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EXISTENCE OF WEAK SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS INVOLVING THE p-LAPLACIAN

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Abstract. This paper shows the existence of nontrivial weak solutions for the quasilinear elliptic equation

$$-\big(\Delta_p u + \Delta_p(u^2)\big) + V(x)|u|^{p-2}u = h(u)$$

in $\mathbb{R}^N$. Here V is a positive continuous potential bounded away from zero and $h(u)$ is a nonlinear term of subcritical type. Using minimax methods, we show the existence of a nontrivial solution in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$ and then show that it decays to zero at infinity when $1 < p < N$.

1. Introduction

This paper is concerned with the quasilinear elliptic equation

$$-L_p u + V(x)|u|^{p-2}u = h(u), \quad u \in W^{1,p}(\mathbb{R}^N), \tag{1.1}$$

where

$$L_p u := \Delta_p u + \Delta_p(u^2)u,$$

and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p \le N$.

Such equations arise in various branches of mathematical physics. For example, solutions of (1.1), in the case $p = 2$, are related to the existence of solitary wave solutions for quasilinear Schrödinger equations

$$i\partial_t\psi = -\Delta\psi + V(x)\psi - \widetilde{h}(|\psi|^2)\psi - \kappa\Delta[\rho(|\psi|^2)]\rho'(|\psi|^2)\psi \tag{1.2}$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $V = V(x)$, $x \in \mathbb{R}^N$, is a given potential, κ is a real constant and $\rho, \widetilde{h}$ are real functions. The semilinear case corresponding to $\kappa = 0$ has been studied extensively in recent years, see for example [2, 12, 16] and references therein. Quasilinear equations of the form (1.2) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of ρ. The case $\rho(s) = s$ was used for the superfluid film equation in plasma physics by Kurihura in [19] (cf. [20]). In the case $\rho(s) = (1 + s)^{1/2}$, equation (1.2) models the self-channeling of a high-power ultra short laser in matter, see [3, 4, 5, 28] and references in [7]. Equation (1.2) also

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appears in plasma physics and fluid mechanics [1, 18, 27, 30], in mechanics [15] and in condensed matter theory [25]. Considering the case $\rho(s) = s$, $\kappa > 0$ and putting

$$\psi(t, x) = \exp(-iFt)u(x), \quad F \in \mathbb{R},$$

we obtain a corresponding equation

$$-\Delta u - \Delta(u^2)u + V(x)u = h(u) \quad \text{in } \mathbb{R}^N \tag{1.3}$$

where we have renamed $V(x) - F$ to be $V(x)$, $h(u) = \widetilde{h}(u^2)u$ and we assume, without loss of generality, that $\kappa = 1$.

Our paper was motivated by the quasilinear Schrödinger equation (1.3), to which much attention has been paid in the past several years. This problem was studied in [6, 10, 22, 23, 24, 26] and references therein. Many important results on the existence of nontrivial solutions of (1.3) were obtained in these papers and give us very good insight into this quasilinear Schrödinger equation. The existence of a positive ground state solution has been proved in [26] by using a constrained minimization argument, which gives a solution of (1.3) with an unknown Lagrange multiplier λ in front of the nonlinear term. In [23], by a change of variables the quasilinear problem was transformed to a semilinear one and an Orlicz space framework was used as the working space, and they were able to prove the existence of positive solutions of (1.3) by the mountain-pass theorem. The same method of change of variables was used recently also in [6], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space and they studied different class of nonlinearities. In [10], for $N = 2$ the authors treated the case where the nonlinearity $h : \mathbb{R} \to \mathbb{R}$ has critical exponential growth, that is, h behaves like $\exp(4\pi s^4) - 1$ as $|s| \to \infty$. They establish an existence result for the problem by combining Ambrosetti-Rabinowitz mountain-pass theorem with a version of the Trudinger-Moser inequality in $\mathbb{R}^2$. In [24], it was established the existence of both one-sign and nodal ground states of soliton type solutions by the Nehari method.

Here, our goal is to prove by variational approach the existence of nontrivial weak solutions of (1.1). A function $u : \mathbb{R}^N \to \mathbb{R}$ is called a weak solution of (1.1) if $u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$ and for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ it holds

$$\begin{aligned} &\int_{\mathbb{R}^N} (1 + 2^{p-1}|u|^p)|\nabla u|^{p-2}\nabla u \nabla\varphi \,\mathrm{d}x + 2^{p-1}\int_{\mathbb{R}^N} |\nabla u|^p |u|^{p-2}u\varphi \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} g(x, u)\varphi \,\mathrm{d}x. \end{aligned} \tag{1.4}$$

where $g(x, u) := h(u) - V(x)|u|^{p-2}u$. We notice that we can not apply directly such methods because the natural functional associated to (1.1) given by

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N} (1 + 2^{p-1}|u|^p)|\nabla u|^p \,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(u)\,\mathrm{d}x,$$

where $H(s) = \int_0^s h(t)\,\mathrm{d}t$, is not well defined in general, for instance, in $W^{1,p}(\mathbb{R}^N)$. For example, if $1 < p < N$ and $u \in C_0^1(\mathbb{R}^N \backslash \{0\})$ is defined by

$$u(x) = |x|^{(p-N)/2p} \quad \text{for } x \in B_1 \backslash \{0\}$$

then we have that $u \in W^{1,p}(\mathbb{R}^N)$, but

$$\int_{\mathbb{R}^N} |u|^p |\nabla u|^p \,\mathrm{d}x = +\infty.$$

To overcome this difficulty, we generalize an argument developed by Liu, Wang and Wang [23] and Colin-Jeanjean [6] for the case $p = 2$. We make the change of variables $v = f^{-1}(u)$, where f is defined by

$$\begin{aligned} f'(t) &= \frac{1}{(1 + 2^{p-1}|f(t)|^p)^{1/p}} && \text{on } [0, +\infty), \\ f(t) &= -f(-t) && \text{on } (-\infty, 0]. \end{aligned} \tag{1.5}$$

Therefore, after the change of variables, from $J(u)$ we obtain the following functional

$$I(v) := J(f(v)) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p \, \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|f(v)|^p \, \mathrm{d}x - \int_{\mathbb{R}^N} H(f(v)) \, \mathrm{d}x \tag{1.6}$$

which is well defined on the space $W^{1,p}(\mathbb{R}^N)$ under the assumptions on the potential $V(x)$ and the nonlinearity $h(s)$ below. The Euler-Lagrange equation associated to the functional I is given by

$$-\Delta_p v = f'(v)g(x, f(v)) \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

In Proposition 2.2, we relate the solutions of (1.7) to the solutions of (1.1).

