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NON-MONOTONE PERIOD FUNCTIONS FOR IMPACT OSCILLATORS

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ABSTRACT. The existence of non-monotone period functions for differential equations of the form

$$\ddot{x} + f(x) + \gamma H(x)g(x) = 0$$

is proved for large γ, where H is the Heaviside function and the functions f and g satisfy certain generic conditions. This result is precipitated by an analysis of the system

$$\ddot{x} + \sin x + \gamma H(x)x^{3/2} = 0,$$

which models the conservative dimensionless impact pendulum utilizing Hertzian contact during impact with a barrier at the downward vertical position.

1. Introduction

The study of period functions is important in applied mathematics; especially, to determine the range of resonances with respect to periodic forcing and for the solution of boundary value problems. Most often, as in the free-swinging pendulum, the period function is monotone. Techniques for detecting monotonicity are the subject of much current research (see, for example, [4, 5, 6, 7, 11, 12]). On the other hand, we know of only one example in the literature where a non-monotone period function occurs for a model equation in applied mathematics (see [9]). While the arguments in this paper are elementary, its main purpose is to report on a new physical model with a non-monotone period function.

Our study began after observing, via numerical simulation, a non-monotone period function for the (conservative) dimensionless impact pendulum model

$$\ddot{x} + \sin x + \gamma H(x)x^{3/2} = 0, \tag{1.1}$$

where γ is a positive parameter and H, here and hereafter, denotes the Heaviside function (see Appendix A or [3] for a derivation of this model). We will prove the existence of non-monotone period functions for the more general differential equation

$$\ddot{x} + f(x) + \gamma H(x)g(x) = 0. \tag{1.2}$$

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In section 2, we state the precise conditions that we require on the functions f and g for equation (1.2) to have a non-monotone period function. Under these assumptions, we prove in section 3 that the period function is decreasing near the rest point at the downward vertical and, as a corollary, the period function is non-monotone. In section 4, our non-monotonicity result is illustrated by numerical integration of the impact pendulum model (1.1).

2. The Model Equation

For the entirety of this paper, we will assume

- (H1) there exist positive constants K_1 and K_2 such that $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, $f \in C^4((-K_1, K_2)) \cap C^1(\mathbb{R})$ and $g \in C^1(\mathbb{R})$;
- (H2) $f(0) = f''(0) = 0$, $f'''(0) < 0$, $f(-K_1) = 0$, $f'(-K_1) < 0$ and $xf(x) > 0$ on $(-K_1, 0) \cup (0, K_2)$;
- (H3) $g'(x) > 0$ on $(0, K_2)$ and $g(0) = g'(0) = 0$; and
- (H4) for G such that $G'(x) = g(x)$ and $G(0) = 0$, there exists a positive constant M such that the inequality
$$R(x) := \frac{G'(x)^2 - 2G(x)G''(x)}{G'(x)^3} \le -M$$
is satisfied for $0 < x < K_2$.

We note that (H1) and (H2) imply $f'(0) > 0$, and we define F such that $F'(x) = f(x)$ and $F(0) = 0$.

The differential equation (1.2) is equivalent to the first-order system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -f(x) - \gamma H(x)g(x), \end{aligned} \tag{2.1}$$

which is in Hamiltonian form with Hamiltonian

$$E(x, y) := \frac{1}{2}y^2 + U(x, \gamma) = \frac{1}{2}y^2 + F(x) + \gamma H(x)G(x). \tag{2.2}$$

Moreover, it has rest points in the phase plane at $(x, y) = (0, 0)$ and $(-K_1, 0)$. We note that for there to be a rest point at $(-K_1, 0)$ no additional requirement on the function g is necessary because the Heaviside function vanishes for negative values of its argument.

