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# PARABOLIC BOUNDARY-VALUE PROBLEMS WITH EQUIVALUED SURFACE ON THE DOMAIN WITH A THIN LAYER

MEIHUA DU, FENGQUAN LI

Abstract. We study the existence, uniqueness and limit behavior of solutions to parabolic boundary-value problems with equivalued surface on a domain with a thin layer.

## 1. Introduction

Motivated by the study of resistivity well-logging in petroleum exploitation, the boundary value problem with equivalued surface, a new kind of boundary value problem for partial differential equations was proposed in 1970's. It is a kind of non-local boundary value problem, which can also be used to give mathematical descriptions for other problems in physics and mechanics (see [7, 8, 10, 12]).

In single-well system of heterogeneous synthetic reservoirs, for the cause of mud contamination in the process of well drilling and well completion, a polluted zone is formed. However, this zone is a thin layer compared with the whole heterogeneous reservoirs (see [2, 5, 6]). In practical calculation, the variation of solution near the thin layer should be quite large, and then in finite element procedure, it is necessary to have a refined partition of elements near the thin layer. This causes a complexity in computation. To get rid of this difficulty, when the thin layer is rather thin, the thin layer can be approximately regarded as an interface and corresponding the boundary value problem with equivalued surface on the thin layer can be approximately replaced by the boundary value problem with equivalued interface. To prove the above conclusion, we need to study existence, uniqueness and limit behavior of solutions for parabolic boundary value problems with equivalued surface on a domain with thin layer. Similar problem for elliptic equation has been studied in [9].

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Here we consider the following parabolic boundary value problems with equivalued surface on the domain with thin layer:

$$
\begin{gathered}
\frac{\partial u}{\partial t}-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}(\tilde{a}_{ij}(x,t)\frac{\partial u}{\partial x_j})=F(x,t)\quad\text{in } \tilde{Q},\\
u=0\quad\text{on } \Sigma,\\
u=\tilde{C}(t)\quad\text{on } \tilde{\Sigma},\\
\int_{\tilde{\Gamma}_1}\frac{\partial u}{\partial n_L}\mathrm{d}s=\int_{\tilde{\Gamma}_2}\frac{\partial u}{\partial n_L}\mathrm{d}s+A(t)\quad\text{a.e. t }\in(0,T),\\
u(x,0)=\tilde{\varphi}_0(x)\quad\text{in } \tilde{\Omega}_1\cup\tilde{\Omega}_2,
\end{gathered}
\tag{1.1}
$$

where $\tilde{Q}=(\tilde{\Omega}_1\cup\tilde{\Omega}_2)\times(0,T)$, $\Sigma=\Gamma\times(0,T)$, $\tilde{C}(t)$ is a function to be determined, $\tilde{\Sigma}=(\tilde{\Gamma}_1\cup\tilde{\Omega}\cup\tilde{\Gamma}_2)\times(0,T)$, $T$ is a fixed positive constant, and the conormal derivative is

$$
\frac{\partial u}{\partial n_L}=\sum_{i,j=1}^{N}\tilde{a}_{ij}(x,t)\frac{\partial u}{\partial x_j}n_i\,. \tag{1.2}
$$