Here we require that the functions $V : \mathbb{R}^N \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy the following conditions:

- (V1) There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$;
- (V2) $\lim_{|x|\to\infty} V(x) = V_\infty$ and $V(x) \leq V_\infty$ for all $x \in \mathbb{R}^N$;
- (H0) h is odd and $h(s) = o(|s|^{p-2}s)$ at the origin;
- (H1) There exists a constant $C > 0$ such that for all $s \in \mathbb{R}$
$$|h(s)| \leq C(1 + |s|^r),$$
where $2p - 1 < r < 2p^* - 1$ if $1 < p < N$ and $r > 2p - 1$ if $p = N$;
- (H2) There exists $\theta \geq 2p$ such that $0 < \theta H(s) \leq sh(s)$ for all $s > 0$ where $H(s) = \int_0^s h(t)\mathrm{d}t$.

The following theorem contains our main result.

Theorem 1.1. *Let $1 < p \leq N$. Assume that* (V1)–(V2) *and* (H0)–(H1) *hold. Then* (1.1) *possesses a nontrivial weak solution $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ provided that one of the following two conditions is satisfied:*

- (a) (H2) *holds with $\theta > 2p$;*
- (b) (H2) *holds with $\theta = 2p$ and $p - 1 < r < p^* - 1$ if $1 < p < N$ or $r > p - 1$ if $p = N$ in* (H1).

Moreover, if $1 < p < N$ we have that $u \in L^\infty(\mathbb{R}^N)$ and $u(x) \to 0$ as $|x| \to \infty$.

The main difficulty in treating this class of quasilinear equations (1.1) is the possible lack of compactness due to the unboundedness of the domain besides the presence of the second order nonhomogeneous term $\Delta_p(u^2)u$ which prevents us to work directly with the functional J. To overcome these difficulties that has arisen from these features, we introduce the change of variables $u = f(v)$ and we reformulate our problem into a new one which has an associated functional I well defined and is of class C^1 on $W^{1,p}(\mathbb{R}^N)$. To find a nontrivial critical point of I, first we prove that I has the mountain-pass geometry. By using a version of the mountain-pass theorem we obtain a Cerami sequence for I that is bounded. This sequence converges weakly in $W^{1,p}(\mathbb{R}^N)$ to a critical point of I and by a lemma due to Lions and some facts related to a auxiliary problem, we show that this critical point is nontrivial. By a result of Tolksdorf [31] we conclude that the critical point belongs to $C^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$.

Our results are an improvement and generalization of some results obtained by Liu, Wang and Wang [23] and Colin-Jeanjean [6] for the case $p=2$. In these works, for example, the authors do not prove that if v is a weak solution of problem (1.7) then $u=f(v)$ is a weak solution of the original equation (1.1). They also do not show that the solutions decay to zero at infinity.

Notation. We use of the following notation:
- C, C_0, C_1, C_2, …denote positive (possibly different) constants.
- B_R denotes the open ball centered at the origin and radius $R>0$.
- $C_0^\infty(\mathbb{R}^N)$ denotes functions infinitely differentiable with compact support in $\mathbb{R}^N$.
- For $1\le p\le\infty$, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with the norms

$$\|u\|_p := \Big(\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x\Big)^{1/p},\quad 1\le p<\infty;$$
$$\|u\|_\infty := \inf\{C>0 : |u(x)|\le C \text{ almost everywhere in } \mathbb{R}^N\}.$$

- $W^{1,p}(\mathbb{R}^N)$ denotes the Sobolev spaces modelled on $L^p(\mathbb{R}^N)$ with its usual norm

$$\|u\| := \big(\|\nabla u\|_p^p + \|u\|_p^p\big)^{1/p}.$$

- $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual X^*.
- The weak convergence in X is denoted by $\rightharpoonup$, and the strong convergence by $\to$.

The outline of the paper is as follows. In Section 2, we give the properties of the change of variables $f(t)$ and some preliminary results. In Section 3, we present an auxiliary problem and some related results and Section 4 is devoted to the proof of Theorem 1.1.

2. Preliminary results

We begin with some preliminary results. Let us collect some properties of the change of variables $f:\mathbb{R}\to\mathbb{R}$ defined in (1.5), which will be usual in the sequel of the paper.

Lemma 2.1. *The function $f(t)$ and its derivative satisfy the following properties:*
1. *f is uniquely defined, C^2 and invertible;*
2. *$|f'(t)|\le 1$ for all $t\in\mathbb{R}$;*
3. *$|f(t)|\le|t|$ for all $t\in\mathbb{R}$;*
4. *$f(t)/t\to 1$ as $t\to 0$;*
5. *$|f(t)|\le 2^{1/2p}|t|^{1/2}$ for all $t\in\mathbb{R}$;*
6. *$f(t)/2\le tf'(t)\le f(t)$ for all $t\ge 0$;*
7. *$f(t)/\sqrt{t}\to a>0$ as $t\to+\infty$.*
8. *there exists a positive constant C such that*

$$|f(t)|\ge\begin{cases} C|t|, & |t|\le 1\\ C|t|^{1/2}, & |t|\ge 1.\end{cases}$$

Proof. To prove (1), it is sufficient to remark that the function

$$y(s) := \frac{1}{(1+2^{p-1}|s|^p)^{1/p}}$$

has bounded derivative. The point (2) is immediate by the definition of f. Inequality (3) is a consequence of (2) and the fact that $f(t)$ is odd and concave function for $t > 0$. Next, we prove (4). As a consequence of the mean value theorem for integrals, we see that

$$f(t) = \int_0^t \frac{1}{(1+2^{p-1}|f(s)|^p)^{1/p}}\,\mathrm{d}s = \frac{t}{(1+2^{p-1}|f(\xi)|^p)^{1/p}}$$

where $\xi \in (0,t)$. Since $f(0)=0$, we get

$$\lim_{t\to 0}\frac{f(t)}{t} = \lim_{\xi\to 0}\frac{1}{(1+2^{p-1}|f(\xi)|^p)^{1/p}} = 1.$$

