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BOUNDARY EIGENCURVE PROBLEMS INVOLVING THE P-LAPLACIAN OPERATOR

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Abstract. In this paper, we show that for each $\lambda \in \mathbb{R}$, there is an increasing sequence of eigenvalues for the nonlinear boundary-value problem

$$\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda\rho(x)|u|^{p-2}u + \mu|u|^{p-2}u \quad \text{on } \partial\Omega\,;$$

also we show that the first eigenvalue is simple and isolated. Some results about their variation, density, and continuous dependence on the parameter λ are obtained.

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1. Introduction and Notation

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, with $N \geq 1$. Let ρ be a function in $L^\infty(\partial\Omega)$ with $\rho \not\equiv 0$ and that can change sign. Let λ, p, μ be real numbers, with $1 < p < \infty$. We are interested in the nonlinear boundary-value problem

$$\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega \tag{1.1}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda\rho(x)|u|^{p-2}u + \mu|u|^{p-2}u \quad \text{on } \partial\Omega. \tag{1.2}$$

Here $\Delta_p u = \nabla(|\nabla u|^{p-2}\nabla u)$, which is known as the p-Laplacian and has attracted a lot of attention because of its applications. It appears in mathematical models for subject such as glaciology, nonlinear diffusion, filtration problem [17], power-low materials [14], non-Newtonian fluids [4], reaction-diffusion problems, flow through porous media, nonlinear elasticity, petroleum extraction, torsional creep problems, etc. For a discussion and some physical background, we refer the reader to [11]. The nonlinear boundary condition (1.2) describes a flux through the boundary $\partial\Omega$ which depends on the solution itself. For a physical motivation of such conditions, see for example [16].

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Observe that in the restrictive cases $\mu = 0$ or $p = 2$, (1.1)–(1.2) becomes linear and it is known as the Steklov problem [8].

Classical Dirichlet problems involving the p-Laplacian have been extensively studied by various authors in the cases: $\lambda = 0$ and $\mu = 0$; we cite the works [1, 2, 3, 11, 18, 19]. For the nonlinear boundary condition (1.2), recently the authors in [9] studied the case when $\mu = 0$ and ρ belongs to $L^s(\partial\Omega)$, which is not necessary essentially bounded, with an additional condition on its sign.

We set

$$\mu_1(\lambda) = \inf\big\{\|v\|_{1,p}^p - \lambda\int_{\partial\Omega}\rho(x)|v|^p d\sigma : v \in W^{1,p}(\Omega), \int_{\partial\Omega}|u|^p d\sigma = 1\big\}, \tag{1.3}$$

where $\|\cdot\|_{1,p}$ denotes the $W^{1,p}(\Omega)$-norm; i.e., $\|v\|_{1,p} = (\|\nabla v\|_p^p + \|v\|_p^p)^{1/p}$ and $\|\cdot\|_p$ is the L^p-norm, with σ is the Lebesgue measure of $\mathbb{R}^{N-1}$. We understand by the principal (or first) eigencurve of the p-Laplacian related to Sobolev trace embedding, the graph of the map $\mu_1 : \lambda \to \mu_1(\lambda)$ from $\mathbb{R}$ into $\mathbb{R}$. In [13], the authors proved the simplicity and isolation of the first eigencurve of Dirichlet p-Laplacian by extending a similar result shown by Binding and Huang in [7].

Our purpose of this paper is to extend some of the results known in the ordinary Dirichlet p-Laplacian, by using suitable Sobolev trace embeddings which lead to a nonlinear eigenvalue problem where the two parameter eigenvalues appear at the nonlinear boundary condition. We show that $\mu_1(\lambda)$ is simple and isolated for any $\lambda \in \mathbb{R}$. Note that to show the simplicity (uniqueness) result, we use a simple convexity argument, by remarking that the energy functional associated with (1.1)–(1.2) is convex in u^p for nonnegative functions u, without use in any way $C^1(\Omega)$ and $L^\infty(\Omega)$ regularities of the eigenfunctions. Here our process is new.

Remark that $\mu_1(0) = \lambda_1$ the optimal reciprocal constant of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$. For the particular case $\mu = 0$ and $\rho \in L^s(\partial\Omega)$ (for a suitable s), the isolation and simplicity of the first eigenvalue of (1.1)–(1.2) are studied by [9].

The main objective of our work is to extend this result to any $\lambda \in \mathbb{R}$, by using new technical methods.

The rest of the paper is organized as follows. In Section 2, we establish some definitions and preliminaries. In Section 3, we use a variational method to prove the existence of a sequence of eigencurves of (1.1)–(1.2). In Section 4, we prove the simplicity and the isolation results of each point of the first eigencurve. Finally, in Section 5, we show some results about variations of the weight as a direct application of the simplicity result.

Definitions. In this paper, all solutions are weak solutions; i.e., $u \in W^{1,p}(\Omega)$ is a solution of (1.1)–(1.2), if for all $v \in W^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx + \int_\Omega |u|^{p-2}uv dx = \int_{\partial\Omega}(\lambda\rho(x) + \mu)|u|^{p-2}uv d\sigma. \tag{1.4}$$

If $u \in W^{1,p}(\Omega)\backslash\{0\}$, then u shall be called an eigenfunction of (1.1)–(1.2) associated with the eigenpair (λ, μ).

Set

$$\mathcal{M} = \big\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega}|u|^p d\sigma = 1\big\}. \tag{1.5}$$

We say that a principal eigenfunction of (1.1)–(1.2), an any eigenfunction $u \in \mathcal{M}$, $u \geq 0$ a.e. on $\bar{\Omega}$ associated to pair $(\lambda, \mu_1(\lambda))$.