System (2.1) has a hyperbolic saddle point at $(-K_1, 0)$; and, there is some number $\gamma_1 > 0$ such that, for $\gamma > \gamma_1$, this saddle point has a corresponding homoclinic orbit surrounding the rest point at the origin and a period annulus containing all other interior orbits. A sample phase portrait for the impact pendulum (1.1) is shown in Fig. 2. In general, the energies of the energy level sets surrounded by the homoclinic loop increase from 0 at the origin to $F(-K_1)$ at the homoclinic orbit.

We will prove that if $\gamma > 0$ is sufficiently large, then there is an open interval of energy levels, bounded below by the energy of the origin, for which the period function is decreasing. Since the period function increases near the homoclinic orbit and is C^1 in the punctured region surrounded by the homoclinic orbit, the period function is non-monotone for sufficiently large $\gamma > 0$. We opt for a simple self-contained proof of this result, which can also be obtained using more general results on the first derivatives of period functions (see, for example, [7]).

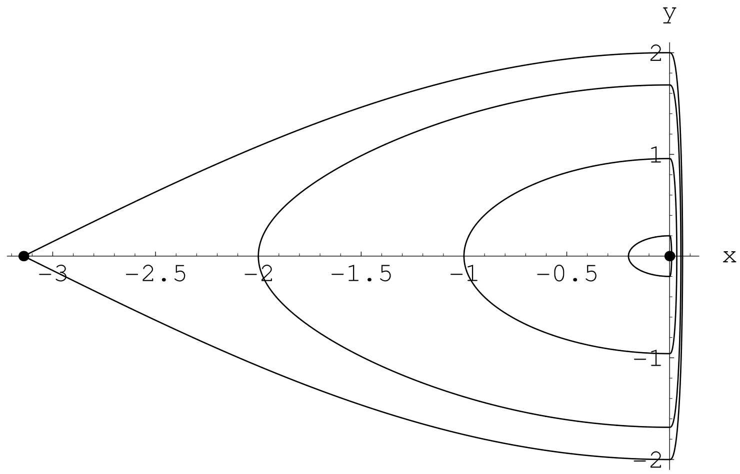

FIGURE 1. Phase portrait for the impact pendulum (1.1) for $\gamma = 5000$.

3. The Period Function Near the Origin

By our hypotheses, F is invertible on the interval $(-K_1, 0]$ and $F + \gamma G$ is invertible on $[0, K_2)$. For each energy level E in the range $(0, F(-K_1))$, let $x_-(E) := x \in (-K_1, 0)$ such that $E = F(x)$ and $x_+(E, \gamma) := x \in (0, K_2)$ such that $E = F(x) + \gamma G(x)$. Because $F^{-1}(0) = 0$ and $x_-(E) = F^{-1}(E)$, it follows that $\lim_{E\to 0} x_-(E) = 0$.

Lemma 3.1. *There exists $\gamma_2 > \gamma_1$ and a positive constant M such that for U as defined in formula* (2.2)*, we have*

$$W(x,\gamma) := \gamma \frac{(U_x(x,\gamma))^2 - 2U(x,\gamma)U_{xx}(x,\gamma)}{(U_x(x,\gamma))^3} \le \frac{-M}{2}$$

for all $\gamma > \gamma_2$ and $0 \le x \le K_2$.

Proof. We have that $W(x, \gamma) \to R(x)$ as $\gamma \to \infty$ and, by hypothesis **H4**, $R(x) \le -M$ on $(0, K_2)$. □

Lemma 3.2. *The function Q given by*

$$Q(s) := \frac{2F''(s)F(s) - (F'(s))^2}{(F'(s))^3},$$

for $s \neq 0$ and $Q(0) = 0$ is class C^1. Moreover, there exist positive constants E_2 and C such that $Q'(s) < 0$ for all $x_-(E) \le s < 0$ and $0 < Q(x_-(E)) \le C\sqrt{E}$ for all $E < E_2$.