To show the item (5), we integrate $f'(t)(1+2^{p-1}|f(t)|^p)^{1/p} = 1$ and we obtain

$$\int_0^t f'(s)(1+2^{p-1}|f(s)|^p)^{1/p}\,\mathrm{d}s = t$$

for $t > 0$. Using the change of variables $y = f(s)$, we get

$$t = \int_0^{f(t)} (1+2^{p-1}y^p)^{1/p}\,\mathrm{d}y \geq 2^{(p-1)/p}\frac{(f(t))^2}{2} = 2^{-1/p}(f(t))^2$$

and thus (5) is proved for $t \geq 0$. For $t < 0$, we use the fact that f is odd. The first inequality in (6) is equivalent to $2t \geq (1+2^{p-1}(f(t))^p)^{1/p}f(t)$. To show this inequality, we study the function $G:\mathbb{R}^+\to\mathbb{R}$, defined by $G(t) = 2t - (1+2^{p-1}(f(t))^p)^{1/p}f(t)$. Since $G(0)=0$ and using the definition of f we obtain for all $t \geq 0$

$$G'(t) = 1 - \frac{2^{p-1}(f(t))^p}{1+2^{p-1}(f(t))^p} = \frac{1}{1+2^{p-1}(f(t))^p} = (f'(t))^p > 0,$$

and the first inequality is proved. The second one is obtained in a similar way. Now, by point (4) it follows that $\lim_{t\to 0^+} f(t)/\sqrt{t} = 0$ and inequality (6) implies that for all $t > 0$

$$\frac{d}{dt}\Big(\frac{f(t)}{\sqrt{t}}\Big) = \frac{2f'(t)t - f(t)}{2t\sqrt{t}} \geq 0.$$

Thus, the function $f(t)/\sqrt{t}$ is nondecreasing for $t > 0$ and this together with estimate (5) shows the item (7). Point (8) is a immediate consequence of the limits (4) and (7). $\square$

We readily deduce that the functional $I : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ is of class C^1 under the conditions (V1)–(V2) and (H1)–(H2). Moreover,

$$\langle I'(v), w\rangle = \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\nabla w\,\mathrm{d}x - \int_{\mathbb{R}^N} g(x,f(v))f'(v)w\,\mathrm{d}x$$

for $v, w \in W^{1,p}(\mathbb{R}^N)$. Thus, the critical points of I correspond exactly to the weak solutions of (1.7). We have the following result that relates the solutions of (1.7) to the solutions of (1.1).

Proposition 2.2. (1) *If $v \in W^{1,p}(\mathbb{R}^N)\cap L^\infty_{\rm loc}(\mathbb{R}^N)$ is a critical point of the functional I, then $u = f(v)$ is a weak solution of* (1.1)*;*

(2) *If v is a classical solution of (1.7) then $u = f(v)$ is a classical solution of* (1.1).

Proof. First, we prove (1). We have that $|u|^p = |f(v)|^p \leq |v|^p$ and $|\nabla u|^p = |f'(v)|^p|\nabla v|^p \leq |\nabla v|^p$. Consequently, $u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$. As v is a critical point of I, we have for all $w \in W^{1,p}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla w \, \mathrm{d}x = \int_{\mathbb{R}^N} g(x, f(v)) f'(v) w \, \mathrm{d}x. \tag{2.1}$$

Since $(f^{-1})'(t) = \frac{1}{f'(f^{-1}(t))}$, it follows that

$$(f^{-1})'(t) = (1 + 2^{p-1}|f(f^{-1}(t))|^p)^{1/p} = (1 + 2^{p-1}|t|^p)^{1/p} \tag{2.2}$$

which implies that

$$\nabla v = (f^{-1})'(u)\nabla u = (1 + 2^{p-1}|u|^p)^{1/p}\nabla u. \tag{2.3}$$

For all $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$f'(v)^{-1}\varphi = (1 + 2^{p-1}|u|^p)^{1/p}\varphi \in W^{1,p}(\mathbb{R}^N)$$

and

$$\nabla(f'(v)^{-1}\varphi) = 2^{p-1}(1 + 2^{p-1}|u|^p)^{(1-p)/p}|u|^{p-2}u\varphi\nabla u + (1 + 2^{p-1}|u|^p)^{1/p}\nabla\varphi \tag{2.4}$$

Taking $w = f'(v)^{-1}\varphi$ in (2.1) and using (2.3)–(2.4), we obtain (1.4) which shows that $u = f(v)$ is a weak solution of (1.1).

Next, we prove (2). We have

$$\Delta_p v = \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\Big) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(|(f^{-1})'(u)\nabla u|^{p-2}(f^{-1})'(u)\frac{\partial u}{\partial x_i}\Big)$$

and deriving

$$\begin{aligned} \Delta_p v = &\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\Big)|(f^{-1})'(u)|^{p-2}(f^{-1})'(u) \\ &+ \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(|(f^{-1})'(u)|^{p-2}(f^{-1})'(u)\Big)|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\,. \end{aligned}$$

Using (2.2), we get

$$\Delta_p v = (1 + 2^{p-1}|u|^p)^{(p-1)/p}\Delta_p u + (p-1)2^{p-1}|u|^{p-2}u\left((1 + 2^{p-1}|u|^p\right)^{-1/p}|\nabla u|^p.$$

Thus,

$$\begin{aligned} &(1 + 2^{p-1}|u|^p)^{(p-1)/p}\Delta_p u + (p-1)2^{p-1}|u|^{p-2}u\left((1 + 2^{p-1}|u|^p\right)^{-1/p}|\nabla u|^p \\ &= -\frac{1}{(1 + 2^{p-1}|u|^p)^{1/p}}g(x,u); \end{aligned}$$

consequently

$$\Delta_p u + 2^{p-1}|u|^p\Delta_p u + (p-1)2^{p-1}|u|^{p-2}u|\nabla u|^p = -g(x,u)$$

Finally, observing that

$$2^{p-1}|u|^p\Delta_p u + (p-1)2^{p-1}|u|^{p-2}u|\nabla u|^p = \Delta_p(u^2)u$$

we conclude that $-\Delta_p u - \Delta_p(u^2)u = g(x,u)$. □

At this moment, it is clear that to obtain a weak solution of (1.1), it is sufficient to obtain a critical point of the functional I in $L^\infty_{\rm loc}(\mathbb{R}^N)$.