Define the energy functionals on $W^{1,p}(\Omega)$ as

$$\Phi_\lambda(u) = \frac{1}{p}\|u\|_{1,p}^p - \frac{\lambda}{p}\int_{\partial\Omega} \rho(x)|u|^p d\sigma = \frac{1}{p}\|u\|_{1,p}^p + \Phi(u), \quad \lambda \in \mathbb{R}$$

and

$$\Psi(u) = \frac{1}{p}\int_{\partial\Omega} |u|^p d\sigma.$$

It is clear that for any $\lambda \in \mathbb{R}$, solutions of (1.1)–(1.2) are the critical points of Φ_λ restricted to $\mathcal{M}$. We shall deal with operators T acting from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))'$. T is said to belong to the class (S_+), if for any sequence v_n weakly convergent to v in $W^{1,p}(\Omega)$, and $\limsup_{n\to+\infty}\langle Tv_n, v_n - v\rangle \le 0$, it follows that $v_n \to v$ strongly in $W^{1,p}(\Omega)$, where $(W^{1,p}(\Omega))'$ is the dual of $W^{1,p}(\Omega)$ with respect to the pairing $\langle\cdot,\cdot\rangle$.

2. Existence Results

We will use Ljusternick-Schnirelmann theory on C^1-manifolds, see [19]. It is clear that for any $\lambda \in \mathbb{R}$, the functional Φ_λ is even and bounded from below on $\mathcal{M}$. Indeed, let $u \in \mathcal{M}$, then

$$\Phi_\lambda(u) \ge \frac{1}{p}(\|u\|_{1,p}^p - |\lambda|\|\rho\|_{\infty,\partial\Omega}).$$

So that

$$\Phi_\lambda(u) \ge \frac{1}{p}(\lambda_1 - |\lambda|\|\rho\|_{\infty,\partial\Omega}) > -\infty, \tag{2.1}$$

where $\lambda_1 = \mu_1(0)$ is the reciprocal of the optimal constant in the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$. By employing the Sobolev trace embedding, we deduce that

- Ψ and Φ are weakly continuous
- Ψ' and Φ' are compact.

The following lemma is the key to show the existence of eigenvalues.

Lemma 2.1. *For each $\lambda \in \mathbb{R}$, we have*

(i) *$(\Phi_\lambda)'$ maps the bounded sets in the bounded sets;*
(ii) *if $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and $(\Phi_\lambda)'(u_n)$ converges strongly in the space $(W^{1,p}(\Omega))'$, then $u_n \to u$ (strongly) in $W^{1,p}(\Omega)$;*
(iii) *the functional Φ_λ satisfies the Palais-Smale condition on $\mathcal{M}$; i.e., for each sequence $(u_n)_n \subset \mathcal{M}$, if $\Phi_\lambda(u_n)$ is bounded and*

$$(\Phi_\lambda)'(u_n) - c_n\Psi'(u_n) \to 0, \tag{2.2}$$

with $c_n = \frac{\langle(\Phi_\lambda)'(u_n),u_n\rangle}{\langle\Psi'(u_n),u_n\rangle}$. Then, $(u_n)_n$ has a convergent subsequence in $W^{1,p}(\Omega)$.

Proof. (i) Let $u, v \in W^{1,p}(\Omega)$. Thus

$$\langle(\Phi_\lambda)'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx + \int_\Omega |u|^{p-2}uv dx + \int_{\partial\Omega} \rho(x)|u|^{p-2}uv d\sigma.$$

By Hölder's inequality, we obtain

$$|\langle(\Phi_\lambda)'(u), v\rangle| \le \Big(\int_\Omega |\nabla u|^{(p-1)p'}dx\Big)^{1/p'}\|\nabla v\|_p + \Big(\int_\Omega |u|^{(p-1)p'}dx\Big)^{1/p'}\|v\|_p$$

$$\begin{aligned} &+ |\lambda| \|\rho\|_{\infty,\partial\Omega} \Big(\int_{\partial\Omega} |u|^{(p-1)p'} d\sigma \Big)^{1/p'} \|v\|_{p,\partial\Omega} \\ &= \|\nabla u\|_p^{p-1} \|\nabla v\|_p + \|u\|_p^{p-1} \|v\|_p + |\lambda| \|\rho\|_{\infty,\partial\Omega} \|u\|_{p,\partial\Omega}^{p-1} \|v\|_{p,\partial\Omega}. \end{aligned}$$

Now, the trace Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ ensures the existence of a constance $c > 0$ such that

$$\|w\|_{p,\partial\Omega} \leq c\|w\|_{1,p}, \quad \text{for each } w \in W^{1,p}(\Omega).$$

Therefore,

$$\|(\Phi_\lambda)'(u)\| \leq \|\nabla u\|_p^{p-1} \|\nabla v\|_p + \|u\|_p^{p-1} \|v\|_p + c^p |\lambda| \|\rho\|_{\infty,\partial\Omega}) \|u\|_{1,p}^{p-1} \|v\|_{1,p}.$$

It is clear that

$$\|\nabla u\|_p^{p-1} \|\nabla v\|_p + \|u\|_p^{p-1} \|v\|_p \leq \|u\|_{1,p}^{p-1} \|v\|_{1,p}.$$

Combining the above inequalities, we conclude that

$$|\langle (\Phi_\lambda)'(u), v \langle| \leq (1 + c^p |\lambda| \|\rho\|_{\infty,\partial\Omega}) \|u\|_{1,p}^{p-1} \|v\|_{1,p},$$

for any $u, v \in W^{1,p}(\Omega)$. It follows that

$$\|(\Phi_\lambda)'(u)\| \leq (1 + c^p |\lambda| \|\rho\|_{\infty,\partial\Omega}) \|u\|_{1,p}^{p-1},$$

where $\|\cdot\|$ denotes the norm of $(W^{1,p}(\Omega))'$. This implies assertion (i).

(ii) We use the condition (S_+) as follows. $(\Phi_\lambda)'(u_n)$ being a convergent sequence strongly to some $f \in (W^{1,p}(\Omega))'$. Thus, we have by calculation

$$\langle Au_n, v \rangle = \langle -\Delta_p u_n, v \rangle + \int_\Omega |u_n|^{p-2} u_n v dx + \int_{\partial\Omega} |\nabla u_n|^{p-2} \nabla u_n \nu v \, d\sigma, \tag{2.3}$$

for any $v \in W^{1,p}(\Omega)$, where A is an operator defined from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))'$ by

$$\langle Au, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx + \int_\Omega |u|^{p-2} uv \, dx.$$

This operator satisfies the condition (S_+) because $-\Delta_p$ does it [13].

