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GLOBAL ATTRACTOR FOR A SEMILINEAR PARABOLIC EQUATION INVOLVING GRUSHIN OPERATOR

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Abstract. The aim of this paper is to prove the existence of a global attractor for a semilinear degenerate parabolic equation involving the Grushin operator.

1. Introduction

In recent years, many works have been devoted to study the existence and nonexistence of solutions to a class of semilinear degenerate elliptic equations and systems involving an operator of Grushin type

$$G_k u = \Delta_x u + |x|^{2k}\Delta_y u, \quad (x, y) \in \Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, k \geqslant 0.$$

The Grushin operator G_k was first introduced in [6]. If $k > 0$ then G_k is not elliptic in domains in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ which contain the origin of $\mathbb{R}^{N_1}$. The local properties of G_k were investigated in [1, 6]. The existence and nonexistence results for the elliptic equation

$$\begin{gathered} G_k u + f(u) = 0, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered}$$

was obtained in [5, 11, 12]. Especially, in [11] the authors prove the Sobolev embedding theorem and show that the critical exponent of the embedding $S_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is $2_k^* = \frac{2N(k)}{N(k)-2}$, where $N(k) = N_1 + (k+1)N_2$. Furthermore, the semilinear elliptic systems with Grushin type operator, which are in the Hamilton form and potential form, were also studied in [3, 4, 9].

In this paper we are interested in the global existence and the long-time behavior of solutions to the following problem

$$\begin{gathered} u_t - G_k u + f(u) + g(x) = 0, \quad x \in \Omega, t > 0 \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ $(N_1, N_2 \geq 1)$, with smooth boundary $\partial\Omega$, $u_0 \in S_0^1(\Omega)$ given, $g \in L^2(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(u) - f(v)| \leqslant C_0|u - v|(1 + |u|^\rho + |v|^\rho)\,, \quad \frac{2}{N(k)-2} < \rho < \frac{4}{N(k)-2}, \tag{1.2}$$

$$F(u) \geq -\frac{\mu}{2}u^2 - C_1, \tag{1.3}$$

$$f(u)u \geq -\mu u^2 - C_2, \tag{1.4}$$

where $C_0, C_1, C_2 \geqslant 0$, F is the primitive $F(y) = \int_0^y f(s)ds$ of f, $\mu < \lambda_1$, λ_1 is the first eigenvalue of the operator $-G_k$ in Ω with homogeneous Dirichlet condition.

Denote $A = -G_k$, the positive and self-adjoint operator with domain of the definition

$$D(A) = \{u \in S_0^1(\Omega) : Au \in L^2(\Omega)\},$$

(see Sec. 2.1) and define the corresponding Nemytski map f by

$$f(u)(x) = f(u(x)), \quad u \in S_0^1(\Omega).$$

Then, (1.1) can be formulated as an abstract evolutionary equation

$$\frac{du}{dt} + Au + f(u) + g = 0, \quad u(0) = u_0. \tag{1.5}$$

The main purpose of this paper is to study the existence of a global solution and of a global attractor for the dynamic system generated by (1.5).

Note that when the exponent ρ in (1.2) satisfies $0 \leqslant \rho \leqslant \frac{2}{N(k)-2}$, the Nemytski f is a locally Lipschitzian map from $S_0^1(\Omega)$ to $L^2(\Omega)$. This combining with the fact that A is a sectorial operator in $L^2(\Omega)$ ensures the existence of a unique classical solution $u \in C([0,T), S_0^1(\Omega)) \cap C((0,T), D(A)) \cap C^1((0,T), L^2(\Omega))$. By computing directly we see that the solution u satisfies

$$\frac{d}{dt}\Phi(u(t)) = -\|u_t(t)\|^2, \tag{1.6}$$

where

$$\Phi(u) = \frac{1}{2}\|u\|^2_{S_0^1} + \int_\Omega (F(u) + gu)\,dx\,dy. \tag{1.7}$$