3. Auxiliary problem

To prove our main result, we shall use results due to do Ó - Medeiros [9] for the equation

$$-\Delta_p v = k(v) \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

The energy functional corresponding to (3.1) is

$$\mathcal{F}(v) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} K(v)\,\mathrm{d}x,$$

where $K(s) := \int_0^s k(t)\mathrm{d}t$. This functional is of class C^1 on $W^{1,p}(\mathbb{R}^N)$ under the assumptions on $k(s)$ below. The authors consider the following conditions on the nonlinearity $k(s)$:

(K0) $k \in C(\mathbb{R},\mathbb{R})$ and is odd;

(K1) When $1 < p < N$ we assume that

$$\lim_{s\to+\infty} \frac{k(s)}{s^{p^*-1}} = 0 \quad \text{where} \quad p^* = \frac{Np}{N-p};$$

when $p = N$ we require, for some $C > 0$ and $\alpha_0 > 0$, that

$$|k(s)| \le C[\exp(\alpha_0 |s|^{N/(N-1)}) - S_{N-2}(\alpha_0, s)],$$

for all $|s| \ge R > 0$, where

$$S_{N-2}(\alpha_0, s) = \sum_{k=0}^{N-2} \frac{\alpha_0^k}{k!} |s|^{kN/N-1};$$

(K2) When $1 < p < N$ we suppose that

$$-\infty < \liminf_{s\to 0^+} \frac{k(s)}{s^{p-1}} \le \limsup_{s\to 0^+} \frac{k(s)}{s^{p-1}} = -\nu < 0$$

and for $p = N$

$$\lim_{s\to 0} \frac{k(s)}{|s|^{N-1}} = -\nu < 0;$$

(K3) There exists $\zeta > 0$ such that $K(\zeta) > 0$.

Let

$$m := \inf\{\mathcal{F}(v) : v \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \text{ is a solution of (3.1)}\}. \tag{3.2}$$

By a *least energy solution (or ground state)* of (3.1) we mean a minimizer of m. Therefore, if w is a minimizer of (3.2) and v is any nontrivial solution of (3.1) then $\mathcal{F}(w) \le \mathcal{F}(v)$.

The following results are proved in [9, Theorems 1.4, 1.6 and 1.8].

Theorem 3.1. *Let $1 < p \le N$ and assume* (K0)–(K2). *Then setting*

$$\Lambda = \{\gamma \in C([0,1], W^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, \quad \mathcal{F}(\gamma(1)) < 0\}, \quad b = \inf_{\gamma\in\Lambda} \max_{0\le t\le 1} \mathcal{F}(\gamma(t)),$$

we have $\Lambda \neq \emptyset$ and $b = m$. Furthermore, for each least energy solution w of (3.1), there exists a path $\gamma \in \Lambda$ such that $w \in \gamma([0,1])$ and

$$\max_{t\in[0,1]} \mathcal{F}(\gamma(t)) = \mathcal{F}(w).$$

Theorem 3.2. *Let $1 < p \le N$. Under the hypotheses* (K0)–(K3)*, problem (3.1) has a least energy solution which is positive.*

Remark 3.3. In [9] it was also proved that under (K0)–(K2) there exist $\alpha > 0,\ \delta > 0$ such that $\mathcal{F}(v) \geq \alpha\|v\|^p$ if $\|v\| \leq \delta$.

4. Proof of Theorem 1.1

To prove Theorem 1.1 we first show that the functional I possesses the mountain-pass geometry. To do this, we shall use some results related to an auxiliary problem.

4.1. Mountain-pass geometry.

Lemma 4.1. *Under the hypotheses* (V1)–(V2) *and* (H0)–(H1)*, the functional I has a mountain-pass geometry.*

Proof. Let the energy functionals associated with the equations $-\Delta_p v = g_0(v)$ and $-\Delta_p v = g_\infty(v)$, respectively, be

$$J_0(v) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p \,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V_0|f(v)|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(v))\,\mathrm{d}x$$

$$J_\infty(v) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p \,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V_\infty|f(v)|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(v))\,\mathrm{d}x,$$

where

$$g_0(v) := f'(v)[h(f(v)) - V_0|f(v)|^{p-2}f(v)],$$
$$g_\infty(v) := f'(v)[h(f(v)) - V_\infty|f(v)|^{p-2}f(v)].$$

Note that $J_0(v) \leq I(v) \leq J_\infty(v)$ for all $v \in W^{1,p}(\mathbb{R}^N)$. It is not difficult to see that the nonlinearity g_0 satisfies the hypotheses (K0)–(K2). Thus, from Remark 3.3, we deduce that there exist $\beta_0 > 0$ and $\delta_0 > 0$ such that

$$I(v) \geq J_0(v) \geq \beta_0\|v\|^p \quad \text{if} \quad \|v\| \leq \delta_0. \tag{4.1}$$

Namely the origin is a strict local minimum for I. Moreover, since g_∞ also satisfies (K0)–(K2), applying Theorem 3.1 to the functional J_∞, we see that there exists $e \in W^{1,p}(\mathbb{R}^N)$ with $\|e\| > \delta_0$ such that $J_\infty(e) < 0$ which implies that $I(e) < 0$. Thus $\Gamma \neq \emptyset$, where

$$\Gamma = \{\gamma \in C([0,1], W^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}.$$

The lemma is proved. □

Remark 4.2. By the condition (H2), there exists $C > 0$ such that $H(s) \geq Cs^\theta$ for $s \geq 1$. In particular, we get $\lim_{s\to+\infty} H(s)/s^p = +\infty$. Thus, there exists $\zeta > 0$ such that $G_0(\zeta) > 0$ and $G_\infty(\zeta) > 0$ where

$$G_\infty(s) = \int_0^s g_\infty(t)\ \mathrm{d}t = H(f(s)) - \frac{V_\infty}{p}|f(s)|^p;$$
$$G_0(s) = \int_0^s g_0(t)\ \mathrm{d}t = H(f(s)) - \frac{V_0}{p}|f(s)|^p.$$

Therefore g_0 and g_∞ also satisfy (K3). As a consequence of Theorem 3.2, the problems

$$-\Delta_p v = g_0(v) \quad \text{and} \quad -\Delta_p v = g_\infty(v) \quad \text{in} \quad \mathbb{R}^N$$

have least energy solutions in $W^{1,p}(\mathbb{R}^N)$ which are positive.