If we take $v = u_n - u$ in (2.3) we obtain

$$\begin{aligned} &\langle Au_n, u_n - v \rangle \\ &= \langle -\Delta_p u_n, u_n - v \rangle + \int_\Omega |u_n|^{p-2} u_n (u_n - u) dx + \int_{\partial\Omega} |\nabla u_n|^{p-2} \nabla u_n \nu (u_n - u) \, d\sigma. \end{aligned}$$

Introducing $(\Phi_\lambda)'(u_n)$, we deduce that

$$\langle Au_n, u_n - u \rangle = \langle (\Phi_\lambda)'(u_n) - f, u_n - u \rangle + \langle f, u_n - u \rangle - \langle (\Phi_\lambda)'(u_n), u_n - u \rangle.$$

Using the compactness of Φ', we find that as $n \to \infty$,

$$\limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle \geq 0.$$

Hence $u_n \to u$ strongly in $W^{1,p}(\Omega)$, in virtue of the condition (S_+).

(iii) From (2.1) we deduce that $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$. Thus, without loss of generality, we can assume that $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ for some function $u \in W^{1,p}(\Omega)$. It follows that $\Psi'(u_n) \to \Psi'(u)$ in $(W^{1,p}(\Omega))'$ and $p\Psi(u) = 1$, because $p\Psi(u_n) = 1, \forall n \in \mathbb{N}^*$. Hence $u \in \mathcal{M}$. Since $(u_n)_n$ is bounded, then (i) ensures that $\{(\Phi_\lambda)'(u_n)\}$ is bounded. By a calculation we obtain via (2.2) that $\{(\Phi_\lambda)'(u_n)\}$ converges strongly in $(W^{1,p}(\Omega))'$. Consequently, from (ii) we conclude that $u_n \to u$ (strongly) in $W^{1,p}(\Omega)$. This achieves the proof of Lemma. $\square$

Set $\Gamma_k = \{K \subset \mathcal{M} : K$ symmetric, compact and $\gamma(K) = k\}$, where $\gamma(K) = k$ is the genus of K; i.e., the smallest integer k such that there is an odd continuous map from K to $\mathbb{R}^k \backslash \{0\}$.

Next, we establish our existence result.

Theorem 2.2. *For each $\lambda \in \mathbb{R}$ and each integer $k \in \mathbb{N}^*$,*

$$\mu_k(\lambda) := \inf_{K \in \Gamma_k} \max_{u \in K} \Phi_\lambda(u)$$

is a critical value of Φ_λ restricted to $\mathcal{M}$. More precisely, there exists $u_k(\lambda) \in \mathcal{M}$ such that

$$\mu_k(\lambda) = p\Phi_\lambda(u_k(\lambda)) = \max_{u \in K} p\Phi_\lambda(u)$$

and $(u_k(\lambda), \mu_k(\lambda))$ is a solution of (1.1)–(1.2). Moreover, $\mu_k(\lambda) \to +\infty$, as $k \to +\infty$.

Proof. In view of [19], we need only to prove that for any $k \in \mathbb{N}^*, \Gamma_k \neq \emptyset$ and the last assertion. Indeed, since $W^{1,p}(\Omega)$ is separable, there exist $(e_i)_{i \geq 1}$ linearly dense in $W^{1,p}(\Omega)$ such that $\operatorname{supp} e_i \cap \operatorname{supp} e_j =$ if $i \neq j$, where $\operatorname{supp} e_i$ denotes the support of e_i. We can suppose that $e_i \in \mathcal{M}$ (if not we take $e'_i = \frac{e_i}{p\Psi(e_i)}$).

Let $k \in \mathbb{N}^*$ and $\mathcal{F}_k = \operatorname{span}\{e_1, e_2, \dots, e_k\}$. $\mathcal{F}_k$ is a vector subspace and $\dim \mathcal{F}_k = k$. If $v \in \mathcal{F}_k$, then there exist $\alpha_1, \dots, \alpha_k$ in $\mathbb{R}$ such that $v = \sum_{i=1}^{i=k} \alpha_i e_i$. Thus

$$\Psi(v) = \sum_{i=1}^{i=k} |\alpha_i|^p \Psi(e_i) = \frac{1}{p} \sum_{i=1}^{i=k} |\alpha_i|^p,$$

because $\Psi(e_i) = 1$, for $i = 1, 2, \dots, k$. It follows that the map $v \to (p\Psi(v))^{1/p}$ is a norm on $\mathcal{F}_k$. Hence, there is a constant $c > 0$ so that

$$c\|v\|_{1,p} \leq (p\Psi(v))^{1/p} \leq \frac{1}{c}\|v\|_{1,p}, \quad \forall v \in \mathcal{F}_k.$$

That is,

$$c\|v\|_{1,p} \leq \Big(\int_{\partial\Omega} |v|^p d\sigma \Big)^{1/p} \leq \frac{1}{c}\|v\|_{1,p}, \quad \forall v \in \mathcal{F}_k.$$

This implies that the set

$$\mathcal{V} = \mathcal{F}_k \cap \big\{ v \in W^{1,p}(\Omega) : \|v\|_{p,\partial\Omega} \leq 1 \big\}$$

is bounded. Because $\mathcal{V} \subset B(0, \frac{1}{c})$, where

$$B(0, \frac{1}{c}) = \{v \in W^{1,p} : \|v\|_{1,p} \leq \frac{1}{c}\}.$$

Moreover $\mathcal{V}$ is a symmetric bounded neighborhood of the origin 0. Consequently, from [19, Proposition 2.3], we deduce that $\gamma(\mathcal{F}_k \cap \mathcal{M}) = k$. Then $\mathcal{F}_k \cap \mathcal{M} \in \Gamma_k$ (because $\mathcal{F}_k \cap \mathcal{M}$ is compact, since it exactly equals to the boundary of $\mathcal{V}$).