Proof. The Taylor expansion of the function F at the origin has the form

$$F(s) = \frac{\lambda^2}{2}s^2 - \frac{\mu^2}{24}s^4 + O(s^5),$$

where λ and μ are positive constants. By substituting this series into the formula for Q and simplifying the resulting expression, we see that Q has a removable singularity at $s = 0$. The regularized Taylor series of Q at $s = 0$ has the form

$$Q(s) = -c^2 s + O(s^3),$$

where c is a positive constant. In particular, $Q'(0) = -c^2$. Thus, there exists $\delta > 0$ such that $-2c^2 \le Q'(s) \le -c^2/2$ for all $s \in (-\delta, 0)$. Also, since $x_-(E) \to 0^-$ as $E \to 0^+$, there exists E_1 such that $-\delta < x_-(E) \le 0$ for all $E < E_1$. So, for all $E < E_1$ and $x_-(E) < s < 0$, we have $Q'(s) < 0$.

By the Mean Value Theorem, there is some $\xi \in (0, s)$ and a positive constant c such that

$$Q(s) = Q(s) - Q(0) = |\,Q'(\xi)||\,s| \le 2c^2|\,s|$$

for all $s \in (-\delta, 0)$. So, $Q(x_-(E)) \le 2c^2|\,x_-(E)|$ for $E < E_1$.

Using the Taylor expansion of F, we also have

$$\lim_{E\to 0^+} \frac{|\,x_-(E)|}{\sqrt{E}} = \lim_{E\to 0^+} \frac{-x_-(E)}{\sqrt{F(x_-(E))}} = \lim_{s\to 0^-} \frac{-s}{\sqrt{\frac{\lambda^2}{2}s^2 + O(s^4)}} = \sqrt{\frac{2}{\lambda^2}}\,.$$

Hence, there exists $E_2 < E_1$ such that $|\,x_-(E)| \le 2\sqrt{E}/\sqrt{\lambda^2}$ for all $E < E_2$.

Combining our results, we have that

$$Q(x_-(E)) \le 2c^2|\,x_-(E)| \le \frac{2c}{\sqrt{\lambda^2}}\sqrt{E} = C\sqrt{E}$$

for all $E < E_2$, where we have consolidated constants. □

Let P be the period function on the period annulus surrounded by the homoclinic loop for system (1.2). The period for the orbit at energy level E is given by $P(E, \gamma)$, and the derivative of P with respect to E is denoted $P'(E, \gamma)$.

The next theorem is our main result.

Theorem 3.3. *Let γ_2 be the number in Lemma 3.1. There exists a positive number E_* such that $P'(E, \gamma) < 0$ for all $\gamma > \gamma_2$ and $0 < E < E_*$.*

Proof. Fix $\gamma > \gamma_2$. For simplicity, we will suppress γ in the expressions $U(x, \gamma)$, $x_+(e, \gamma)$ and $P(E, \gamma)$.

Using the Hamiltonian (2.2) and integrating along orbits, we arrive at the familiar formula for the period of the orbit at energy level E (see [2]):

$$P(E) = \frac{2}{\sqrt{2}} \int_{x_-(E)}^{x_+(E)} \frac{dx}{\sqrt{E - U(x)}}\,.$$

The change of variables $s = h(x)$, where $h(x) = \operatorname{sgn}(x)\sqrt{2U(x)}$, transforms the integral into

$$P(E) = \frac{2}{\sqrt{2}} \int_{-\sqrt{2E}}^{\sqrt{2E}} \frac{s}{U'(h^{-1}(s))\sqrt{E - \frac{s^2}{2}}}\,ds\,.$$

After another change of variables, $s = \sqrt{2E}\sin\theta$, the period function is represented by

$$P(E) = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{h'(h^{-1}(\sqrt{2E}\sin\theta))} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (h^{-1})'(\sqrt{2E}\sin\theta)\,d\theta. \tag{3.1}$$

