EXISTENCE OF SOLUTIONS FOR IMPULSIVE NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

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ABSTRACT. In this paper an existence result for initial value problems for first order impulsive neutral functional differential equations with multiple delay is proved under weak conditions.

1. INTRODUCTION

Impulsive differential equations have become more important in recent years in some mathematical models of processes and phenomena studied in physics, optimal control, chemotherapy, biotechnology, population dynamics and ecology. The reader is referred to monographs [1, 2, 3] and references therein.

In this paper we study the existence of solutions for initial value problems for first order neutral functional differential equations, with multiple delays and with impulsive effects, of the form

$$\frac{d}{dt}[x(t) - f(t, x_t)] = g(t, x_t) + \sum_{i=1}^{p} x(t - \tau_i),$$

$$\text{a.e. } t \in J = [0, 1], t \neq t_k, k = 1, \ldots, m, \tag{1.1}$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \ldots, m, \tag{1.2}$$

$$x_0 = \phi, \tag{1.3}$$

where $f, g : J \times \mathcal{D} \to \mathbb{R}^n$ are given functions, $\mathcal{D}$ consists of functions $\psi : J_0 \to \mathbb{R}^n$ such that ψ is continuous everywhere except for a finite number of points s at which $\psi(s^-)$ and $\psi(s^+)$ exist with $\psi(s^-) = \psi(s)$, $J_0 = [-r, 0]$, $r = \max\{\tau_i : i = 1, \ldots, p\}$, $\phi \in \mathcal{D}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$ and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $k = 1, 2, \ldots, m$.

Our method of study is to convert the initial value problem (1.1)-(1.3) into equivalent integral equation and apply the Schaefer's fixed point theorem.

In the literature the existence of solutions for impulsive differential equations is studied under restrictive conditions on the impulses I_k, $k = 1, \ldots, m$. In many results, in addition to continuity, boundedness condition is often assumed, which

is not fulfilled in some important cases such as for linear impulses. Here, the only condition on the I_k, $k = 1, \ldots, m$, is continuity.

Throughout this paper, the terminology and notation are those used in functional analysis.

By $L^1(J, \mathbb{R}^n)$ we denote the Banach space of measurable functions $x : J \to \mathbb{R}^n$ which are Lebesgue integrable, normed by

$$\|x\|_{L^1} = \int_0^1 |x(t)| dt.$$

For $\psi \in \mathcal{D}$, the norm of ψ is defined by

$$\|\psi\|_{\mathcal{D}} = \sup\{|\psi(\theta)| : \theta \in J_0\}.$$

In order to define the solution of problem (1.1)-(1.3), we introduce the space $PC(J, \mathbb{R}^n)$ consisting of functions $x : J \to \mathbb{R}^n$ such that x is continuous everywhere except for $t = t_k$ at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, \ldots, m$. If we set $\Omega = \{x : J_1 \to \mathbb{R}^n \ / \ x \in \mathcal{D} \cap PC(J, \mathbb{R}^n)\}$, where $J_1 = [-r, 1]$, then Ω is a Banach space normed by

$$\|x\| = \sup\{|x(t)| : t \in J_1\}, \quad x \in \Omega.$$

Obviously, for any $x \in \Omega$ and any $t \in J$, the history function x_t defined by $x_t(\theta) = x(t + \theta)$, for $\theta \in J_0$, belongs to $\mathcal{D}$.

Also we denote by $AC((t_k, t_{k+1}), \mathbb{R}^n)$ the space of all absolutely continuous functions $x : (t_k, t_{k+1}) \to \mathbb{R}^n$, $k = 0, \ldots, m$.

A function $x \in \Omega \cap AC((t_k, t_{k+1}), \mathbb{R}^n)$, $k = 0, \ldots, m$, is said to be a solution of problem (1.1)-(1.3) if $x - f(\cdot, x_.)$ is absolutely continuous on $J \setminus \{t_1, \ldots, t_m\}$ and x satisfies the differential equation (1.1) a.e. on $J \setminus \{t_1, \ldots, t_m\}$ and the conditions (1.2)-(1.3).

Our main result will be proved using the following fixed point theorem due to Schaefer [4] (see also [5, page 29]).