To complete the proof, it suffices to show that for any $\lambda \in \mathbb{R}$, $\mu_k(\lambda) \to +\infty$, as $k \to +\infty$. Indeed, let $(e_n, e_j^*)_{n,j}$ be a biorthogonal system such that $e_n \in W^{1,p}(\Omega)$, $e_j^* \in (W^{1,p}(\Omega))'$, the $(e_n)_n$ are dense in $W^{1,p}(\Omega)$; and the $(e_j^*)_j$ are total in $(W^{1,p}(\Omega))'$. Set for any $k \in \mathbb{N}^*$

$$\mathcal{F}_{k-1}^{\perp} = \overline{\operatorname{span}(e_{k+1}, e_{k+2}, e_{k+3}, \dots)}.$$

Observe that for any for any $K \in \Gamma_k$, $K \cap \mathcal{F}_{k-1}^{\perp} \neq \emptyset$ (by [19, (g) of Proposition 2.3]). Now, we claim that

$$t_k := \inf_{K\in\Gamma_k} \sup_{K\cap\mathcal{F}_{k-1}^{\perp}} p\Phi_\lambda(u) \to +\infty, \quad \text{as } k \to +\infty.$$

Indeed, to obtain the contradiction, assume for k large enough that there is $u_k \in \mathcal{F}_{k-1}^{\perp}$ with $\int_{\partial\Omega} |u_k|^p d\sigma = 1$ such that

$$t_k \le p\Phi_\lambda(u_k) \le M,$$

for some $M > 0$ independent of k. Therefore,

$$\|u_k\|_{1,p}^p - \lambda \int_{\partial\Omega} \rho(x)|u_k|^p d\sigma \le M.$$

Hence

$$\|u_k\|_{1,p}^p \le M + \lambda\|\rho\|_{\infty,\partial\Omega} < \infty.$$

This implies that $(u_k)_k$ is bounded in $W^{1,p}(\Omega)$. For a subsequence of $(u_k)_k$ if necessary, we can suppose that (u_k) converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\partial\Omega)$. By our choice of $\mathcal{F}_{k-1}^{\perp}$, we have $u_k \rightharpoonup 0$ in $W^{1,p}(\Omega)$. Because $\langle e_n^*, e_k\rangle$, for all $k \ge n$. This contradicts the fact that $\int_{\partial\Omega} |u_k|^p d\sigma = 1$, for all k and the the claim is proved.

Finally, since $\mu_k(\lambda) \ge t_k$ we conclude that $\mu_k(\lambda) \to +\infty$, as $k \to +\infty$ and the proof is complete. □

3. Simplicity and isolation of $\mu_1(\lambda)$

3.1. **Simplicity.** First, observe that solutions of (1.1)–(1.2), by an well-known advanced regularity, belong to $C^{1,\alpha}(\bar{\Omega})$, see [20].

Lemma 3.1. *Eigenfunctions associated to $\mu_1(\lambda)$ are either positive or negative in Ω. Moreover if $u \in C^{1,\alpha}(\Omega)$ then u has definite sign in $\bar{\Omega}$.*

Proof. Let u be an eigenfunction associated to $\mu_1(\lambda)$,. Since $\Phi_\lambda(|u|) \le \Phi_\lambda(u)$ and $\Psi(|u|) = \Psi(u)$, it follows from (1.3) that $|u|$ is also an eigenfunction associated to $\mu_1(\lambda)$. Using Harnack's inequality, cf. [14], we deduce that $|u| > 0$ in Ω and by continuity we conclude that has definite sign in $\bar{\Omega}$. In fact $|u| > 0$ in $\bar{\Omega}$ because $\frac{\partial u}{\partial \nu}(x_0) < 0$ for any $x_0 \in \partial\Omega$ with $u(x_0) = 0$, by applying Hopf's Lemma, see [21]. □

Theorem 3.2 (Uniqueness)**.** *For any $\lambda \in \mathbb{R}$, the eigenvalue $\mu_1(\lambda)$ defined by* (1.3) *is a simple; i.e., the set of the eigenfunctions associated with $(\lambda, \mu_1(\lambda))$ is $\{tu_1(\lambda) : t \in \mathbb{R}\}$, where $u_1(\lambda)$ denotes the principal eigenfunction associated with $(\lambda, \mu_1(\lambda))$.*

Proof. By Theorem 2.2 it is clear that $\mu_1(\lambda)$ is an eigenvalue of the problem (1.1)–(1.2) for any $\lambda \in \mathbb{R}$. Let u and v be two eigenfunctions associated to $(\lambda, \mu_1(\lambda))$, such that $u, v \in \mathcal{M}$. Thus in virtue of Lemma 3.1 we can assume that u and v are positives.

Note that $W^{1,p}(\Omega) \ni w \to \|\nabla w\|_p^p$; $w \to \int_{\partial\Omega} |w|^p d\sigma$ and $w \to \int_{\partial\Omega} \rho(x)|w|^p d\sigma$ are linear functionals in w^p, for $w^p \ge 0$. Hence if we consider $w = \left(\frac{u^p+v^p}{2}\right)^{1/p}$, then it belongs to $W^{1,p}(\Omega)$ and $\int_{\partial\Omega} |w|^p d\sigma = 1$. Consequently, w is admissible in the definition of $\mu_1(\lambda)$. On the other hand, by the convexity of $\chi \to |\chi|^p$ we have by calculation the following inequalities

$$
\begin{aligned}
\int_\Omega |\nabla w|^p dx &= \frac{1}{2}\int_\Omega |u^{p-1}\nabla u + v^{p-1}\nabla v|^p (u^p+v^p)^{1-p} dx \\
&= \frac{1}{2}\int_\Omega \Big|\frac{u^p}{u^p+v^p}\frac{\nabla u}{u} + \frac{v^p}{v^p+u^p}\frac{\nabla v}{v}\Big|^p (u^p+v^p)^{1-p} dx \\
&\le \frac{1}{2}\int_\Omega \Big(\frac{u^p}{u^p+v^p}\Big|\frac{\nabla u}{u}\Big|^p + \frac{v^p}{v^p+u^p}\Big|\frac{\nabla v}{v}\Big|^p\Big) dx \\
&\le \frac{1}{2}\int_\Omega (|\nabla u|^p + |\nabla v|^p) dx.
\end{aligned}
\tag{3.1}
$$

By the choice of u and v, we deduce that

$$
\left|t\frac{\nabla u}{u} + (1-t)\frac{\nabla v}{v}\right|^p = t\left|\frac{\nabla u}{u}\right|^p + (1-t)\left|\frac{\nabla v}{v}\right|^p, \tag{3.2}
$$

with $t = u^p/(u^p+v^p)$.