By differentiating and splitting the integral in two pieces, we have

$$\begin{aligned} P'(E) &= \sqrt{\frac{2}{E}} \int_{-\frac{\pi}{2}}^{0} (h^{-1})''(\sqrt{2E}\sin\theta)\sin\theta \, d\theta \\ &\quad + \sqrt{\frac{2}{E}} \int_{0}^{\frac{\pi}{2}} (h^{-1})''(\sqrt{2E}\sin\theta)\sin\theta \, d\theta \\ &= I + II. \end{aligned}$$

Lemma 3.4. *Let E_2 be as in Lemma 3.2. If $0 < E < E_2$, then $0 < I < C$.*

Proof. Rewrite I as

$$I = \sqrt{\frac{2}{E}} \int_{-\frac{\pi}{2}}^{0} (h^{-1})''(h(\tau(\theta)))\sin\theta \, d\theta,$$

where $\tau(\theta) := F^{-1}(E\sin^2\theta)$, which is a well-defined function because F is restricted to $(-K_1, 0]$. Using the formula $(h^{-1})''(h(s)) = -h''(s)/(h'(s))^3$ and making the change of variables $s = \tau(\theta)$, we have

$$I = \sqrt{\frac{2}{E}} \int_{x_-(E)}^{0} \frac{-h''(s)\sin(\tau^{-1}(s))}{(h'(s))^3\tau'(\tau^{-1}(s))} \, ds\,.$$

Substituting in the formula

$$\tau^{-1}(s) = \sin^{-1}(-\sqrt{F(s)/E}\,)$$

and using the definitions of h and τ, the last expression for I simplifies to

$$I = \frac{1}{\sqrt{2E}} \int_{x_-(E)}^{0} Q(s)\frac{-F'(s)}{\sqrt{E - F(s)}} \, ds\,.$$

By Lemma 3.2 and the inequalities $Q(s) > 0$ and $F'(s) < 0$ on $(x_-(E), 0)$, it follows that I is positive and

$$I \le \frac{Q(x_-(E))}{\sqrt{2E}} \int_{x_-(E)}^{0} \frac{-F'(s)}{\sqrt{E - F(s)}} \, ds = \frac{\sqrt{2}Q(x_-(E))}{\sqrt{E}} \le \frac{\sqrt{2}\,C\sqrt{E}}{\sqrt{E}} = C\,,$$

where we have consolidated constants. ☐

Lemma 3.5. *Integral II is negative and $|\,II| > C/(\gamma\sqrt{E}\,)$.*

Proof. By defining $\tau(\theta) := U^{-1}(E\sin^2\theta)$ and proceeding as in Lemma 3.4, we express II in the form

$$II = \frac{1}{\sqrt{2E}} \int_{0}^{x_+(E)} \frac{(U'(s))^2 - 2U(s)U''(s)}{(U'(s))^3} \frac{U'(s)}{\sqrt{E - U(s)}} \, ds\,.$$

By Lemma 3.1 and the inequality $U'(s) \ge 0$ on $(0, x_+(E))$, the integrand is always negative. Thus, $II < 0$ and

$$|\,II| > \frac{M}{2\sqrt{2E}\gamma} \int_{0}^{x_+(E)} \frac{U'(s)}{\sqrt{E - U(s)}} \, ds = \frac{2M\sqrt{E}}{2\sqrt{2E}\gamma} = \frac{C}{\gamma\sqrt{E}}\,,$$

where we have consolidated constants. ☐

To complete the proof of the theorem, we choose $E_* = \min(E_2, C_2^2/(\gamma^2 C_1^2))$ so that $P'(E,\gamma) = I + II < 0$ whenever $0 < E < E_*$ and $\gamma < \gamma_2$. ☐

Corollary 3.6. *Let γ_2 be as in Theorem 3.3. If $\gamma > \gamma_2$, then the period function for system* (1.2) *is non-monotone and has at least one critical point in the interval* $[E_*, F(-K_1))$.