Theorem 1.1. *Let X be a normed space and let $\Gamma : X \to X$ be a completely continuous map, that is, it is a continuous mapping which is compact on each bounded subset of X. If the set $\mathcal{E} = \{x \in X : \lambda x = \Gamma x \text{ for some } \lambda > 1\}$ is bounded, then Γ has a fixed point.*

2. Existence result

In this section we state and prove our existence result for problem (1.1)-(1.3), using the following conditions:

(H1) The function $f : J \times \mathcal{D} \to \mathbb{R}^n$ is such that

$$|f(t, x)| \leq c_1 \|x\|_{\mathcal{D}} + c_2 \text{ for all } t \in J \text{ and all } x \in \mathcal{D}$$

where $0 \leq c_1 < 1$ and $c_2 \geq 0$ are some constants.

(H2) The function $g : J \times \mathcal{D} \to \mathbb{R}^n$ is Carathéodory, that is,
  (i) $t \mapsto g(t, x)$ is measurable for each $x \in \mathcal{D}$,
  (ii) $x \mapsto g(t, x)$ is continuous for a.e. $t \in J$.

(H3) There exist a function $q \in L^1(J, \mathbb{R})$ with $q(t) > 0$ for a.e. $t \in J$ and a continuous and nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$|g(t, x)| \leq q(t)\psi(\|x\|_{\mathcal{D}}) \quad \text{for a.e. } t \in J \text{ and each } x \in \mathcal{D}$$

with

$$\int_C^\infty \frac{ds}{s + \psi(s)} = \infty \tag{2.1}$$

where

$$C = \frac{1}{1 - c_1}\Big[\|\phi\|_{\mathcal{D}}\Big(1 + c_1 + \sum_{i=1}^p \tau_i\Big) + 2c_2\Big].$$

(H4) The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, m$, are continuous.

Theorem 2.1. *Under assumptions* (H1)–(H4)*, the initial value problem* (1.1)–(1.3) *has a solution on* J_1.

Proof. Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $\Gamma : \Omega \to \Omega$ defined by

$$\Gamma x(t) = \begin{cases} \phi(t) & \text{for } t \in J_0, \\ \phi(0) - f(0, \phi(0)) + f(t, x_t) + \int_0^t g(s, x_s)ds \\ + \sum_{i=1}^p \int_{-\tau_i}^0 \phi(s)ds + \sum_{i=1}^p \int_0^{t-\tau_i} x(s)ds \\ + \sum_{0 < t_k < t} I_k(x(t_k^-)) & \text{for } t \in J. \end{cases}$$

We shall show that the operator Γ satisfies the conditions of Theorem 1.1 with $X = \Omega$. For better readability, we break the proof into a sequence of steps.

Step 1. We show that Γ has bounded values for bounded sets in Ω. To show this, let B be a bounded set in Ω. Then there exists a real number $\rho > 0$ such that $\|x\| \leq \rho$, for all $x \in B$.

Let $x \in B$ and $t \in J$. After some standard calculations we get

$$|\Gamma x(t)| \leq \|\phi\|_{\mathcal{D}}\Big(1 + c_1 + \sum_{i=1}^p \tau_i\Big) + 2c_2 + c_1\|x_t\|_{\mathcal{D}} + \int_0^1 q(s)\psi(\|x_s\|_{\mathcal{D}})ds$$

$$+ p\int_0^1 |x(s)|ds + \sum_{k=1}^m |I_k(x(t_k^-))|$$

$$\leq \|\phi\|_{\mathcal{D}}\Big(1 + c_1 + \sum_{i=1}^p \tau_i\Big) + 2c_2 + (c_1 + p)\rho$$

$$+ \psi(\rho)\|q\|_{L^1} + \sum_{k=1}^m \sup\{|I_k(u)| : |u| \leq \rho\} =: \eta.$$

If $t \in J_0$, then $|\Gamma x(t)| \leq \|\phi\|_{\mathcal{D}}$ and the previous inequality holds. Hence

$$\|\Gamma x\| \leq \eta, \quad \text{for all} \quad x \in B,$$

that is, Γ is bounded on bounded subsets of Ω.