Now, we claim that Now, we claim that $u = v$ a.e. on $\overline{\Omega}$. Indeed, consider the auxiliary function

$$
F(\chi_1,\chi_2) = |t\chi_1 + (1-t)\chi_2|^p - t\,|\chi_1|^p + (1-t)\,|\chi_2|^p\,.
$$

Since $t \neq 0$, critical points of F are solutions of the system

$$
\frac{\partial F(\chi_1,\chi_2)}{\partial \chi_1} = pt\left(|t\chi_1 + (1-t)\chi_2|^{p-2}\,(t\chi_1 - |\chi_1|^{p-2}\,\chi_1\right) = 0; \tag{3.3}
$$

$$
\frac{\partial F(\chi_1,\chi_2)}{\partial \chi_2} = p(t-1)\left(|t\chi_1 + (1-t)\chi_2|^{p-2}\,(t\chi_1 - |\chi_2|^{p-2}\,\chi_2\right) = 0. \tag{3.4}
$$

Thus (3.2), (3.3) and (3.4) imply that $(\chi_1 = \frac{\nabla u}{u}, (\chi_2 = \frac{\nabla v}{v})$ is a solution of the above system. Therefore,

$$
\left|\frac{\nabla u}{u}\right|^{p-2}\frac{\nabla u}{u} = \left|\frac{\nabla v}{v}\right|^{p-2}\frac{\nabla v}{v}.
$$

Hence

$$
\frac{\nabla u}{u} = \frac{\nabla v}{v} \text{ a.e. in } \overline{\Omega}.
$$

This implies easily that $u = cv$ for some positive constant c. By normalization we conclude that $c = 1$. The proof is completed. ☐

Remark 3.3. Various proofs of the uniqueness result were given in Direchlet p-Laplacian case by using $C^{1,\alpha}$-regularity and L^∞-estimation of the first eigenfunctions and by applying either Picone's identity [1]; or Diaz-Saá's inequality [2, 10, 12], and or an abstract inequality [15].

3.2. **Isolation.**

Proposition 3.4. *For each $\lambda \in \mathbb{R}$, $\mu_1(\lambda)$ is the only eigenvalue associated with λ, having an eigenfunction that does not change sign on the boundary $\partial\Omega$.*

Proof. Fix $\lambda \in \mathbb{R}$ and let $u_1(\lambda)$ be the principal eigenfunction associated with $(\lambda, \mu_1(\lambda))$. Suppose that there exists an eigenfunction v corresponding to a pair

(λ,μ) with $v \geq 0$ on $\partial\Omega$ and $v \in \mathcal{M}$. By the Maximum Principle, $v > 0$ on $\overline{\Omega}$. For simplify of notation, set $u = u_1(\lambda)$. Let $\epsilon > 0$ be small enough, and write

$$u_\epsilon = u + \epsilon, \quad v_\epsilon = v + \epsilon, \tag{3.5}$$

$$\phi(u_\epsilon, v_\epsilon) = \frac{u_\epsilon^p - v_\epsilon^p}{u_\epsilon^{p-1}}. \tag{3.6}$$

It is clear that $\phi(u_\epsilon, v_\epsilon) \in W^{1,p}(\Omega)$ and it is an admissible test function in (1.1)–(1.2). Thus we obtain

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi(u_\epsilon, v_\epsilon)dx + \int_\Omega u^{p-1}\phi(u_\epsilon, v_\epsilon)dx \\ &= \int_{\partial\Omega} (\lambda\rho(x) + \mu_1(\lambda))u^{p-1}\phi(u_\epsilon, v_\epsilon)d\sigma \end{aligned} \tag{3.7}$$

and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\phi(u_\epsilon, v_\epsilon)dx + \int_\Omega v^{p-1}\phi(u_\epsilon, v_\epsilon)dx = \int_{\partial\Omega} (\lambda\rho(x) + \mu))v^{p-1}\phi(u_\epsilon, v_\epsilon)d\sigma. \tag{3.8}$$

From (3.7) and (3.8), we deduce by calculations that

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi(u_\epsilon, v_\epsilon)\,dx + \int_\Omega |\nabla v|^{p-2}\nabla v \nabla\phi(u_\epsilon, v_\epsilon)dx + \int_\Omega |v|^{p-2}v\phi(u_\epsilon, v_\epsilon)\,dx \\ &= \int_{\partial\Omega} \lambda\rho(x)\Big(\big(\frac{u}{u_\epsilon}\big)^{p-1} - \big(\frac{v}{v_\epsilon}\big)^{p-1}\Big)(u_\epsilon^p - v_\epsilon^p)d\sigma \\ &\quad + \mu_1(\lambda)\int_{\partial\Omega} u^{p-1}\big[u_\epsilon - \big(\frac{v_\epsilon}{u_\epsilon}\big)^{p-1}v_\epsilon\big]d\sigma + \mu\int_{\partial\Omega} u^{p-1}\big[v_\epsilon - \big(\frac{u_\epsilon}{v_\epsilon}\big)^{p-1}u_\epsilon\big]d\sigma. \end{aligned} \tag{3.9}$$

On the other hand, by a long calculation again, we obtain

$$\nabla\phi(u_\epsilon, v_\epsilon) = \big\{1 + (p-1)\big(\frac{v_\epsilon}{u_\epsilon}\big)^p\big\}\nabla u_\epsilon - p\big(\frac{v_\epsilon}{u_\epsilon}\big)^{p-1}\nabla v_\epsilon \tag{3.10}$$

and

$$\int_\Omega \big[u^{p-1}\phi(u_\epsilon, v_\epsilon) + v^{p-1}\phi(u_\epsilon, v_\epsilon)\big]\,dx = \int_\Omega \big[\big(\frac{u}{u_\epsilon}\big)^{p-1} - \big(\frac{v}{v_\epsilon}\big)^{p-1}\big](u_\epsilon^p - v_\epsilon^p)dx. \tag{3.11}$$