The equality (1.6) implies that Φ is a strict Lyapunov functional. Then the proof of existence of a global solution is quite straightforward by using the strictly Lyapunov functional Φ. Therefore, in this paper we will focus on the case of $\frac{2}{N(k)-2} < \rho < \frac{4}{N(k)-2}$. Firstly, under the assumption (1.3), one can check that the Nemytski f is a locally Lipschitzian map from $S_0^1(\Omega)$ to $L^q(\Omega)$, $q = \frac{2_k^*}{\rho+1}$. Secondly, by the fixed point method, we prove the existence of a unique local mild solution u, i.e. $u \in C([0,T), S_0^1(\Omega))$ is the solution of the following integral equation

$$u(t) = e^{-At}u_0 - \int_0^T e^{-A(t-s)}(f(u(s) + g)ds.$$

In this case, however, it is not easy to show that $\Phi(u)$ is a strict Lyapunov functional. Indeed, the equality (1.6) is obtained, at least formally, by taking the scalar product of the equation with u_t. Note that we only have $u_t \in S^{-1}(\Omega)$, the dual space of $S_0^1(\Omega)$, and so one cannot multiply the equation by u_t. Hence we have to study the regularity of u_t. We show that, in particular, $u_t \in S_0^1(\Omega)$. This enables us to use the natural Lyapunov functional $\Phi(u)$ and condition (1.3) to prove that the solution exists globally in time. Besides that, we also show that orbits of bounded sets are bounded. Finally, by proving the asymptotically compact property of the semigroup $S(t)$ generated by (1.5) and using the dissipativeness condition (1.4) for proving the boundedness of the set E of equilibrium points, we obtain the existence of a global attractor $\mathcal{A}$ in $S_0^1(\Omega)$. Furthermore, we show that every solution tends to the set of equiblirium points as $t \to +\infty$. We state our main result in the following theorem.

Theorem 1.1. *Under the assumptions* (1.2)-(1.4)*, problem* (1.5) *defines a semigroup $S(t) : S_0^1(\Omega) \to S_0^1(\Omega)$, which possesses a compact connected global attractor $\mathcal{A} = W^u(E)$ in the space $S_0^1(\Omega)$. Furthermore, for each $u_0 \in S_0^1(\Omega)$, the corresponding solution $u(t)$ tends to the set E of equiblirium points in $S_0^1(\Omega)$ as $t \to +\infty$.*

This result can be extended to some more generalized systems with the slight modifications on hypotheses and functional spaces, which are described in Remark 3.1. The rest of the paper is organized as follows. The next section recalls some notations and results related to Grushin operator and semigroup. Section 3 is devoted to deal with problem (1.1), for which the existence of the global solution and the global attractor is proved.

2. Preliminary Results

2.1. **Functional Spaces and Operators.** We begin by recall some results in [11]. Denote by $S^1(\Omega)$ the set of all functions $u \in L^2(\Omega)$ such that $\frac{\partial u}{\partial x_i}, |x|^k \frac{\partial u}{\partial y_j} \in L^2(\Omega)$, $i = 1, \dots, N_1$, $j = 1, \dots, N_2$, with the norm

$$\|u\|_{S^1(\Omega)} = \Big(\int_\Omega \big(|u|^2 + |\nabla_x u|^2 + |x|^{2k} |\nabla_y u|^2\big)\, dx\, dy\Big)^{1/2},$$

where $\nabla_x u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_{N_1}})$, $\nabla_y u = (\frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_{N_2}})$. The $S_0^1(\Omega)$ is defined as the closure of $C_0^1(\Omega)$ in $S^1(\Omega)$.

The following embedding inequality was proved in [11]

$$\Big(\int_\Omega |u|^p\, dx\, dy\Big)^{1/p} \leqslant C\Big(\int_\Omega \big(|u|^2 + |\nabla_x u|^2 + |x|^{2k} |\nabla_y u|^2\big)\, dx\, dy\Big)^{1/2},$$

where $2 \leqslant p \leqslant \frac{2N(k)}{N(k)-2} - \tau$, $C > 0$, $k \geqslant 0$, provided small number τ. Moreover, the number $2_k^* = \frac{2N(k)}{N(k)-2}$ is the critical Sobolev exponent of the embedding $S_0^1(\Omega) \hookrightarrow L^p(\Omega)$ and when $2 \leqslant p < 2_k^*$, the embedding is compact.