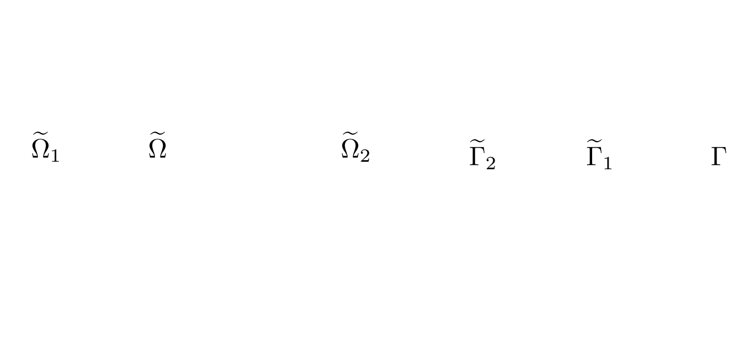


Figure 1. The compostion of $\Omega$ with thin layer $\widetilde{\Omega}$

Let $\Omega\subset\mathbb{R}^N(N\geq 2)$ be a bounded domain with smooth outside boundary $\Gamma$ (see Fig.1). Suppose that $\Omega$ is composed of three non-overlapping subdomains $\tilde{\Omega}_1$, $\tilde{\Omega}$ and $\tilde{\Omega}_2$, and $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are the interfaces of $\tilde{\Omega}$ with $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ respectively. The unit normal $\vec{n}=(n_1,n_2,\dots,n_N)$ takes the inward and outward directions (or vice versa) for the domain $\tilde{\Omega}$ on $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$.

This paper is organized as follows: In section 2, we will prove the existence and uniqueness of weak solution to problem (1.1). In section 3, we will discuss parabolic boundary value problem (3.1) with equivalued interface. In section 4, the limit behavior of solutions to problems (1.1) will be studied.

## 2. Existence and uniqueness of weak solution to problem (1.1)

In this section, we discuss the existence and uniqueness of weak solution to problem (1.1). We first state the following assumption:

(H0) The functions $\tilde{a}_{ij}$ are piecewise smooth in $Q$ and $\tilde{a}_{ij} = \tilde{a}_{ji}$; there exist two constants $\alpha, \beta > 0$ such that

$$\alpha|\xi|^2 \le \sum_{i,j=1}^{N} \tilde{a}_{ij}(x,t)\xi_i\xi_j \le \beta|\xi|^2, \quad \forall\, \xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N, \tag{2.1}$$

a.e. $(x,t) \in Q$, where $Q = \Omega \times (0,T)$. We also assume $F \in L^2(Q)$, $A \in L^2(0,T)$, $\tilde{\varphi}_0 \in L^2(\Omega)$.

Let

$$V = \{v \in H_0^1(\Omega) : v|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2} = \text{constant}\},$$
$$U = \{v \in \mathring{W}_2^{1,1}(Q) : v(x,T) = 0, v|_{\tilde{\Sigma}} = C(t)\},$$

where $C(t)$ is arbitrary function of $t$.

**Definition 2.1.** A measurable function $u \in L^2(0,T;V)$ is called a weak solution to problem (1.1), if for any $\psi \in U$,

$$\begin{aligned} &-\int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u \frac{\partial \psi}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega} \tilde{a}_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \psi}{\partial x_i} \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F\psi \mathrm{d}x\mathrm{d}t + \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \tilde{\varphi}_0(x)\psi(x,0)\mathrm{d}x + \int_0^T A(t)\psi|_{\tilde{\Sigma}}\mathrm{d}t. \end{aligned} \tag{2.2}$$

Now we can state the existence and uniqueness of weak solutions to (1.1).

**Theorem 2.2.** *Assume* (H0), $\tilde{\varphi}_0 \in L^2(\Omega)$, $F \in L^2(Q)$, $A \in L^2(0,T)$. *Then* (1.1) *admits a unique solution* $u \in L^2(0,T;V)$ *in the sense of Definition* (2.1).

*Proof.* (1) Existence: We will first consider the problem:

$$\begin{gathered} \frac{\partial \tilde{u}}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(\tilde{a}_{ij}(x,t)\frac{\partial \tilde{u}}{\partial x_j}) = F(x,t) \quad \text{in } \tilde{Q}, \\ \tilde{u} = 0 \quad \text{on } \Sigma, \\ \tilde{u} = \tilde{C}(t) \quad \text{on } (\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2) \times (0,T), \\ \int_{\tilde{\Gamma}_1} \frac{\partial \tilde{u}}{\partial n_L} \mathrm{d}s = \int_{\tilde{\Gamma}_2} \frac{\partial \tilde{u}}{\partial n_L} \mathrm{d}s + A(t) \quad \text{a.e. t } \in (0,T), \\ \tilde{u}(x,0) = \tilde{\varphi}_0(x) \quad \text{in } \tilde{\Omega}_1 \cup \tilde{\Omega}_2. \end{gathered} \tag{2.3}$$