Step 2. Next we show that Γ maps bounded sets into equicontinuous sets. Let B be, as in Step 1, a bounded set and $x \in B$. Let t and $h \neq 0$ be such that $t, t + h \in J\backslash\{t_1, \ldots, t_m\}$. It is not difficult to get

$$|\Gamma x(t + h) - \Gamma x(t)|$$

$$\leq |f(t + h, x_{t+h}) - f(t, x_t)| + \psi(\rho)\int_t^{t+h} q(s)ds + p\rho h + \sum_{t < t_k < t+h} |I_k(x(t_k^-))|.$$

As $h \to 0$ the right-hand side of the above inequality tends to zero. This proves the equicontinuity on $J \setminus \{t_1, \ldots, t_m\}$.

It remains to examine the equicontinuity at $t = t_i$, $i = 1, \ldots, m$. Let $t = t_i$ for some $i \in \{1, \ldots, m\}$ and let $h \neq 0$ be such that $\{t_k : k \neq i\} \cap [t_i - |h|, t_i + |h|] = \emptyset$. Then we have

$$|\Gamma x(t_i + h) - \Gamma x(t_i)| \leq |f(t_i + h, x_{t_i+h}) - f(t_i, x_{t_i})| + \psi(\rho) \int_{t_i}^{t_i+h} q(s)ds + p\rho h.$$

The right-hand side of the above inequality tends to zero as $h \to 0$. The equicontinuity on J_0 follows from the uniform continuity of ϕ on this interval.

Step 3. Now we show that Γ is continuous. Let $\{x_n\} \subset \Omega$ be a sequence such that $x_n \to x$. We will show that $\Gamma x_n \to \Gamma x$. For $t \in J$, we obtain

$$\begin{aligned}
|\Gamma x_n(t) - \Gamma x(t)| &\leq |f(t, x_{nt}) - f(t, x_t)| + \int_0^1 |g(s, x_{ns}) - g(s, x_s)|ds \\
&\quad + p \int_0^1 |x_n(s)) - x(s)|ds + \sum_{k=1}^m |I_k(x_n(t_k^-)) - I_k(x(t_k^-))|.
\end{aligned} \tag{2.2}$$

Using (H3) it can easily shown that the function $t \mapsto g(t, x_{nt}) - g(t, x_t)$ is Lebesgue integrable. By the continuity of f and I_k, $k = 1, \ldots, m$, and the dominated convergence theorem, the right-hand side of inequality (2.2) tends to zero as $n \to \infty$; which completes the proof that Γ is continuous.

As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we conclude that Γ is completely continuous.

Step 4. Finally we show that the set $\mathcal{E} = \{x \in \Omega : \lambda x = \Gamma x \text{ for some } \lambda > 1\}$ is bounded. Let $x \in \mathcal{E}$ and let $\lambda > 1$ be such that $\lambda x = \Gamma x$. Then $x|_{[-r, t_1]}$ satisfies, for each $t \in [0, t_1]$,

$$\begin{aligned}
x(t) = \lambda^{-1}\Big[&\phi(0) - f(0, \phi(0)) + f(t, x_t) + \int_0^t g(s, x_s)ds \\
&+ \sum_{i=1}^p \int_{-\tau_i}^0 \phi(s)ds + \sum_{i=1}^p \int_0^{t-\tau_i} x(s)ds \Big].
\end{aligned}$$

It is straightforward to verify that

$$\begin{aligned}
|x(t)| \leq \|\phi\|_{\mathcal{D}}\Big(1 + c_1 + \sum_{i=1}^p \tau_i\Big) + 2c_2 + c_1\|x_t\|_{\mathcal{D}} \\
+ \int_0^t [q(s)\psi(\|x_s\|_{\mathcal{D}})ds + p|x(s)|]ds.
\end{aligned} \tag{2.3}$$

Introduce the function $v_1(t) = \max\{|x(s)| : s \in [-r, t]\}$, for $t \in [0, t_1]$. We have $|x(t)|, \|x_t\|_{\mathcal{D}} \leq v_1(t)$ for all $t \in [0, t_1]$ and there is $t^* \in [-r, t]$ such that $v_1(t) = |x(t^*)|$. If $t^* < 0$, we have $v_1(t) \leq \|\phi\|_{\mathcal{D}}$ for all $t \in [0, t_1]$. Now, if $t^* \geq 0$, from (2.3) it follows that, for $t \in [0, t_1]$,