Denote $X = L^2(\Omega)$ and $(.,.)$ be the scalar product in X, the operator $A = -G_k$ is positive and self-adjoint, with domain defined by

$$D(A) = \{u \in S_0^1(\Omega) : Au \in X\}.$$

The space $D(A)$ is a Hilbert space endowed with the usual graph scalar product. Moreover, there exists a complete system of eigensolutions (e_j, λ_j) such that

$$\begin{gathered} (e_j, e_k) = \delta_{jk}, \quad -G_k e_j = \lambda_j e_j, \quad j = 1, 2, \dots \\ 0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots, \quad \lambda_j \to \infty, \quad \text{as } j \to \infty. \end{gathered}$$

For any $\theta \in \mathbb{R}$, denote X^θ as the space of formal series $\sum_{k=1}^\infty c_k e_k$ such that

$$\sum_{k=1}^\infty c_k^2 \lambda_k^{2\theta} < \infty.$$

$\{X^\theta\}_{\theta\in\mathbb{R}}$ is called the family of fractional power spaces of A.

Let $\theta \in \mathbb{R}$, we define the operator A^θ as following

$$A^\theta(\sum_{k=1}^{\infty} c_k e_k) = \sum_{k=1}^{\infty} c_k \lambda_k^\theta e_k$$

for any formal series $\sum_{k=1}^\infty c_k e_k$. Hence we can consider A^θ as the operator from X^η to $X^{\eta-\theta}$, and we have

$$A^\theta(X^\eta) = X^{\eta-\theta}, \quad A^{\theta_1+\theta_2} = A^{\theta_1}A^{\theta_2}.$$

From the above definition, one can see that

$$X^1 = \{u \in S_0^1(\Omega) : Au \in X = L^2(\Omega)\},$$
$$X^{1/2} = S_0^1(\Omega), \quad X^0 = X = L^2(\Omega),$$

and for any $\theta \in \mathbb{R}$, X^θ is a separable Hilbert space endowed with the inner scalar product

$$(u, v)_{X^\theta} = (A^\theta u, A^\theta v), \quad \|u\|_{X^\theta} = \|A^\theta u\|_X.$$

One can see that X^θ is continuously imbedded into X^η if $\theta \geq \eta$, moreover, this imbedding is compact if $\theta > \eta$.

Note that, for every $\theta > 0$, operator $A^{-\theta} : X \to X^\theta$ defined above can be represented as following

$$A^{-\theta}u = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-At} u \, dt, \quad u \in X.$$

Thus, $A^\theta : X^\theta \to X$ is the inverse of $A^{-\theta} : X \to X^\theta$, X^θ is also the dual space of $X^{-\theta}$.

We have the following basic result [8, Theorem 1.4.3].

Theorem 2.1. *Suppose that A is sectorial and Re $\sigma(A) > \delta > 0$. For $\theta \geq 0$, there exists a positive number $C_\theta < \infty$ such that*

$$\|A^\theta e^{-At}\| \leqslant C_\theta t^{-\theta} e^{-\delta t} \text{ for all } t > 0,$$

and if $0 < \theta \leqslant 1$, $x \in X^\theta$,

$$\|(e^{-At} - I)x\| \leqslant \frac{1}{\theta} C_{1-\theta} t^\theta \|A^\theta x\|.$$

From this theorem, in particular, we have some results which we will use in the next section

$$\|e^{-At}\| \leqslant Me^{-\delta t}, \quad \text{for all } t \geq 0.$$
$$\|(e^{-At} - \mathrm{I})x\| \leqslant Ct\|Ax\| \quad \text{for any } x \in X^1,\ t \geq 0.$$
$$e^{-tA}x \in \bigcap_{\theta\in\mathbb{R}} X^\theta, \quad \text{for any } x \in X^\eta, t > 0.$$
$$\|e^{-tA}x\|_{X^{1/2}} \leqslant C_\gamma t^{-1/2-\gamma} \|x\|_{X^{-\gamma}}, \quad \text{for any } x \in X^{-\gamma},\ t > 0,\ \gamma \in (0, 1/2).$$

2.2. **Existence of Global Attractors.** For convenience of the readers, we summarize some definitions and results of theory of infinite dimensional dynamical dissipative systems in [2, 7, 10] which we will use.

Let X be a metric space (not necessarily complete) with metric d. If $C \subset X$ and $b \in X$ we set $\rho(b, C) := \inf_{c \in C} d(b, c)$, and if $B \subset X, C \subset X$ we set $\operatorname{dist}(B, C) := \sup_{b \in B} \rho(b, C)$. Let $S(t)$ be a *continuous semigroup* on the metric space X.