Let

$$V_1 = \{v \in H^1(\tilde{\Omega}_1 \cup \tilde{\Omega}_2) : v|_{\Gamma} = 0,\ v|_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2} = \text{constant}\}. \tag{2.4}$$

Here we will use the Galerkin method (cf [4, 13]). Taking a basis $\{\omega_k\}_{k=1}^{\infty}$ of $V_1$ that is complete and orthonormal in $L^2(\tilde{\Omega}_1 \cup \tilde{\Omega}_2)$. For any fixed $m$, let $S_m = \operatorname{span}\{\omega_1, \omega_2, \dots, \omega_m\}$.

We set $\tilde{u}_m = \sum_{k=1}^m c_{km}\omega_k$, then Galerkin equations read as follows:

$$\int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \frac{\partial \tilde{u}_m}{\partial t} \omega_k \mathrm{d}x + \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \sum_{i,j=1}^{N} \tilde{a}_{ij} \frac{\partial \omega_k}{\partial x_i} \frac{\partial \tilde{u}_m}{\partial x_j} \mathrm{d}x = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F\omega_k \mathrm{d}x + A(t)\omega_k|_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2}, \tag{2.5}$$

$$\tilde{u}_m(x,0) = \sum_{k=1}^{m} (\tilde{\varphi}_0, \omega_k)\omega_k = \sum_{k=1}^{m} c_{k0}\omega_k = \tilde{\varphi}_{0m}(x). \tag{2.6}$$

Namely, for almost all $t \in (0,T)$,

$$\frac{d}{dt}c_{km}(t)+\sum_{l=1}^{m}c_{lm}(t)\int_{\tilde\Omega_1\cup\tilde\Omega_2}\sum_{i,j=1}^{N}\tilde a_{ij}\frac{\partial\omega_l}{\partial x_j}\frac{\partial\omega_k}{\partial x_i}\mathrm{d}x=\int_{\tilde\Omega_1\cup\tilde\Omega_2}F\omega_k\mathrm{d}x+A(t)\omega_k|_{\tilde\Gamma_1\cup\tilde\Gamma_2}, \tag{2.7}$$

$$c_{km}(0)=(\tilde\varphi_0,\omega_k)=c_{k0}. \tag{2.8}$$

By the theory of system of ordinary differential equations, problem (2.7)–(2.8) has a unique solution vector $(c_{1m},c_{2m},\dots,c_{mm})$. Multiplying (2.7) by $c_{km}(t)$ and summing over $k$, we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\|\tilde u_m(\cdot,t)\|^2_{L^2(\tilde\Omega_1\cup\tilde\Omega_2)}+\int_{\tilde\Omega_1\cup\tilde\Omega_2}\sum_{i,j=1}^{N}\tilde a_{ij}\frac{\partial\tilde u_m}{\partial x_j}\frac{\partial\tilde u_m}{\partial x_i}\,\mathrm{d}x\\&=\int_{\tilde\Omega_1\cup\tilde\Omega_2}F\tilde u_m\mathrm{d}x+A(t)\tilde u_m|_{\tilde\Gamma_1\cup\tilde\Gamma_2}.\end{aligned}$$

Integrating over $(0,\tau)$ with respect to $t$, we have

$$\begin{aligned}&\frac12\int_{\tilde\Omega_1\cup\tilde\Omega_2}\tilde u_m^2(x,\tau)\,\mathrm{d}x+\int_0^\tau\int_{\tilde\Omega_1\cup\tilde\Omega_2}\sum_{i,j=1}^{N}\tilde a_{ij}\frac{\partial\tilde u_m}{\partial x_j}\frac{\partial\tilde u_m}{\partial x_i}\,\mathrm{d}x\mathrm{d}t\\&=\int_0^\tau\int_{\tilde\Omega_1\cup\tilde\Omega_2}F\tilde u_m\,\mathrm{d}x\mathrm{d}t+\int_0^\tau A(t)\tilde u_m|_{\tilde\Gamma_1\cup\tilde\Gamma_2}\,\mathrm{d}t+\frac12\int_{\tilde\Omega_1\cup\tilde\Omega_2}\tilde\varphi_{0m}^2\,\mathrm{d}x.\end{aligned}$$