Therefore, (3.9), (3.10) and (3.11) yield

$$\begin{aligned} &\int_\Omega \Big[\big\{1 + (p-1)\big(\frac{v_\epsilon}{u_\epsilon}\big)^p\big\}|\nabla u_\epsilon|^p + \big\{1 + (p-1)\big(\frac{u_\epsilon}{v_\epsilon}\big)^p\big\}|\nabla v_\epsilon|^p\Big]dx \\ &\quad + \int_\Omega \Big[-p\big(\frac{v_\epsilon}{u_\epsilon}\big)^{p-1}|\nabla v_\epsilon|^{p-2}\nabla u_\epsilon \nabla v_\epsilon + p\big(\frac{u_\epsilon}{v_\epsilon}\big)^{p-1}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla v_\epsilon\Big]dx \\ &= J_\epsilon + K_\epsilon - I_\epsilon, \end{aligned} \tag{3.12}$$

with

$$I_\epsilon = \int_\Omega \Big(\big(\frac{u}{u_\epsilon}\big)^{p-1} - \big(\frac{v}{v_\epsilon}\big)^{p-1}\Big)\,(u_\epsilon^p - v_\epsilon^p)\,dx\,, \tag{3.13}$$

$$J_\epsilon = \lambda\int_{\partial\Omega} \rho(x)\Big(\big(\frac{u}{u+\epsilon}\big)^{p-1} - \big(\frac{v}{v+\epsilon}\big)^{p-1}\Big)\,(u_\epsilon^p - v_\epsilon^p)\,d\sigma\,, \tag{3.14}$$

$$K_\epsilon = \mu_1(\lambda)\int_{\partial\Omega} u^{p-1}\Big[u_\epsilon - \big(\frac{v_\epsilon}{u_\epsilon}\big)^{p-1}v_\epsilon\Big]d\sigma + \mu\int_{\partial\Omega} u^{p-1}\Big[v_\epsilon - \big(\frac{u_\epsilon}{v_\epsilon}\big)^{p-1}u_\epsilon\Big]d\sigma. \tag{3.15}$$

It is clear that $I_\epsilon \geq 0$. Now, thanks to the inequalities of Lindqvist [15], we can distinguish tow cases according to p.

First case: $p \geq 2$. From (3.12) we have

$$J_\epsilon + K_\epsilon \geq \frac{1}{2^{p-2}-1} \int_\Omega \left(\frac{1}{(u+1)^p} + \frac{1}{(v+1)^p} \right) |u\nabla v - v\nabla u|^p dx \geq 0. \tag{3.16}$$

Second case: $1 < p < 2$.

$$J_\epsilon + K_\epsilon \geq c(p) \int_\Omega \frac{uv(u^p + v^p)}{(v|\nabla u| + u|\nabla v| + 1)^{2-p}} |u\nabla v - v\nabla u|^2 dx \geq 0, \tag{3.17}$$

where the constant $c(p) > 0$ independent of u, v, λ and $\mu_1(\lambda)$.

The Dominated Convergence Theorem implies

$$\lim_{\epsilon \to 0^+} J_\epsilon = \lim_{\epsilon \to 0^+} K_\epsilon = (\mu_1(\lambda) - \mu) \int_{\partial\Omega} (u^p - v^p) d\sigma = 0,$$

because

$$\int_{\partial\Omega} u^p d\sigma = \int_{\partial\Omega} v^p d\sigma = 1. \tag{3.18}$$

Now, letting $\epsilon \to 0^+$ in (3.16) and (3.17), we arrive at $u\nabla v = v\nabla u$ a.e. on Ω. Thus

$$\nabla \left(\frac{u}{v} \right) = 0 \quad \text{a.e. on } \Omega.$$

Hence, there exists $t > 0$ such that $u = tv$ a.e. on Ω. By continuity $u = v$ a.e. in $\overline{\Omega}$; and by the normalization (3.18) we deduce that $t = 1$ and $u = v$ a.e. on $\partial\Omega$. This implies that $u = v$ a.e. on $\overline{\Omega}$. Finally, we conclude that $\mu = \mu_1(\lambda)$. Which completes the prof. □

Remark 3.5. Proposition 3.4 can also be shown by using *Picone's identity*. A similar result was given by [9] in the restrictive case $\lambda = 0$.

Corollary 3.6. *For each $\lambda \in \mathbb{R}$, if u is an eigenfunction associated with a pair (λ, μ) and $\mu \neq \mu_1(\lambda)$, then u changes sign on the boundary $\partial\Omega$. Moreover, we have the estimate*

$$\min(|\partial\Omega^-|, |\partial\Omega^+|) \geq c_{p^*}^{-N} (|\lambda| \|\rho\|_{\infty,\partial\Omega} + |\mu|)^{-\eta}, \tag{3.19}$$

where

$$\eta = \begin{cases} \frac{N}{p} & \text{if } 1 < p < N \\ 2 & \text{if } p > N; \end{cases}$$

c_{p^*} is the best constant in the Sobolev trace embedding $W^{1,p}(\Omega)$ in $L^{p^*}(\partial\Omega)$; and $|\partial\Omega^\pm|$ denotes the $(N-1)$-dimensional measure of $\partial\Omega^\pm$. Here $p^* = \frac{p(N-1)}{N-p}$ is the critical Sobolev exponent.

Proof. Set $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. It follows from (1.4), where we put $v = u^-$, that

$$\int_\Omega |\nabla u^-|^p dx + \int_\Omega |u^-|^p dx = \int_{\partial\Omega} (\lambda\rho(x) + \mu) |u^-|^p d\sigma.$$

Thus

$$\begin{aligned} \|u^-\|_{1,p} &\leq (|\lambda| \|\rho\|_{\infty,\partial\Omega} + |\mu|) \int_{\partial\Omega^-} |u^-|^p d\sigma \\ &\leq (|\lambda| \|\rho\|_{\infty,\partial\Omega} + |\mu|) \, |\partial\Omega^-|^{p/N} \Big(\int_{\partial\Omega} |u^-|^{p^*} \Big)^{p/p^*}. \end{aligned}$$

By the Sobolev embedding $W^{1,p}(\partial\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$, we deduce that
$$|\partial\Omega^-| \geq c_{p^*}^{-N}\left(|\lambda|\|\rho\|_{\infty,\partial\Omega} + |\mu|\right)^{-\eta}.$$
For $\partial\Omega^+$ the same estimate follows by taking $v = u^+$ in (1.4). Hence (3.19) follows. $\square$

Remark 3.7. (i) The right-hand side of (3.19) is positive because $\rho \not\equiv 0$ and if $\lambda = 0$ then μ shall be an eigenvalue of p-Laplacian related to trace embedding, so $\mu - \lambda_1 > 0$, with λ_1 is the first eigenvalue of (1.1)–(1.2) in the case ($\lambda = 0$).