A set $A \subset X$ is *invariant* if $S(t)A = A$, for any $t \geqslant 0$.

The positive orbit of $x \in X$ is the set $\gamma^+(x) = \{S(t)x | t \geqslant 0\}$. If $B \subset X$, the *positive orbit* of B is the set

$$\gamma^+(B) = \cup_{t \geqslant 0} S(t)B = \cup_{z \in B} \gamma^+(z).$$

More generally, for $\tau \geqslant 0$, we define the orbit after the time τ of B by

$$\gamma^+_\tau(B) = \gamma^+(S(\tau)B).$$

The subset $A \subset X$ *attracts* a set B if $\operatorname{dist}(S(t)B, A) \to 0$ as $t \to \infty$.

The subset A is a *global attractor* if A is closed, bounded, invariant, and attracts all bounded sets.

The semigroup $S(t)$ is *asymptotically compact* if, for any bounded subset B of X such that $\gamma^+_\tau(B)$ is bounded for some $\tau \geqslant 0$, every set of the form $\{S(t_n)z_n\}$, with $z_n \in B$ and $t_n \geqslant \tau, t_n \to +\infty$ as $n \to \infty$, is relatively compact.

A continuous semigroup $S(t)$ is a *continuous gradient system* if there exists a function $\Phi \in C^0(X, \mathbb{R})$ such that $\Phi(S(t)u) \leqslant \Phi(u)$, for all $t \geqslant 0$, for all $u \in X$, and the relation $\Phi(S(t)u) = \Phi(u)$, for all $t \geqslant 0$ implies that u is an equilibrium point, i.e. $S(t)u = u$ for all $t \geqslant 0$. The function Φ is called a strict Lyapunov functional.

Let E be the set of equilibrium points for the semigroup $S(t)$. We give the definition of the unstable set of E by

$$W^u(E) = \{y \in X : S(-t)y \text{ is defined for } t \geqslant 0 \text{ and } S(-t)y \to E \text{ as } t \to \infty\}.$$

From [10, Proposition 2.19 and Theorem 4.6], we have the following result.

Theorem 2.2. *Let $S(t), t \geqslant 0$, be an asymptotically compact gradient system, which has the property that, for any bounded set $B \subset X$, there exist $\tau \geqslant 0$ such that $\gamma^+_\tau(B)$ is bounded. If the set of equilibrium points E is bounded, then $S(t)$ has a compact global attractor $\mathcal{A}$ and $\mathcal{A} = W^u(E)$. Moreover, if X is a Banach space, then $\mathcal{A}$ is connected.*

If the global attractor $\mathcal{A}$ exists, then (see cite[page 21]Chu) it contains a *global minimal attractor* $\mathcal{M}$ which is defined as a minimal closed positively invariant set possessing the property

$$\lim_{t \to +\infty} \operatorname{dist}(S(t)y, \mathcal{M}) = 0 \quad \text{for every } y \in X.$$

Moreover, if $\mathcal{M}$ is compact then it is invariant and $\mathcal{M} = \cup_{z \in V} \omega(z)$.

2.3. **Singular Gronwall Inequality.** To prove the existence of a local solution and the asymptotic compactness of the semigroup generated by (1.5), we need the following lemma (see [7, Chapter 7]).

Lemma 2.1. *Assume that $\varphi(t)$ is a continuous nonnegative function on the interval $(0, T)$ such that*

$$\varphi(t) \leqslant c_0 t^{-\gamma_0} + c_1 \int_0^t (t-s)^{-\gamma_1} \varphi(s) ds, \quad t \in (0, T),$$

where $c_0, c_1 \geqslant 0$ and $0 \leqslant \gamma_0, \gamma_1 < 1$. Then there exists a constant $K = K(\gamma_1, c_1, T)$ such that

$$\varphi(t) \leqslant \frac{c_0}{1-\gamma_0} t^{-\gamma_0} K(\gamma_1, c_1, T), \quad t \in (0, T).$$

3. Main Results

3.1. Global solution.

Lemma 3.1. *For every $p \in (2, 2_k^*)$, there is a positive real $\gamma \in [0, 1/2)$ such that X^γ is continuously embedded in $L^p(\Omega)$.*