4.2. **Cerami sequences.** We recall that a sequence (v_n) in $W^{1,p}(\mathbb{R}^N)$ is called *Cerami sequence* for I at the level c if

$$I(v_n) \to c \quad \text{and} \quad \|I'(v_n)\|(1+\|v_n\|) \to 0 \quad \text{as } n \to \infty.$$

We have the following lemma:

Lemma 4.3. *Suppose that* (V1)–(V2) *and* (H0)–(H2) *hold. Then each Cerami sequence for I at the level $c > 0$ is bounded in $W^{1,p}(\mathbb{R}^N)$.*

Proof. First, we will show that if a sequence (v_n) in $W^{1,p}(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|f(v_n)|^p \,\mathrm{d}x \le C \tag{4.2}$$

for some constant $C > 0$, then it is bounded in $W^{1,p}(\mathbb{R}^N)$. Indeed, we need just to prove that $\int_{\mathbb{R}^N} |v_n|^p \,\mathrm{d}x$ is bounded. We write

$$\int_{\mathbb{R}^N} |v_n|^p \,\mathrm{d}x = \int_{\{|v_n|\le 1\}} |v_n|^p \,\mathrm{d}x + \int_{\{|v_n|>1\}} |v_n|^p \,\mathrm{d}x.$$

By (8) and Remark 4.2 there exists $C > 0$ such that $H(f(s)) \ge Cs^p$ for all $s \ge 1$. This implies that

$$\int_{\{|v_n|>1\}} |v_n|^p \,\mathrm{d}x \le \frac{1}{C}\int_{\{|v_n|>1\}} H(f(v_n))\,\mathrm{d}x \le \frac{1}{C}\int_{\mathbb{R}^N} H(f(v_n))\,\mathrm{d}x.$$

Again using (8) in Lemma 2.1, it follows that

$$\int_{\{|v_n|\le 1\}} |v_n|^p \,\mathrm{d}x \le \frac{1}{C^p}\int_{\{|v_n|\le 1\}} |f(v_n)|^p \,\mathrm{d}x \le \frac{1}{C^p V_0}\int_{\mathbb{R}^N} V(x)|f(v_n)|^p \,\mathrm{d}x.$$

These estimates prove that (v_n) is bounded in $W^{1,p}(\mathbb{R}^N)$.

Now let (v_n) be in $W^{1,p}(\mathbb{R}^N)$ an arbitrary Cerami sequence for I at the level $c > 0$. We have that

$$\frac{1}{p}\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|f(v_n)|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(v_n))\,\mathrm{d}x = c + o_n(1) \tag{4.3}$$

and for all $\varphi \in W^{1,p}(\mathbb{R}^N)$

$$\begin{aligned}\langle I'(v_n),\varphi\rangle &= \int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{|f(v_n)|^{p-2}f(v_n)\varphi}{(1+2^{p-1}|f(v_n)|^p)^{1/p}}\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} \frac{h(f(v_n))\varphi}{(1+2^{p-1}|f(v_n)|^p)^{1/p}}\,\mathrm{d}x\end{aligned} \tag{4.4}$$

Considering the function $\varphi_n(x) := (1+2^{p-1}|f(v_n(x))|^p)^{1/p} f(v_n(x))$ and using points (3) and (6) in Lemma 2.1, we obtain that $|\varphi_n| \le |v_n|$ and

$$|\nabla\varphi_n| = \left(1 + \frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\right)|\nabla v_n| \le 2|\nabla v_n|.$$

Thus $\|\varphi_n\| \le 2\|v_n\|$. Taking $\varphi = \varphi_n$ in (4.4) and since (v_n) is a Cerami sequence, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(1 + \frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\right)|\nabla v_n|^p \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} V(x)|f(v_n)|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} h(f(v_n))f(v_n)\,\mathrm{d}x \\ &= \langle I'(v_n),\varphi_n\rangle = o_n(1).\end{aligned} \tag{4.5}$$

From (4.3), (4.5) and (H2) it follows that

$$\int_{\mathbb{R}^N}\Big[\frac{1}{p}-\frac{1}{\theta}\Big(1+\frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\Big)\Big]|\nabla v_n|^p\,\mathrm{d}x+\frac{1}{2p}\int_{\mathbb{R}^N}V(x)|f(v_n)|^p\,\mathrm{d}x \le c+o_n(1). \tag{4.6}$$

If $\theta > 2p$, we get

$$\Big(\frac{\theta-2p}{p\theta}\Big)\int_{\mathbb{R}^N}|\nabla v_n|^p\,\mathrm{d}x+\frac{1}{2p}\int_{\mathbb{R}^N}V(x)|f(v_n)|^p\,\mathrm{d}x\le c+o_n(1)$$

which shows that (4.2) holds and thus (v_n) is bounded. Now if $\theta = 2p$ we deduced from (4.6)

$$\frac{1}{2p}\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|f(v_n)|^p}\,\mathrm{d}x+\frac{1}{2p}\int_{\mathbb{R}^N}V(x)|f(v_n)|^p\,\mathrm{d}x\le c+o_n(1). \tag{4.7}$$

Denoting $u_n = f(v_n)$, we have that $|\nabla v_n|^p = (1 + 2^{p-1}|f(v_n)|^p)|\nabla u_n|^p$ and (4.7) implies that

$$\frac{1}{2p}\int_{\mathbb{R}^N}|\nabla u_n|^p\,\mathrm{d}x+\frac{1}{2p}\int_{\mathbb{R}^N}V(x)|u_n|^p\,\mathrm{d}x\le c+o_n(1). \tag{4.8}$$

From (4.8) we achieved that (u_n) is bounded in $W^{1,p}(\mathbb{R}^N)$. Using the hypotheses (H0)–(H1), we get

$$H(s)\le |s|^p+C|s|^{r+1} \tag{4.9}$$

and by Sobolev embedding $\int_{\mathbb{R}^N} H(f(v_n))\,\mathrm{d}x = \int_{\mathbb{R}^N} H(u_n)\,\mathrm{d}x$ is bounded, where we are supposing that the condition (b) in Theorem 1.1 holds. Hence, using (4.3) we obtain (4.2). Thus (v_n) is bounded in $W^{1,p}(\mathbb{R}^N)$ and this concludes the proof. $\square$