$$v_1(t) \leq \|\phi\|_{\mathcal{D}}\Big(1 + c_1 + \sum_{i=1}^p \tau_i\Big) + 2c_2 + c_1 v_1(t) + \int_0^t [q(s)\psi(v_1(s)) + pv_1(s)]ds$$

and hence

$$v_1(t) \leq C_1^1 + C_1^2 \int_0^t Q(s)[\psi(v_1(s)) + v_1(s)]ds$$

where

$$C_1^1 = C_1^2 \Big[\|\phi\|_{\mathcal{D}} \Big(1 + c_1 + \sum_{i=1}^{p} \tau_i \Big) + 2c_2 \Big], \quad C_1^2 = \frac{1}{1 - c_1}$$

and $Q(t) = \max\{q(t), p\}$, for $t \in [0, t_1]$. Set

$$w_1(t) = C_1^1 + C_1^2 \int_0^t Q(s)[\psi(v_1(s)) + v_1(s)]ds, \quad \text{for } t \in [0, t_1].$$

Then we have $v_1(t) \leq w_1(t)$ for all $t \in [0, t_1]$. A direct differentiation of w_1 yields

$$w_1'(t) \leq Q(t)[\psi(w_1(t)) + w_1(t)], \quad \text{a.e. } t \in [0, t_1]$$

$$w_1(0) = C_1^{(1)}.$$

By integration, this gives

$$\int_0^t \frac{w_1'(s)}{\psi(w_1(s)) + w_1(s)} \, ds \leq \int_0^t Q(s)ds \leq \|Q\|_{L^1}, \quad t \in [0, t_1]. \tag{2.4}$$

By a change of variables, inequality (2.4) implies

$$\int_{C_1^1}^{w_1(t)} \frac{ds}{\psi(s) + s} \leq \|Q\|_{L^1}, \quad t \in [0, t_1].$$

By (2.1) and the mean value theorem, there is a constant $M_1 = M_1(t_1) > 0$ such that $w_1(t) \leq M_1$ for all $t \in [0, t_1]$, and therefore $v_1(t) \leq M_1$, for all $t \in [0, t_1]$. At last, we choose M_1 such that $\|\phi\|_{\mathcal{D}} \leq M_1$ to get

$$\max\{|x(t)| : t \in [-r, t_1]\} = v_1(t_1) \leq M_1.$$

Now, consider $x|_{[-r, t_2]}$. It satisfies, for each $t \in [0, t_2]$,

$$x(t) = \lambda^{-1} \Big[\phi(0) - f(0, \phi(0)) + f(t, x_t) + \int_0^t g(s, x_s)ds$$

$$+ \sum_{i=1}^{p} \int_{-\tau_i}^{0} \phi(s)ds + \sum_{i=1}^{p} \int_0^{t-\tau_i} x(s)ds + I_1(x(t_1)) \Big].$$

Therefore,

$$|x(t)| \leq \|\phi\|_{\mathcal{D}} \Big(1 + c_1 + \sum_{i=1}^{p} \tau_i \Big) + 2c_2 + c_1 \|x_t\|_{\mathcal{D}}$$

$$+ \int_0^t [q(s)\psi(\|x_s\|_{\mathcal{D}}) + p|x(s)|]ds + \sup\{|I_1(u)| : |u| \leq M_1\}. \tag{2.5}$$

Denote $v_2(t) = \max\{|x(s)| : s \in [-r, t]\}$, for $t \in [0, t_2]$. Then, for each $t \in [0, t_2]$, we have $|x(t)|, \|x_t\|_{\mathcal{D}} \leq v_2(t)$. Let $t^* \in [-r, t]$ be such that $v_2(t) = |x(t^*)|$. In the case $t^* < 0$, we have $v_2(t) \leq \|\phi\|_{\mathcal{D}}$ for all $t \in [0, t_2]$. Now, if $t^* \geq 0$, then by (2.5) we have, for $t \in [0, t_2]$,

$$v_2(t) \leq \|\phi\|_{\mathcal{D}} \Big(1 + c_1 + \sum_{i=1}^{p} \tau_i \Big) + 2c_2 + c_1 v_2(t) + \int_0^t [q(s)\psi(v_2(s)) + pv_2(s)]ds$$

$$+ \sup\{|I_1(u)| : |u| \leq M_1\};$$

that is,

$$v_2(t) \leq C_2^1 + C_2^2 \int_0^t Q(s)[\psi(v_2(s)) + v_2(s)]ds$$

where

$$C_2^1 = C_2^2 \Big[\|\phi\|_{\mathcal{D}} \Big(1 + c_1 + \sum_{i=1}^{p} \tau_i \Big) + 2c_2 + \sup\{|I_1(u)| : |u| \le M_1\} \Big],$$

$C_2^2 = 1/(1 - c_1)$, and $Q(t) = \max\{q(t), p\}$, for $t \in [0, t_2]$.