Proof. Using Holder inequality we have

$$\|u\|_{L^p} \leqslant \|u\|_{X^0}^{\delta} \|u\|_{L^{2_k^*}}^{1-\delta}, \quad \text{where } \delta = \frac{2(2_k^* - p)}{p(2_k^* - 2)}.$$

Hence

$$\|u\|_{L^p} \leqslant C \|u\|_X^{\delta} \|u\|_{X^{1/2}}^{1-\delta}. \tag{3.1}$$

By the interpolation of fractional power spaces

$$\|u\|_{X^{1/2}} \leqslant \|u\|_X^{1/2} \|u\|_{X^1}^{1/2}, \quad \forall u \in X^1. \tag{3.2}$$

Let B be the inclusion map from $X^{1/2}$ to $Y = L^p(\Omega)$, it follows from (3.1) and (3.2) that

$$\|Bu\|_Y \leqslant \|u\|_X^{\delta} \big(C \|u\|_X^{1/2} \|u\|_{X^1}^{1/2}\big)^{1-\delta} = C_1 \|u\|_{X^1}^{\overline{\delta}} \|u\|_X^{1-\overline{\delta}} = C_1 \|Au\|_X^{\overline{\delta}} \|u\|_X^{1-\overline{\delta}},$$

where $\overline{\delta} = \frac{1}{2}(1-\delta) < \frac{1}{2}$. By [8, page 28] we obtain that B has a unique extension to a continuous linear operator from X^γ to Y for every γ satisfying $\overline{\delta} < \gamma < \frac{1}{2}$. Lemma 3.1 is proved. □

Putting $q = \frac{2_k^*}{\rho+1}$, $p = \frac{q}{q-1}$. Since $\frac{2}{N(k)-2} < \rho < \frac{4}{N(k)-2}$ then $1 < q < 2$. Thus,

$$p = \frac{q}{q-1} = \frac{2_k^*}{2_k^* - 1} \in (2, 2_k^*).$$

It is inferred from Lemma 3.1 that there exists $\gamma \in (0, 1/2)$ such that X^γ is continuously embedded in $L^p(\Omega)$. Hence $L^q(\Omega) = (L^p(\Omega))'$ is continuously embedded in $X^{-\gamma}$.

Lemma 3.2. *Assume that f satisfies the condition (1.2), then $f : X^{1/2} \longrightarrow L^q(\Omega)$ is Lipschitzian on every bounded subset of $X^{1/2}$.*

Proof. Let $u \in X^{1/2}$, it follows from (1.2) that $|f(u)| \leqslant C(1 + |u|^{\rho+1})$. Hence

$$\begin{aligned} \int_\Omega |f(u)|^q \,dx\,dy &\leqslant C \int_\Omega \Big(1 + |u|^{q(\rho+1)}\Big) \,dx\,dy \\ &= C \int_\Omega \Big(1 + |u|^{2_k^*}\Big) \,dx\,dy < +\infty \end{aligned}$$

since $X^{1/2}$ is continuously embedded into $L^{2_k^*}(\Omega)$. This shows that f is a map from $X^{1/2}$ to $L^q(\Omega)$.

Let $u, v \in X^{1/2}$ and $\|u\|_{X^{1/2}} \leqslant r, \|v\|_{X^{1/2}} \leqslant r$, we have

$$\int_\Omega |f(u)-f(v)|^q\,dx\,dy \leqslant C\int_\Omega |u-v|^q\big(1+|u|^{q\rho}+|v|^{q\rho}\big)\,dx\,dy$$
$$\leqslant C\int_\Omega |u-v|^q\,dx\,dy + C\int_\Omega |u|^{q\rho}|u-v|^q\,dx\,dy + C\int_\Omega |v|^{q\rho}|u-v|^q\,dx\,dy.$$

Applying Holder's inequality, we have

$$\int_\Omega |u|^{q\rho}|u-v|^q\,dx\,dy \leqslant \|u\|_{2_k^*}^{q\rho}\|u-v\|_{2_k^*}^q,$$
$$\int_\Omega |v|^{q\rho}|u-v|^q\,dx\,dy \leqslant \|v\|_{2_k^*}^{q\rho}\|u-v\|_{2_k^*}^q.$$