4.3. **Existence of nontrivial critical points for I.** Since I has the mountain-pass geometry, we know (see, for example, [8] and [11]) that I possesses a Cerami sequence (v_n) at the level

$$c=\inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I(\gamma(t))>0.$$

By Lemma 4.3, (v_n) is bounded. Thus, we can assume that, up to a subsequence, $v_n \rightharpoonup v$ in $W^{1,p}(\mathbb{R}^N)$. We claim that $I'(v) = 0$. Indeed, since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we only need to show that $\langle I'(v), \psi\rangle = 0$ for all $\psi \in C_0^\infty(\mathbb{R}^N)$. Observe that

$$\begin{aligned}
&\langle I'(v_n),\psi\rangle-\langle I'(v),\psi\rangle\\
&=\int_{\mathbb{R}^N}\big(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v\big)\,\nabla\psi\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}\Big(\frac{|f(v_n)|^{p-2}f(v_n)}{(1+2^{p-1}|f(v_n)|^p)^{1/p}}-\frac{|f(v)|^{p-2}f(v)}{(1+2^{p-1}|f(v)|^p)^{1/p}}\Big)V(x)\psi\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}\Big(\frac{h(f(v))}{(1+2^{p-1}|f(v)|^p)^{1/p}}-\frac{h(f(v_n))}{(1+2^{p-1}|f(v_n)|^p)^{1/p}}\Big)\psi\,\mathrm{d}x.
\end{aligned}$$

Using the fact that $v_n \to v$ in $L^q_{\rm loc}(\mathbb{R}^N)$ for $q \in [1, p^*)$ if $1 < p < N$, $q \ge 1$ if $p = N$, by the Lebesgue dominated convergence theorem and (H0)–(H1), it follows that

$$\langle I'(v_n),\psi\rangle-\langle I'(v),\psi\rangle\to 0.$$

Since $I'(v_n) \to 0$, we conclude that $I'(v) = 0$. Now, we show that $v \ne 0$. Let us assume, by contradiction, that $v = 0$. We claim that (v_n) is also a Cerami sequence for the functional J_∞, defined previously, at the level c. In fact, using that $V(x) \to V_\infty$ as $|x| \to \infty$, $v_n \to 0$ in $L^p_{\rm loc}(\mathbb{R}^N)$ and (3) in Lemma 2.1 we have

$$J_\infty(v_n) - I(v_n) = \frac{1}{p}\int_{\mathbb{R}^N} (V_\infty - V(x))|f(v_n)|^p \,\mathrm{d}x \to 0.$$

Moreover, by the previous arguments, we obtain

$$\begin{aligned}\|J'_\infty(v_n) - I'(v_n)\| &= \sup_{\|u\|\le 1} |\langle J'_\infty(v_n), u\rangle - \langle I'(v_n), u\rangle| \\ &\le \sup_{\|u\|\le 1} \int_{\mathbb{R}^N} |f(v_n)|^{p-1}|V_\infty - V(x)||u| \,\mathrm{d}x \\ &\le \Big(\int_{\mathbb{R}^N} |f(v_n)|^p |V_\infty - V(x)|^{p/(p-1)} \,\mathrm{d}x\Big)^{(p-1)/p} \to 0,\end{aligned}$$

as $n \to \infty$ which implies

$$\|J'_\infty(v_n)\|(1 + \|v_n\|) \le \|J'_\infty(v_n) - I'(v_n)\|(1 + \|v_n\|) + \|I'(v_n)\|(1 + \|v_n\|) \to 0$$

as $n \to \infty$. Next we will prove that

Claim 1. There exist $\alpha > 0$, $R > 0$ and (y_n) in $\mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n)} |v_n|^p \,\mathrm{d}x \ge \alpha.$$

Verification. We suppose that the claim is not true. Therefore, it holds that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^p \,\mathrm{d}x = 0, \quad \forall\, R > 0.$$

By [21, Lemma I.1], we have $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (p, p^*)$ if $1 < p < N$ and $q > p$ if $p = N$. From (H0)–(H1), for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all $s \in \mathbb{R}$

$$h(f(s))f(s) \le \epsilon|f(s)|^p + C_\epsilon|f(s)|^{r+1}.$$

From this estimate, using (3) and (5) in Lemma 2.1, for $v \in W^{1,p}(\mathbb{R}^N)$ we get

$$\int_{\mathbb{R}^N} h(f(v))f(v) \,\mathrm{d}x \le \epsilon \int_{\mathbb{R}^N} |v|^p \,\mathrm{d}x + C_\epsilon \int_{\mathbb{R}^N} |v|^{r+1} \,\mathrm{d}x \tag{4.10}$$

$$\int_{\mathbb{R}^N} h(f(v))f(v) \,\mathrm{d}x \le \epsilon \int_{\mathbb{R}^N} |v|^p \,\mathrm{d}x + C_\epsilon \int_{\mathbb{R}^N} |v|^{(r+1)/2} \,\mathrm{d}x. \tag{4.11}$$

We use inequality (4.10) when $\theta = 2p$ and (4.11) when $\theta > 2p$. We are going consider only the case $\theta > 2p$ because the other one is similar. By (6) in Lemma 2.1 and (4.11) we see that for all $\epsilon > 0$

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \,\mathrm{d}x &\le \lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))f(v_n) \,\mathrm{d}x \\ &\le \lim_{n\to\infty} \Big(\epsilon \int_{\mathbb{R}^N} |v_n|^p \,\mathrm{d}x + C_\epsilon \int_{\mathbb{R}^N} |v_n|^{(r+1)/2} \,\mathrm{d}x\Big) \\ &\le \epsilon \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^p \,\mathrm{d}x\end{aligned}$$

because $(r+1)/2 \in (p, p^*)$ if $1 < p < N$ or $(r+1)/2 > p$ if $p = N$. We then obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))f(v_n) \,\mathrm{d}x = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \,\mathrm{d}x = 0. \tag{4.12}$$

Since $\langle I'(v_n), v_n\rangle \to 0$, it follows that
$$\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|f(v_n)|^{p-2} f(v_n)f'(v_n)v_n \,\mathrm{d}x \to 0.$$
Using again (6) in Lemma 2.1 we get
$$\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|f(v_n)|^p \,\mathrm{d}x \to 0.$$
By the first limit in (4.12) and (H2), we conclude that
$$\lim_{n\to\infty} \int_{\mathbb{R}^N} H(f(v_n)) \,\mathrm{d}x = 0.$$
This implies that $I(v_n) \to 0$ in contradiction with the fact that $I(v_n) \to c > 0$ and the Claim is proved.