(ii) As an easy consequence of Corollary 3.6, we get that the number of the nodal components of each eigenfunction of (1.1)–(1.2) is finite.

Using Proposition 3.4 and Corollary 3.6, we can state the following important result.

Theorem 3.8. *For each $\lambda \in \mathbb{R}$, $\mu_1(\lambda)$ is isolated.*

4. Variations of the weight

Let $\mu_1(\lambda) = \mu_1(\rho)$ and $u_1(\lambda) = u_1(\rho)$ (for indicating the dependance of the weight ρ).

Theorem 4.1. *For each $\lambda \in \mathbb{R}$, if $(\rho_k)_k$ is a sequence of functions in $L^\infty(\partial\Omega)$ that converges to ρ and $\rho \not\equiv 0$, then*
$$\lim_{k\to\infty} \mu_1(\rho_k) = \mu_1(\rho), \tag{4.1}$$
$$\lim_{k\to\infty} \|u_1(\rho_k) - u_1(\rho)\|_{1,p}^p = 0. \tag{4.2}$$

Proof. If $\lambda = 0$, the result is evident because $\mu_1(\rho_k) = \mu_1(\rho)$, for all $k \in \mathbb{N}^*$. If $\lambda \neq 0$, then for $v \in \mathcal{M}$,
$$|\lambda \int_{\partial\Omega} (\rho_k - \rho)|v|^p d\sigma| \leq |\lambda|\|\rho_k - \rho\|_{\infty,\partial\Omega}.$$
Using the convergence of ρ_k to ρ in $L^\infty(\partial\Omega)$, for all $\epsilon > 0$, there exists $k_\epsilon \in \mathbb{N}$ such that for all $k \geq k_\epsilon$,
$$|\lambda \int_{\partial\Omega} (|(\rho_k - \rho)|v|^p d\sigma| \leq |\lambda|\frac{\epsilon}{|\lambda|} = \epsilon.$$
This implies
$$\lambda \int_{\partial\Omega} \rho|v|^p d\sigma \leq \epsilon + \lambda \int_{\partial\Omega} \rho_k|v|^p d\sigma, \tag{4.3}$$
$$\lambda \int_{\partial\Omega} \rho_k|v|^p d\sigma \leq \epsilon + \lambda \int_{\partial\Omega} \rho|v|^p d\sigma, \tag{4.4}$$
for $v \in \mathcal{M}$, $\epsilon > 0$ and $k \geq k_\epsilon$.

On the other hand, we have $\rho \not\equiv 0$. We take k_ϵ large enough so that $\rho_k \not\equiv 0$. Thus
$$\mu_1(\rho_k) \leq \|v\|_{1,p}^p - \lambda \int_{\partial\Omega} \rho_k|v|^p d\sigma.$$
Combining (4.3) and (4.4), we obtain
$$\mu_1(\rho_k) \leq \|v\|_{1,p}^p - \lambda \int_{\partial\Omega} \rho|v|^p d\sigma + \epsilon.$$

Passing to the infimum over $v \in \mathcal{M}$, we find

$$\mu_1(\rho_k) \le \mu_1(\rho) + \epsilon, \quad \mu_1(\rho) \le \mu_1(\rho_k) + \epsilon, \quad \forall \epsilon > 0, \ \forall k > k_\epsilon.$$

Hence, we conclude the convergence (4.1).

For the strong convergence (4.2) we argue as follows. We have for k large enough, $\rho_k \not\equiv 0$ and

$$\mu_k(\rho_k) = \|u_1(\rho_k)\|_{1,p}^p - \lambda \int_{\partial\Omega} \rho_k (u_1(\rho_k))^p d\sigma. \tag{4.5}$$

Thus

$$\|u_1(\rho_k)\|_{1,p}^p \le |\mu_1(\rho_k)| + |\lambda| \|\rho_k\|_{\infty,\partial\Omega}.$$

From (4.1) and the convergence of ρ_k to ρ in $L^\infty(\partial\Omega)$, we deduce that $(u_1(\rho_k))_k$ is a bounded sequence in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is reflexive and compactly embedded in $L^p(\partial\Omega)$ we can extract a subsequence of $(u_1(\rho_k))_k$ again labelled by k, such that $u_1(\rho_k) \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and $u_1(\rho_k) \to u$ (strongly) in $L^p(\partial\Omega)$, as $k \to \infty$. We can also suppose that $u_1(\rho_k) \to u$ in $L^p(\Omega)$. Passing to a subsequence if necessary, we can assume that $u_1(\rho_k) \to u$ a.e. in $\bar{\Omega}$. Thus $u \ge 0$ a.e. in $\overline{\Omega}$. We will prove that $u \equiv u_1(\rho)$. To do this, using the Dominated Convergence Theorem in $\partial\Omega$, we deduce that

$$\int_{\partial\Omega} \rho_k (u_1(\rho_k))^p d\sigma \to \int_{\partial\Omega} \rho u^p d\sigma,$$

as $k \to \infty$. By (4.5), (4.1) and the lower weak semi-continuity of the norm we obtain that

$$\|u\|_{1,p}^p \le \mu_1(\rho) + \lambda \int_{\partial\Omega} \rho u^p d\sigma. \tag{4.6}$$

The normalization $\int_{\partial\Omega} u^p d\sigma = 1$ is proved. Moreover, $u \ge 0$ a.e. in $\bar{\Omega}$, because $u_1(\rho_k) > 0$ in $\overline{\Omega}$ Thus u is an admissible function in the variational definition of $\mu_1(\lambda)$. So

$$\mu_1(\lambda) \le \|u\|_{1,p}^p - \lambda \int_{\partial\Omega} \rho u^p d\sigma.$$

This and (4.6) yield

$$\mu_1(\rho) = \|u\|_{1,p}^p - \lambda \int_{\partial\Omega} \rho u^p d\sigma. \tag{4.7}$$

By the uniqueness of the principal eigenfunction associated to $\mu_1(\lambda)$, we must have $u \equiv u_1(\rho)$. Consequently the limit function $u_1(\rho)$ is independent of the choice of the (sub)sequence. Hence, $u_1(\rho_k)$ converges to $u_1(\rho)$ at least in $L^p(\partial\Omega)$ and in $L^p(\Omega)$. To complete the proof of (4.2), it suffices to use the Clarckson's inequalities related to uniform convexity of $W^{1,p}(\Omega)$. For this we distinguish two cases.