Since $S_0^1(\Omega)$ is continuously embedded into $L^{2_k^*}(\Omega)$ and $1 < q < 2_k^*$, there exists a positive number $M_1(r)$ such that

$$\|f(u)-f(v)\|_{L^q} \leqslant M_1(r)\|u-v\|_{X^{1/2}}.$$

This completes the proof. □

Since $L^q(\Omega)$ is continuously embedded in $X^{-\gamma}$, we can consider that f, as a map from $X^{1/2}$ to $X^{-\gamma}$, is Lipschitzian continuous on every bounded subset of $X^{1/2}$. ¿From this property of f and properties of the semigroup e^{-tA} generated by the operator $-A$ (see Sec. 2.1), using the arguments as in [8, Chapter 3], we obtain the following proposition on the existence and the smoothness of the local mild solution.

Proposition 3.1. *Assume that f satisfies the condition* (1.2)*. Then for any $R > 0$ and $u_0 \in X^{1/2}$ such that $\|u_0\|_{X^{1/2}} \leqslant R$, there exists $T = T(R) > 0$ small enough such that the problem* (1.5) *has a unique mild solution $u \in C([0,T); X^{1/2})$. Moreover, u is differentiable on $(0,T)$ and $u_t(t) \in X^\delta$ for any $\delta \in (1/2, 1-\gamma)$, and for all $t \in (0,T)$.*

Denote by $\langle , \rangle$ the pairing between $X^{-1/2}$ and $X^{1/2}$. From (1.5) we have

$$\langle u_t, u_t\rangle + \langle Au, u_t\rangle + \langle f(u), u_t\rangle + \langle g, u_t\rangle = 0.$$

Hence

$$\|u_t\|_X^2 + \frac{1}{2}\frac{d}{dt}\|u\|_{X^{1/2}}^2 + \frac{d}{dt}\int_\Omega (F(u)+gu)dx = 0.$$

Putting

$$\Phi(u) = \frac{1}{2}\|u\|_{X^{1/2}}^2 + \int_\Omega (F(u)+gu)dx \tag{3.3}$$

we obtain

$$\frac{d}{dt}\Phi(u(t)) = -\|u_t(t)\|_X^2,\ t \in (0,T). \tag{3.4}$$

Theorem 3.2. *Assume that f satisfies conditions* (1.2), (1.3)*. Then for any $u_0 \in X^{1/2}$, the problem* (1.5) *has a unique global solution $u \in C([0,\infty), X^{1/2})$.*

Proof. Suppose that the solution u is defined on the maximal interval $[0, t_{\max})$. Using hypothesis (1.3) and Cauchy inequality we get

$$\Phi(u(t)) \geq \frac{1}{2}\|u(t)\|_{X^{1/2}}^2 - \frac{\mu}{2}\|u(t)\|^2 - C(\Omega) - \varepsilon\|u(t)\|^2 - \frac{1}{4\varepsilon}\|g\|^2.$$

By choosing ε small enough such that $\mu + 2\varepsilon < \lambda_1$ we obtain

$$\Phi(u(0)) \geq \Phi(u(t)) \geq \frac{1}{2}(1 - \frac{\mu + 2\varepsilon}{\lambda_1})\|u\|^2_{X^{1/2}} - C.$$

Hence

$$\|u(t)\|_{X^{1/2}} \leqslant M, \ \forall t \in [0, t_{\max}).$$

This implies $t_{\max} = +\infty$. Indeed, let $t_{\max} < +\infty$ and $\limsup_{t \to t^-_{\max}} \|u(t)\|_{X^{1/2}} < +\infty$. Then there exists a sequence $(t_n)_{n \geqslant 1}$ and a constant K such that $t_n \to t^-_{\max}$, as $n \to +\infty$ and $\|u(t_n)\|_{X^{1/2}} < K, \ n = 1, 2, \dots$. As we have already shown above, for each $n \in \mathbb{N}$ there exists a unique solution of the problem (1.5) with initial data $u(t_n)$ on $[t_n, t_n + T^*]$, where $T^* > 0$ depending on K and independent of $n \in \mathbb{N}$. Thus, we can get $t_{\max} < t_n + T^*$, for $n \in \mathbb{N}$ large enough. This contradicts the maximality of $t_{\max}$ and the proof of Theorem 3.2 is completed. □

3.2. **Global attractor.** Using the arguments as in proof of Theorem 3.2, we see that, for all R, u_0 with $\|u_0\|_{X^{1/2}} \leqslant R$, there exists a number $M > 0$ only depending on R such that $\|u(t)\|_{X^{1/2}} \leqslant M$ for all $t > 0$. In other words, the orbits of bounded sets are bounded.