Now, define $\widetilde{v}_n(x) = v_n(x + y_n)$. As (v_n) is a Cerami sequence for J_∞, it is not difficult to see that $\widetilde{v}_n$ is also a Cerami sequence for J_∞. Proceeding as in the case of (v_n), up to a subsequence, we obtain $\widetilde{v}_n \rightharpoonup \widetilde{v}$ with $J'_\infty(\widetilde{v}) = 0$. As $\widetilde{v}_n \to \widetilde{v}$ in $L^p(B_R)$, by Claim 1 we conclude that
$$\int_{B_R} |\widetilde{v}|^p \,\mathrm{d}x = \lim_{n\to\infty} \int_{B_R} |\widetilde{v}_n|^p \,\mathrm{d}x = \lim_{n\to\infty} \int_{B_R(y_n)} |v_n|^p \,\mathrm{d}x \geq \alpha.$$
what implies that $\widetilde{v} \neq 0$.
By (6) in Lemma 2.1, for all n we obtain
$$f^2(\widetilde{v}_n) - f(\widetilde{v}_n)f'(\widetilde{v}_n)\widetilde{v}_n \geq 0$$
which implies
$$|f(\widetilde{v}_n)|^p - |f(\widetilde{v}_n)|^{p-2} f(\widetilde{v}_n)f'(\widetilde{v}_n)\widetilde{v}_n \geq 0.$$
Furthermore, from the condition (H2) we conclude for all n that
$$\frac{1}{p}h(f(\widetilde{v}_n))f'(\widetilde{v}_n)\widetilde{v}_n - H(f(\widetilde{v}_n)) \geq \frac{1}{2p}h(f(\widetilde{v}_n))f(\widetilde{v}_n) - H(f(\widetilde{v}_n)) \geq 0.$$
Thus, from Fatou's lemma and since $\widetilde{v}_n$ is a Cerami sequence for J_∞, we obtain
$$\begin{aligned}
c &= \lim_{n\to\infty} \Big[J_\infty(\widetilde{v}_n) - \frac{1}{p}\langle J'_\infty(\widetilde{v}_n), \widetilde{v}_n\rangle\Big] \\
&= \limsup_{n\to\infty} \frac{1}{p} \int_{\mathbb{R}^N} V_\infty \big[|f(\widetilde{v}_n)|^p - |f(\widetilde{v}_n)|^{p-2} f(\widetilde{v}_n)f'(\widetilde{v}_n)\widetilde{v}_n\big] \,\mathrm{d}x \\
&\quad + \limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big[\frac{1}{p}h(f(\widetilde{v}_n))f'(\widetilde{v}_n)\widetilde{v}_n - H(f(\widetilde{v}_n))\Big] \,\mathrm{d}x \\
&\geq \frac{1}{p} \int_{\mathbb{R}^N} V_\infty \big[|f(\widetilde{v})|^p - |f(\widetilde{v})|^{p-2} f(\widetilde{v})f'(\widetilde{v})\widetilde{v}\big] \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \Big[\frac{1}{p}h(f(\widetilde{v}))f'(\widetilde{v})\widetilde{v} - H(f(\widetilde{v}))\Big] \,\mathrm{d}x \\
&= J_\infty(\widetilde{v}) - \frac{1}{p}\langle J'_\infty(\widetilde{v}), \widetilde{v}\rangle = J_\infty(\widetilde{v}).
\end{aligned}$$
Therefore, $\widetilde{v} \neq 0$ is a critical point of J_∞ satisfying $J_\infty(\widetilde{v}) \leq c$. We deduce that the least energy level m_∞ for J_∞ satisfies $m_\infty \leq c$. We denote by $\widetilde{w}$ a least energy solution of the equation $-\Delta_p v = g_\infty(v)$ (see Remark 4.2). Now applying Theorem

3.1 to the functional J_∞ we can find a path $\gamma \in C([0,1], W^{1,p}(\mathbb{R}^N))$ such that $\gamma(0) = 0$, $J_\infty(\gamma(1)) < 0$, $\widetilde{w} \in \gamma([0,1])$ and

$$\max_{t\in[0,1]} J_\infty(\gamma(t)) = J_\infty(\widetilde{w}).$$

We can assume that $V \not\equiv V_\infty$ in (V2), otherwise there is nothing to prove. Thus

$$I(\gamma(t)) < J_\infty(\gamma(t)), \quad \forall\, t \in (0,1]$$

and hence

$$c \le \max_{t\in[0,1]} I(\gamma(t)) < \max_{t\in[0,1]} J_\infty(\gamma(t)) = J_\infty(\widetilde{w}) \le m_\infty \le c$$

which is a contradiction. Therefore, v is a nontrivial critical point of I.

4.4. **L^∞-estimate and decay to zero at infinity.** We know that for all $w \in W^{1,p}(\mathbb{R}^N)$

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla w \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|f(v)|^{p-2} f(v) f'(v) w \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} h(f(v)) f'(v) w \,\mathrm{d}x. \end{aligned} \tag{4.13}$$

Now, let us assume that $1 < p < N$. Without loss of generality, we are going suppose that $v \ge 0$. Otherwise, we work with the positive and negative parts of v. For each $k > 0$ we define

$$v_k = \begin{cases} v & \text{if } v \le k \\ k & \text{if } v \ge k, \end{cases}$$
$$\vartheta_k = v_k^{p(\beta-1)} v, \quad w_k = v v_k^{\beta-1}$$

with $\beta > 1$ to be determined later. Taking ϑ_k as a test function in (4.13), using that

$$h(f(v)) \le \frac{V_0}{2} f(v) + C f(v)^r,$$

and condition (V1) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} v_k^{p(\beta-1)} |\nabla v|^p \,\mathrm{d}x + p(\beta-1) \int_{\mathbb{R}^N} v_k^{p(\beta-1)-1} v \nabla v_k \nabla v \,\mathrm{d}x \\ &\le C \int_{\mathbb{R}^N} f(v)^r f'(v) v v_k^{p(\beta-1)} \,\mathrm{d}x. \end{aligned}$$