Corollary 3.7. *Let γ_2 be as in Theorem 3.3. If $\gamma > \gamma_2$ and $x_c > 0$ is sufficiently small, then the period function for the system*

$$\ddot{x} + f(x) + \gamma H(x - x_c)g(x - x_c) = 0 \tag{3.2}$$

is non-monotone and has at least two critical points.

Proof. Let $T(x_0, x_c)$ be the period of the orbit with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ for equation (3.2) and define

$$T(0, x_c) = \lim_{x_0 \to 0} T(x_0, x_c).$$

Since T is continuous and the function $x_0 \mapsto T(x_0, 0)$ is decreasing near the origin, there exists $\bar{x}_0$ near 0 such that $T(\bar{x}_0, 0) < T(0, 0)$. Hence, $T(\bar{x}_0, \bar{x}_c) < T(0, 0)$ for $\bar{x}_c$ sufficiently small.

Alternatively, in a neighborhood of the origin, for $x_c > 0$, the period function $T(x_0, \bar{x}_c)$ must coincide with the period function of $\ddot{x} + f(x) = 0$, which is increasing near the origin.

Because the periods of periodic orbits are unbounded in a neighborhood of the homoclinic loop boundary of the period annulus under consideration, we have the desired result. □

4. The Impact Pendulum

The non-dimensional system $\ddot{x} + \sin x + \gamma H(x)x^{\frac{3}{2}} = 0$ is a Hertzian contact model (see [8]) for an undamped unforced pendulum striking an elastic barrier at its downward vertical position. The constant γ corresponds to the elastic modulus of the barrier (see [10]).

The functions $f(x) = \sin x$ and $g(x) = x^{3/2}$ satisfy the assumptions in section 2 for Theorem 3.3, which states that there exists a region near the rest point at the barrier where the period function is decreasing. Using numerical integration techniques with $\gamma = 3.57 \times 10^8$ (an approximate value for an aluminum barrier), we are able to integrate the system numerically and graph its period function.

A plot of the period function near $E = 0$ is given in Fig. 2, which confirms that the period function is decreasing near $E = 0$. The interval of decrease is small in this case because γ is large.

Numerical experiments suggest that a version of the decreasing period phenomenon persists in case the wall is positioned at some positive angle relative to the downward vertical. The Hertzian contact model for the impact pendulum with wall angle $x_c > 0$ is

$$\ddot{x} + \sin x + \gamma H(x - x_c)(x - x_c)^{3/2} = 0. \tag{4.1}$$

Theorem 3.3 does not apply to the impact pendulum in this case because our hypotheses are not satisfied. In fact, due to the smoothness of the period function and its positive derivative in the region of small oscillation with no impacts, there must exist an interval containing the contact point on which the period function increases. Our numerical experiments verify this fact and suggest that the period function will decrease for an interval corresponding to more energetic impacts, reach

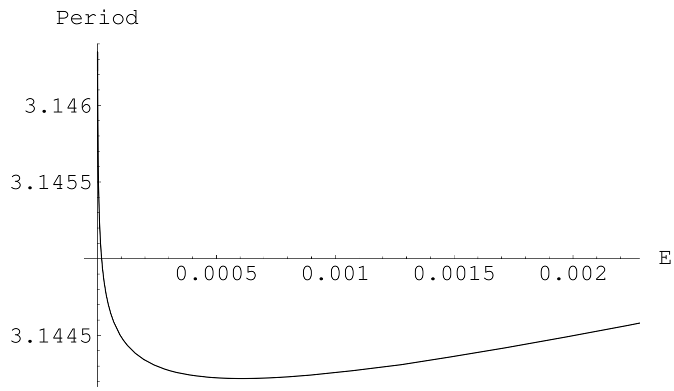

FIGURE 2. Period function for the impact pendulum (1.1) with $\gamma = 3.57 \times 10^8$.