Theorem 3.3. *Under conditions* (1.2)-(1.4)*, the semigroup* $S(t)$ *generated by* (1.5) *has a compact connected global attractor* $\mathcal{A} = W^u(E)$ *in* $X^{1/2}$.

Proof. Firstly, from (3.4) and the proof of Theorem 3.2 we see that $\gamma^+(B)$ is bounded for any bounded subset B of $X^{1/2}$ and the function Φ defined by (3.3) is a strict Lyapunov functional.

Notice that the set of equilibrium points

$$E = \{z \in X^{1/2} \ |Az + f(z) + g = 0\}.$$

Let $z \in E$, we have

$$0 = \|z\|^2_{X^{1/2}} + \int_\Omega (f(z)z + gz)dx.$$

Using hypothesis (1.4) and Cauchy inequality we obtain that

$$\|z\|_{X^{1/2}} \leqslant M, \quad \text{for all } z \in E,$$

i.e. E is bounded in $X^{1/2}$. Thus, in order to prove the existence of the global attractor, we only need to prove that $S(t)$ is asymptotically compact in $X^{1/2}$.

Let $(u_n)_{n \geq 1}$ be a bounded sequence in $X^{1/2}$ and $t_n \to +\infty$. Fix $T > 0$, since $\{u_n\}$ is bounded and orbits of bounded sets are bounded, $\{S(t_n - T)u_n\}$ is bounded in $X^{1/2}$. Since $X^{1/2}$ is compactly embedded in X, there is subsequence $\{S(t_{n_k} - T)u_{n_k}\}$ and $v \in X^{1/2}$ such that $v_k = S(t_{n_k} - T)u_{n_k} \rightharpoonup v$ weakly in $X^{1/2}$ and $v_k \to v$ strongly in X as $k \to \infty$. We will prove that $S(t_{n_k})u_{n_k} = S(T)v_k$ converges strongly to $S(T)v$ in $X^{1/2}$, and thus $S(t)$ is asymptotically compact.

Denote $v_k(t) = S(t)v_k, v(t) = S(t)v$, we have

$$v_k(t) = e^{-At}v_k - \int_0^t e^{-A(t-s)}(f(v_k(s)) + g)ds,$$

$$v(t) = e^{-At}v - \int_0^t e^{-A(t-s)}(f(v(s)) + g)ds.$$

Hence

$$\|v_k(t) - v(t)\|_{X^{1/2}} \leqslant C_1 t^{-1/2}\|v_k - v\| + C_2 \int_0^t (t-s)^{-1/2-\gamma}\|v_k(s) - v(s)\|_{X^{1/2}}ds.$$

By the singular Gronwall inequality (see Lemma 2.1), there is a constant C such that, for $t \in (0,T]$,

$$\|v_k(t) - v(t)\|_{X^{1/2}} \leqslant Ct^{-1/2}\|v_k - v\|,$$

in particular,

$$\|v_k(T) - v(T)\|_{X^{1/2}} \leqslant CT^{-1/2}\|v_k - v\|.$$

Since $v_k \to v$ in $L^2(\Omega)$, $v_k(T) \to v(T)$ in $X^{1/2}$ as $k \to +\infty$. This implies that $S(t)$ is asymptotically compact. Applying Theorem 2.2, we obtain the conclusion of the theorem. □

The following proposition describes the asymptotic behavior of solutions of (1.5) as $t \to +\infty$.

Proposition 3.4. *Under the conditions* (1.2)–(1.4)*, the semigroup* $S(t), t \geqslant 0$*, generated by* (1.5) *has a global minimal attractor* $\mathcal{M}$*, given by* $\mathcal{M} = E$*, in the space* $X^{1/2}$*. In particular, we have*

$$\lim_{t\to+\infty} dist(S(t)y, E) = 0 \quad \text{for every } y \in X^{1/2}.$$

Proof. The existence of $\mathcal{M}$ follows directly from the fact that the semigroup $S(t)$ has a compact global attractor (see Sect. 2.2). To prove the second statement, we will show that $\mathcal{M} = E$.