First case: $p \ge 2$. We have

$$\begin{aligned} &\int_\Omega \Big| \frac{\nabla u_1(\rho_k) - \nabla u_1(\rho)}{2} \Big|^p dx + \int_\Omega \Big| \frac{\nabla u_1(\rho_k) + \nabla u_1(\rho)}{2} \Big|^p dx \\ &\le \frac{1}{2} \int_\Omega |\nabla u_1(\rho_k)|^p dx + \frac{1}{2} \int_\Omega |\nabla u_1(\rho)|^p dx \end{aligned}$$

and

$$\mu_1(\rho_k) \int_{\partial\Omega} \Big(\frac{u_1(\rho_k) + u_1(\rho)}{2} \Big)^p d\sigma$$

$$\leq \int_\Omega \Big|\frac{\nabla u_1(\rho_k)+\nabla u_1(\rho)}{2}\Big|^p dx - \lambda \int_{\partial\Omega} \rho_k \Big(\frac{u_1(\rho_k)+u_1(\rho)}{2}\Big)^p d\sigma.$$

Moreover,

$$\int_\Omega \Big|\frac{u_1(\rho_k)-u_1(\rho)}{2}\Big|^p dx \leq \int_\Omega \Big|\frac{u_1(\rho_k)+u_1(\rho)}{2}\Big|^p dx + \frac{1}{2}\|u_1(\rho_k)\|_p^p + \frac{1}{2}\|u_1(\rho)\|_p^p.$$

Hence

$$\begin{aligned}
&\|u_1(\rho_k)-u_1(\rho)\|_{1,p}^p \\
&\leq -\mu_1(\rho_k)\int_{\partial\Omega}\Big(\frac{u_1(\rho_k)+u_1(\rho)}{2}\Big)^p d\sigma - \lambda\int_{\partial\Omega}\rho_k\Big(\frac{u_1(\rho_k)+u_1(\rho)}{2}\Big)^p d\sigma \\
&\quad + \frac{1}{2}\Big(\mu_1(\rho_k) - \lambda\int_{\partial\Omega}\rho_k(x)u_1(\rho_k)d\sigma\Big) + \frac{1}{2}\Big(\mu_1(\rho) - \lambda\int_{\partial\Omega}\rho u_1^p d\sigma\Big).
\end{aligned}$$

Then, by using the Dominated Convergence Theorem we deduce that

$$\limsup_{k\to+\infty}\|u_1(\rho_k)-u_1(\rho)\|_{1,p}^p = 0.$$

Second case: $1 < p < 2$. In this case, we have

$$\begin{aligned}
&\Big\{\int_\Omega\Big|\frac{\nabla u_1(\rho_k)-\nabla u_1(\rho)}{2}\Big|^p dx\Big\}^{\frac{1}{p-1}} + \Big\{\int_\Omega\Big|\frac{\nabla u_1(\rho_k)+\nabla u_1(\rho)}{2}\Big|^p dx\Big\}^{\frac{1}{p-1}} \\
&\leq \Big\{\frac{1}{2}\int_\Omega|\nabla u_1(\rho_k)|^p dx + \frac{1}{2}\int_\Omega|\nabla u_1(\rho)|^p dx\Big\}^{\frac{1}{p-1}}
\end{aligned}$$

and

$$\begin{aligned}
&\mu_1(\rho_k)\int_{\partial\Omega}\Big(\frac{u_1(\rho_k)+u_1(\rho)}{2}\Big)^p d\sigma \\
&\leq \int_\Omega\Big|\frac{\nabla u_1(\rho_k)+\nabla u_1(\rho)}{2}\Big|^p - \lambda\int_{\partial\Omega}\rho_k\Big(\frac{u_1(\rho_k)+u_1(\rho)}{2}\Big)^p d\sigma.
\end{aligned}$$

Hence, by the definitions of $\mu_1(\rho_k)$ and $\mu_1(\rho)$; and the second Clarckson's inequality we obtain the convergence (4.2). □

Corollary 4.2. *For any bounded domain Ω, the function $\lambda \to \mu_1(\lambda)$ is differentiable on $\mathbb{R}$ and the function $\lambda \to u(\lambda)$ is continuous from $\mathbb{R}$ into $W^{1,p}(\Omega)$. More precisely*

$$\mu_1'(\lambda_0) = -\int_{\partial\Omega}\rho(x)(u_1(\lambda_0))^p d\sigma, \quad \forall\lambda_0\in\mathbb{R}.$$

Proof. Denote by $\mu_1(\lambda,\rho)$ the principal eigenvalue associated with λ and the weight ρ and by $u_1(\lambda,\rho)$ the principal eigenfunction corresponding. Suppose that $\lambda_k \to \lambda_0$ in $\mathbb{R}$, then $h_k = \lambda_k\rho \to \lambda_0\rho = h$ in $L^\infty(\partial\Omega)$. From Theorem 4.1 we deduce that

$$\begin{gathered}
\mu_1(\lambda_k) = \mu_1(1,h_k) \to \mu_1(1,h) = \mu_1(\lambda_0), \\
u_1(\lambda_k) = u_1(1,h_k) \to u_1(1,h) = u_1(\lambda_0) \text{ in } W^{1,p}(\Omega).
\end{gathered}$$

For the differentiability, it suffices to use the variational characterization of $\mu_1(\lambda)$ and of $\mu_1(\lambda_0)$, so that

$$(\lambda-\lambda_0)\int_{\partial\Omega}\rho(x)(u_1(\lambda))^p d\sigma \leq \mu_1(\lambda)-\mu_1(\lambda_0) \leq (\lambda_0-\lambda)\int_{\partial\Omega}(u_1(\lambda_0))^p d\sigma,$$

for all $\lambda,\lambda_0\in\mathbb{R}$. This completes the proof. □

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