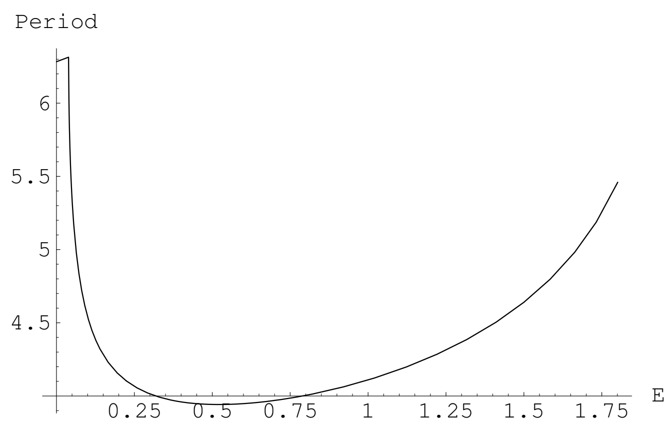

FIGURE 3. Period function for the impact pendulum (4.1) with $x_c = 0.281$ and $\gamma = 3.57 \times 10^8$.

a minimum value, and then increase as the energies of the periodic orbits approach the energy of the homoclinic loop. This scenario is illustrated in Fig. 3.

A natural prediction (cf. [1, Ch. 5]) is that harmonic motions of the periodically forced and damped pendulum with impacts will correspond to low-order resonances between the forcing period and the available periods of the conservative impact pendulum studied in this paper. In experiments, where only relatively small oscillations are feasible, the interval of available periods is the interval corresponding to the local maximum and local minimum in Fig. 3. By approximating these values, the range of (1 : 1)-period locking (harmonic motions) has been predicted and verified by physical experiments (see [3]).

Appendix A. Derivation of the Impact Pendulum Model

Consider a pendulum that encounters a barrier when the pendulum's angular position x (measured counterclockwise relative to the downward vertical) is x_c.

The kinetic energy for the pendulum is

$$T = \frac{1}{2}m\Big[\Big(L\dot{x}\cos x\Big)^2 + \Big(L\dot{x}\sin x\Big)^2\Big],$$

where m is the pendulum mass and L is the pendulum effective length. Here, the pendulum effective length refers to the distance between the pendulum pivot point and the pendulum center of mass. Because the mass in this physical system is distributed along the pendulum shaft and bob, the effective length differs from the total length, l, which is the distance from the pivot to the sphere center of mass. The pendulum's potential energy when not in contact with the barrier is

$$V = mgL(1 - \cos x).$$

Using Lagrange's equation

$$\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{x}}\Big) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0,$$

the equation of motion for the pendulum during the contact and non-contact regimes is

$$\ddot{x} + \omega^2 \sin x + \frac{l}{mL^2} H(x - x_c) F_c(x - x_c) = 0. \tag{A.1}$$

where H is the Heaviside function, F_c is the contact force function that occurs at distance l from the pendulum pivot point, and $\omega^2 = g/L$ is the square of the pendulum's natural frequency.

The Hertzian contact force is given by

$$F_c(x) = \frac{4}{3}E\sqrt{R}(l\sin x)^{3/2},$$

where E is the elastic modulus of the barrier and R is the radius of the sphere that impacts the barrier (see [8]).

The equation of motion (A.1) is made non-dimensional by changing the time-scale via $t \mapsto t/\omega$. After simplifying and replacing the sine function in the contact term by the first term of its Taylor series centered at x_c (which is justified by the small penetration depth), we obtain the smooth dimensionless model equation

$$\ddot{x} + \sin x + \gamma(x - x_c)^{3/2} H(x - x_c) = 0,$$

where

$$\gamma = \frac{4l^{5/2}ER^{1/2}}{3\omega^2 mL^2}.$$

While the equation of motion incorporates the discontinuous Heaviside function, we note that the contact term is class C^1 due to the presence of the Hertzian penetration function given in the model equation (A.2) by $(x - x_c)^{3/2}$.

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