It is obvious that $E \subset \mathcal{M}$. We now prove that $\mathcal{M} \subset E$. Indeed, since $M = \bigcup_{z\in X^{1/2}} \omega(z)$, it suffices to show that $\omega(z) \subset E$, for all $z \in X^{1/2}$. Taking $a \in \omega(z)$ arbitrarily, by the definition of $\omega(z)$, there exists a real sequence $\{t_n\}, t_n \to +\infty$, such that $S(t_n)z = u(t_n) \to a$. Hence and since the Lyapunov functional Φ is bounded below, it implies that

$$\Phi(a) = \lim_{t\to+\infty} \Phi(u(t_n)) = \inf\{\Phi(S(t)z) = \Phi(u(t))|t \geqslant 0\};$$

i.e. Φ is constant on $\omega(z)$. Therefore, by the nonincreasing property of Lyapunov function along the orbit $S(t)z$ and the positively invariant property of $\omega(z)$, we conclude that $a \in E$. This completes the proof. □

Proof of Theorem 1.1. The conclusion follows from Theorems 3.2, 3.3 and Proposition 3.4. □

3.3. **Remarks.** (1) One can extend the results of this paper to the equation in the following form

$$u_t - G_{\alpha_1,\dots,\alpha_m}u + f(u) + g(x) = 0, \quad x \in \Omega, t > 0,$$

where Ω is a bounded smooth domain in $\mathbb{R}^{N_0}\times\mathbb{R}^{N_1}\times\cdots\times\mathbb{R}^{N_m}$, $(x_0, x_1, \dots, x_m) \in \Omega$, $G_{\alpha_1,\dots,\alpha_m} = \Delta_{x_0} + |x_0|^{2\alpha_1}\Delta_{x_1} + \cdots + |x_0|^{2\alpha_m}\Delta_{x_m}$. Note that in this case we still are able to define the space $S_0^1(\Omega)$ and have the embedding theorem which are similar to those in Sec. 2.1 (for more details, see citeTh-T). Here $N(k) = N_0 + (\alpha_1 + 1)N_1 + \dots + (\alpha_m + 1)N_m$.

(2) One can extend the results of this paper to the case the system in the potential form

$$U_t - L_{\alpha_1,\dots,\alpha_m}U + \nabla F(U) + G(x) = 0,$$

where Ω is a smooth bounded domain in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, $U = (u_1, \dots, u_m)$ is the unknown function, $L_{\alpha_1,\dots,\alpha_m} = \operatorname{diag}(G_{\alpha_1}, \dots, G_{\alpha_m})$ $G = (g_1, \dots, g_m) \in (L^2(\Omega))^m$ given and $F : \mathbb{R}^m \to \mathbb{R}$ satisfies the following conditions

$$\Big|\frac{\partial F}{\partial u_i}(u) - \frac{\partial F}{\partial u_i}(v)\Big| \leqslant c_0\Big(1 + \sum_{i=1}^m |u_i|^\rho + \sum_{i=1}^m |v_i|^\rho\Big) \sum_{j=1}^m |u_j - v_j|,$$

$$F(u_1, \dots, u_m) \geqslant -\frac{\mu}{2}(u_1^2 + \dots + u_m^2) - C_1$$

$$U\nabla F(U) = u_1\frac{\partial F}{\partial u_1} + \dots + u_m\frac{\partial F}{\partial u_m} \geqslant -\mu(u_1^2 + \dots + u_m^2) - C_2,$$

where $0 \leqslant \rho < \frac{4}{N(\alpha)-2}$, $N(\alpha) = \max\{N(\alpha_1), \dots, N(\alpha_m)\}$, $N(\alpha_i) = N_1 + (\alpha_i + 1)N_2$; $\mu < \lambda_1$, λ_1 is the minimum of the set containing first eigenvalues of $-G_{\alpha_1}, \dots, -G_{\alpha_m}$ in Ω with homogeneous Dirichlet condition; C_1 and C_2 are the nonnegative constants.

In the case $\alpha_1 = \dots = \alpha_m = \alpha$, one can also extend these results to the more general system

$$U_t - DL_\alpha U + \nabla F(U) + G(x) = 0,$$

where F, G as above; D is a positively definite, symmetric, $m \times m$ real matrix; $L_\alpha = \operatorname{diag}(G_\alpha, \dots, G_\alpha)$.

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