gx + hy). \tag{7.3}$$

So taking into consideration Lemma 2.1 we obtain that the conditions $\mu_2 = 0$ and $\mu_1 \neq 0$ imply $f = -c$. Then we get the following family of systems

$$\dot{x} = cx + dy + gx^2 + hxy, \quad \dot{y} = -c^2x/d - cy + gxy + hy^2,$$

which up to a time rescaling depends on three parameters.
The case $d = 0$. In this case we obtain $cf = 0$. Then for systems (7.1) with $a = b = d = cf = 0$ we have $\mu_4 = \mu_3 = 0$ and for $c = 0$ (respectively, $f = 0$) we calculate $\mu_2 = f(fg - eh)(gx + hy)x$ and $\mu_1 = h(fg - eh)(gx + hy)$ (respectively $\mu_2 = ch(gx + hy)(cy - ex)$ and $\mu_1 = -h(cg + eh)(gx + hy)$). As we can see in both

cases the conditions $\mu_2 = 0$ and $\mu_1 \neq 0$ imply $c = f = 0$ and we get the family of systems

$$\dot{x} = x(gx + hy), \quad \dot{y} = ex + gxy + hy^2,$$

which up to a time rescaling depends on two parameters.

7.2. **Systems with finite singularities of total multiplicity 2.** By Lemma 2.1 for this class of systems the conditions $\mu_0 = \mu_1 = 0$ and $\mu_2 \neq 0$ have to be fulfilled.

7.2.1. *Systems with two finite real simple singular points.* In this case we can consider systems (7.2) possessing two singularities $(0,0)$ and $(1,0)$. For systems (7.2) we obtain $\mu_1 = -c(cd + ch - fh)(cx - hy)$ and $\mu_2 = -c(cx - hy)\big[f(c - f)x + (cd + df + ch - fh)y\big]$. Hence the conditions $\mu_1 = 0$ and $\mu_2 \neq 0$ yield $c \neq 0$ and $d = h(f - c)/c$. This leads to the following family of systems

$$\dot{x} = cx + h(f - c)y/c - cx^2 + hxy, \quad \dot{y} = y(f - cx + hy),$$

which up to a time rescaling depends on two parameters.

7.2.2. *Systems with two finite complex singular points.* In this case according to Lemma 3.1 via an admissible affine transformation a quadratic system can be brought to the canonical form (3.6) with the singularities $\widetilde{M_{1,2}}(0, \pm i)$. For these systems the identity $C_2 = 0$ yields $a = l = 0$, $m = g/2$ and $h = b/2$. Thus we get the family of systems

$$\dot{x} = cx + gx^2 + bxy, \quad \dot{y} = b + ex + gxy + by^2, \tag{7.4}$$

for which the conditions $\mu_0 = \mu_1 = 0$ and $\mu_2 \neq 0$ must hold. For systems (7.4) calculations yield $\mu_0 = 0$, $\mu_1 = -b(be + cg)(gx + by)$ and $\mu_2 = b(gx + by)\big[(bg - ce)x + (b^2 + c^2)y\big]$. Thus $b \neq 0$ and the condition $\mu_1 = 0$ implies $e = -cg/b$. Hence we get a family of systems which up to a time rescaling depends on two parameters.

7.2.3. *Systems with one double real finite singular point.* In this case due to a translation we obtain systems (7.1) with $a = b = 0$ (then $\mu_4 = 0$) and by Lemmas 2.1 and 2.2 we must force $\mu_3 = \mu_1 = 0$ and $\mu_2 \neq 0$ in order to have a point of multiplicity 2 at the origin and no more finite singularities. Thus for systems (7.1) with $a = b = 0$ we shall set $cf - de = 0$ in order to have a multiple point at the origin.

The case $d \neq 0$. Then we have $e = cf/d$ (in this case $\mu_3 = 0$) and we get the values of the polynomials μ_1 and μ_2 given in the formulas (7.3). Since $\mu_2 \neq 0$ the condition $\mu_1 = 0$ implies $g = -fh/d$. Thus we get the following family of systems

$$\dot{x} = cx + dy - fhx^2/d + hxy, \quad \dot{y} = cfx/d + fy - fhxy/d + hy^2,$$

which up to a time rescaling depends on three parameters.

The case $d = 0$. In this case the point $(0,0)$ will be a double singular point for systems (7.1) with $a = b = 0$ if $cf = 0$. For $c = 0$ we get the family of systems

$$\dot{x} = gx^2 + hxy, \quad \dot{y} = ex + fy + gxy + hy^2,$$

for which $\mu_0 = \mu_3 = \mu_4 = 0$, $\mu_1 = h(fg - eh)(gx + hy)$ and $\mu_2 = f(fg - eh)x(gx + hy)$. Hence the conditions $\mu_1 = 0$, $\mu_2 \neq 0$ imply $h = 0$ and we get a family of systems which up to a time rescaling depends on two parameters.