If we set

$$w_2(t) = C_2^1 + C_2^2 \int_0^t Q(s)[\psi(v_2(s)) + v_2(s)]ds, \quad \text{for } t \in [0, t_2],$$

then $v_2(t) \le w_2(t)$ for all $t \in [0, t_2]$ and

$$w_2'(t) \le Q(t)[\psi(w_2(t)) + w_2(t)], \quad \text{a.e. } t \in [0, t_2]$$
$$w_2(0) = C_2^1.$$

This yields

$$\int_0^t \frac{w_2'(s)}{\psi(w_2(s)) + w_2(s)} \, ds \le \int_0^t Q(s)ds \le \|Q\|_{L^1}, \quad t \in [0, t_2]$$

which implies

$$\int_{C_2^1}^{w_2(t)} \frac{ds}{\psi(s) + s} \le \|Q\|_{L^1}, \quad t \in [0, t_2].$$

Again, by (2.1) and the mean value theorem, there is a constant $M_2 = M_2(t_1, t_2) > 0$ such that $w_2(t) \le M_2$ for all $t \in [0, t_2]$, and then $v_2(t) \le M_2$, for all $t \in [0, t_2]$. Finally, if we choose M_2 such that $\|\phi\|_{\mathcal{D}} \le M_2$, we get

$$\max\{|x(t)| : t \in [-r, t_2]\} = v_2(t_2) \le M_2.$$

Continue this process for $x|_{[-r,t_3]}, \dots, x|_{J_1}$, we obtain that there exists a constant $M = M(t_1, \dots, t_m) > 0$ such that

$$\|x\| \le M.$$

This finish to show that the set $\mathcal{E}$ is bounded in Ω.

As a result the conclusion of Theorem 1.1 holds and consequently the initial value problem (1.1)-(1.3) has a solution x on J_1. This completes the proof. $\square$

We conclude this paper with a discussion on two special cases. In each one, some of the conditions in Theorem 2.1 can be either removed or weakened.

Case 1: Consider the initial value problem for first order impulsive functional differential equations with multiple delays

$$x'(t) = g(t, x_t) + \sum_{i=1}^{p} x(t - \tau_i), \text{ a.e. } t \in J = [0, 1], t \ne t_k, k = 1, \dots, m, \quad (2.6)$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \dots, m, \quad (2.7)$$

$$x_0 = \phi, \quad (2.8)$$

derived from problem (1.1)-(1.3) when $f \equiv 0$. In this case one obtains the next existence result which is an immediate corollary of Theorem 2.1.

Theorem 2.2. *Under conditions* (H2)–(H4), *the initial value problem* (2.6)–(2.8) *has a solution on* J_1 *if constant* C *in* (H3) *is replaced by*

$$C = \|\phi\|_{\mathcal{D}}\Big(1 + \sum_{i=1}^{p} \tau_i\Big).$$

Case 2: Without the second term in the right hand side of (2.6), problem (2.6)-(2.8) is an initial value problem for first order impulsive functional differential equations

$$x'(t) = g(t, x_t), \ \text{a.e. } t \in J = [0,1], t \neq t_k, k = 1, \ldots, m, \tag{2.9}$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \ldots, m, \tag{2.10}$$

$$x_0 = \phi. \tag{2.11}$$

The corresponding existence result is as bellow. Its proof is omitted because it is the same as the proof of Theorem 2.1.

Theorem 2.3. *Under conditions* (H2)–(H4), *the initial value problem* (2.9)–(2.11) *has a solution on* J_1 *if relation* (2.1) *in* (H3) *is replaced by*

$$\int_{\|\phi\|_{\mathcal{D}}}^{\infty} \frac{ds}{\psi(s)} = \infty.$$

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