Because the second summand in the left side of the inequality above is not negative and using (5) and (6) in Lemma 2.1 we see that

$$\int_{\mathbb{R}^N} v_k^{p(\beta-1)} |\nabla v|^p \,\mathrm{d}x \le C \int_{\mathbb{R}^N} v^{(r+1)/2} v_k^{p(\beta-1)} \,\mathrm{d}x = C \int_{\mathbb{R}^N} v^{\widetilde{r}-p} w_k^p \,\mathrm{d}x \tag{4.14}$$

where $\widetilde{r} := (r+1)/2$. By the Gagliardo-Nirenberg inequality and (4.14), we obtain

$$\begin{aligned}
\Big(\int_{\mathbb{R}^N} w_k^{p^*}\,\mathrm{d}x\Big)^{p/p^*} &\le C_1 \int_{\mathbb{R}^N} |\nabla w_k|^p\,\mathrm{d}x \\
&\le C_2 \int_{\mathbb{R}^N} v_k^{p(\beta-1)}|\nabla v|^p\,\mathrm{d}x + C_3(\beta-1)^p \int_{\mathbb{R}^N} v^p v_k^{p(\beta-2)}|\nabla v_k|^p\,\mathrm{d}x \\
&\le C_4\beta^p \int_{\mathbb{R}^N} v_k^{p(\beta-1)}|\nabla v|^p\,\mathrm{d}x \\
&\le C_5\beta^p \int_{\mathbb{R}^N} v^{\widetilde{r}-p} w_k^p\,\mathrm{d}x,
\end{aligned}$$

where we have used that $v_k \le v$, $1 \le \beta^p$ and $(\beta-1)^p \le \beta^p$. Using the Hölder inequality,

$$\Big(\int_{\mathbb{R}^N} w_k^{p^*}\,\mathrm{d}x\Big)^{p/p^*} \le \beta^p C_5 \Big(\int_{\mathbb{R}^N} v^{p^*}\,\mathrm{d}x\Big)^{(\widetilde{r}-p)/p^*} \Big(\int_{\mathbb{R}^N} w_k^{pp^*/(p^*-\widetilde{r}+p)}\,\mathrm{d}x\Big)^{(p^*-\widetilde{r}+p)/p^*}.$$

Since that $|w_k| \le |u|^\beta$, by the continuity of the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ we get

$$\Big(\int_{\mathbb{R}^N} |v v_k^{\beta-1}|^{p^*}\,\mathrm{d}x\Big)^{p/p^*} \le \beta^p C_6 \|v\|^{\widetilde{r}-p} \Big(\int_{\mathbb{R}^N} v^{\beta pp^*/(p^*-\widetilde{r}+p)}\,\mathrm{d}x\Big)^{(p^*-\widetilde{r}+p)/p^*}.$$

Choosing $\beta = 1 + (p^* - \widetilde{r})/p$ we have $\beta pp^*/(p^* - \widetilde{r} + p) = p^*$. Thus,

$$\Big(\int_{\mathbb{R}^N} |v v_k^{\beta-1}|^{p^*}\,\mathrm{d}x\Big)^{p/p^*} \le \beta^p C_6 \|v\|^{\widetilde{r}-p} \|v\|_{\beta\alpha^*}^{p\beta},$$

where $\alpha^* = pp^*/(p^* - \widetilde{r} + p)$. By the Fatou's lemma,

$$\|v\|_{\beta p^*} \le (\beta^p C_6 \|v\|^{\widetilde{r}-p})^{1/p\beta} \|v\|_{\beta\alpha^*}. \tag{4.15}$$

For each $m = 0, 1, 2, \dots.$ let us define $\beta_{m+1}\alpha^* := p^*\beta_m$ with $\beta_0 := \beta$. Using the previous argument for β_1, by (4.15) we have

$$\begin{aligned}
\|u\|_{\beta_1 p^*} &\le (\beta_1^p C_6 \|u\|^{\widetilde{r}-p})^{1/p\beta_1} \|u\|_{\beta_1\alpha^*} \\
&\le (\beta_1^p C_6 \|u\|^{\widetilde{r}-p})^{1/p\beta_1} (\beta^p C_6 \|u\|^{r-p})^{1/p\beta} \|u\|_{\beta\alpha^*} \\
&\le (C_6 \|u\|^{\widetilde{r}-p})^{1/p\beta + 1/p\beta_1} (\beta)^{1/\beta} (\beta_1)^{1/\beta_1} \|u\|_{p*}.
\end{aligned}$$

Observing that $\beta_m = \chi^m \beta$ where $\chi = p^*/\alpha^*$, by iteration we obtain

$$\|u\|_{\beta_m p^*} \le (C_6 \|u\|^{\widetilde{r}-p})^{1/p\beta \sum_{i=0}^m \chi^{-i}} \beta^{1/\beta \sum_{i=0}^m \chi^{-i}} \chi^{1/\beta \sum_{i=0}^m i\chi^{-i}} \|u\|_{p*}.$$

Since $\chi > 1$ and $\lim_{m\to\infty} 1/(p\beta) \sum_{i=0}^m \chi^{-i} = 1/(p^* - \widetilde{r})$, we can take the limit as $m \to \infty$ to conclude that $v \in L^\infty(\mathbb{R}^N)$ and

$$\|v\|_\infty \le C_7 \|v\|^{(p^*-p)/(p^*-\widetilde{r})}.$$

In the case $p = N$, by using Theorem 1 in [29], we can conclude that v is locally bounded in $\mathbb{R}^N$. Resuming, in both cases, as a consequence of a result due to Tolksdorf [31], we obtain that $v \in C^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$, $\alpha \in (0, 1)$.

Next, for $1 < p < N$ we prove that $v(x) \to 0$ as $|x| \to \infty$. Since $v \in L^\infty(\mathbb{R}^N)$, by (V1), property (6) in Lemma 2.1 and (4.13) we conclude that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla\varphi\,\mathrm{d}x \le C \int_{\mathbb{R}^N} (1 + |v|^{p-1})\varphi\,\mathrm{d}x$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$. Thus, by Theorem 1.3 in [32] we have for any $x \in \mathbb{R}^N$,

$$\sup_{y \in B_1(x)} v(y) \leq C\|v\|_{L^p(B_2(x))}.$$

In particular, $v(x) \leq C\|v\|_{L^p(B_2(x))}$ and since

$$\|v\|_{L^p(B_2(x))} \to 0 \quad \text{as} \quad |x| \to \infty$$

we conclude that $v(x) \to 0$ as $|x| \to \infty$.

To finalize the proof of Theorem 1.1, we use (1) in Proposition 2.2 to conclude that $u = f(v)$ is a nontrivial weak solution of (1.1) in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$ and since $|u| = |f(v)| \leq |v|$, we have $u(x) \to 0$ as $|x| \to \infty$.

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