In the case $f = 0$ we obtain $\mu_0 = \mu_3 = \mu_4 = 0$, $\mu_1 = -h(cg + eh)(gx + hy)$ and $\mu_2 = ch(-ex + cy)(gx + hy)$. So from $\mu_1 = 0$, $\mu_2 \neq 0$ we obtain $h \neq 0$, $e = -cg/h$ and we again have a family of systems which up to a time rescaling depends on two parameters.

7.3. **Systems with finite singularities of total multiplicity less than or equal to 1.**

7.3.1. *Systems with one simple finite singular point.* In this case due to a translation we obtain systems (7.1) with $a = b = 0$ (then $\mu_4 = 0$) and by Lemma 2.1 we must force $\mu_1 = \mu_2 = 0$ and $\mu_3 \neq 0$ in order to have a simple singular point at the origin and no more finite singularities. Thus for systems (7.1) with $a = b = 0$ we calculate

$$\mu_0 = \mu_4 = 0, \quad \mu_1 = (dg^2 - cgh + fgh - eh^2)(gx + hy). \tag{7.5}$$

The case $g \neq 0$. Then $\mu_1 = 0$ yields $d = h(cg - fg + eh)/g^2$ and calculations yield

$$\mu_2 = \frac{1}{g^2}(eh - fg)(cg - fg + 2eh)(gx + hy)^2,$$
$$\mu_3 = \frac{1}{g^4}(eh - fg)(cg + eh)(gx + hy)^2[egx + (fg - cg - eh)y].$$

Hence the conditions $\mu_2 = 0$ and $\mu_3 \neq 0$ imply $f = (cg + 2eh)/g$ and we get the family of systems

$$\dot{x} = cx - eh^2y/g^2 + gx^2 + hxy, \quad \dot{y} = ex + (cg + 2eh)y/g + gxy + hy^2,$$

which up to a time rescaling depends on three parameters.

The case $g = 0$. Then $\mu_1 = -eh^3y$ and $\mu_3 = h(de - cf)y[ex^2 + (f - c)cxy - dy^2]$, and since $\mu_3 \neq 0$ the condition $\mu_1 = 0$ implies $e = 0$. Then $\mu_2 = c(c - f)h^2y^2$, $\mu_3 = cfhy^2(cx - fx + dy)$ and hence the conditions $\mu_2 = 0$ and $\mu_3 \neq 0$ yield $f = c$. In such a way in this case we obtain the family of systems

$$\dot{x} = cx + dy + hxy, \quad \dot{y} = cy + hy^2,$$

which up to a time rescaling depends on two parameters.

7.3.2. *Systems without finite singularities.* By Lemma 2.1 for systems (7.1) (for which $\mu_0 = 0$) we must force the conditions $\mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$. For these systems we have the value of μ_1 indicated in formulas (7.5). Therefore following the above way we consider two cases $g \neq 0$ and $g = 0$.

The case $g \neq 0$. Then $\mu_1 = 0$ yields $d = h(cg - fg + eh)/g^2$ and we get $\mu_2 = \big[ag^3 - f(c - f)g^2 + gh(ce - 3ef + bg) + 2e^2h^2\big](gx + hy)^2/g^2$, and evidently the relation $\mu_2 = 0$ yields $a = \big[f(c - f)g^2 - gh(ce - 3ef + bg) - 2e^2h^2\big]/g^3$. Then we have $\mu_3 = -\frac{1}{g^4}(cg - 2fg + 3eh)(bg^2 - efg + e^2h)(gx + hy)^2$ and $\mu_4 = (bg^2 - efg + e^2h)W_7(x, y)$, and the conditions $\mu_3 = 0$ and $\mu_4 \neq 0$ imply $c = (2fg - 3eh)/g$. Thus we get the family of systems

$$\dot{x} = \frac{(fg - eh)^2 - bg^2h}{g^3} + \frac{2fg - 3eh}{g}x + \frac{h(fg - 2eh)}{g^2}y + gx^2 + hxy,$$
$$\dot{y} = b + ex + fy + gxy + hy^2,$$

which up to a time rescaling depends on four parameters.

The case $g = 0$. Then $\mu_1 = -eh^3y$ and $\mu_4 = hW_8(b, c, e, f, g, h, x, y)$, and since $\mu_4 \neq 0$ the condition $\mu_1 = 0$ gives $e = 0$. In this case from $\mu_2 = h^2(c^2 - cf + bh)y^2 = 0$ we obtain $b = c(f - c)/h$ and then we have $\mu_3 = h(2c - f)(cd - ah)y^3$ and $\mu_4 =$

$(cd - ah)W_9(b, c, e, f, g, h, x, y)$. Hence, since $\mu_4 \neq 0$ the condition $\mu_3 = 0$ implies $f = 2c$ and we get the family of systems

$$\dot{x} = a + cx + dy + hxy, \quad \dot{y} = (c + hy)^2/h,$$

which up to a time rescaling depends on three parameters.

Since all the possible cases were examined the Main Theorem is proved.

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