Let $\tilde Q_\tau=(\tilde\Omega_1\cup\tilde\Omega_2)\times(0,\tau)$, by (H0), Hölder's inequality and the trace theorem, we have

$$\begin{aligned}&\frac12\int_{\tilde\Omega_1\cup\tilde\Omega_2}\tilde u_m^2(x,\tau)\,\mathrm{d}x+\alpha\|D\tilde u_m\|^2_{L^2(\tilde Q_\tau)}\\&\le\|F\|_{L^2(\tilde Q_\tau)}\|\tilde u_m\|_{L^2(\tilde Q_\tau)}+\frac12\|\tilde\varphi_0\|^2_{L^2(\Omega)}+\frac{1}{|\tilde\Gamma_2|^{1/2}}\int_0^\tau|A(t)|\Big(\int_{\tilde\Gamma_2}\tilde u_m^2\,\mathrm{d}s\Big)^{1/2}\mathrm{d}t\\&\le\|F\|_{L^2(\tilde Q_T)}\|\tilde u_m\|_{L^2(\tilde Q_\tau)}+\frac12\|\tilde\varphi_0\|^2_{L^2(\Omega)}+\frac{C}{|\tilde\Gamma_2|^{1/2}}\|A(t)\|_{L^2(0,T)}(\|\tilde u_m\|_{L^2(\tilde Q_\tau)}\\&\quad+\|D\tilde u_m\|_{L^2(\tilde Q_\tau)})\\&\le\Big(\|F\|_{L^2(\tilde Q_T)}+\frac{C}{|\tilde\Gamma_2|^{1/2}}\|A(t)\|_{L^2(0,T)}\Big)\|\tilde u_m\|_{L^2(\tilde Q_\tau)}+\frac12\|\tilde\varphi_0\|^2_{L^2(\Omega)}\\&\quad+\frac{C}{|\tilde\Gamma_2|^{1/2}}\|A(t)\|_{L^2(0,T)}\|D\tilde u_m\|_{L^2(\tilde Q_\tau)}.\end{aligned}$$

By Young's inequality and Gronwall's inequality, we obtain

$$\|\tilde u_m\|_{L^2(0,T;V_1)}\le C, \tag{2.9}$$

$$\|\tilde u_m\|_{L^\infty(0,T;L^2(\tilde\Omega_1\cup\tilde\Omega_2))}\le C, \tag{2.10}$$

where $C$ is a positive constant independent of $m$.

Integrating (2.5) over $(t, t+\Delta t)$, we get

$$\begin{aligned} &c_{km}(t+\Delta t) - c_{km}(t) + \int_t^{t+\Delta t} \int_{\tilde\Omega_1\cup\tilde\Omega_2} \sum_{i,j=1}^N \tilde a_{ij} \frac{\partial \tilde u_m}{\partial x_j} \frac{\partial \omega_k}{\partial x_i}\, \mathrm{d}x\mathrm{d}\tau \\ &= \int_t^{t+\Delta t} \int_{\tilde\Omega_1\cup\tilde\Omega_2} F\omega_k \,\mathrm{d}x\mathrm{d}\tau + \int_0^\tau A(\tau)\omega_k|_{\tilde\Gamma_1\cup\tilde\Gamma_2}\,\mathrm{d}\tau. \end{aligned}$$

Condition (H0), (2.9) and trace theorem imply

$$|c_{km}(t+\Delta t) - c_{km}(t)| \le C(1 + ||F||_{L^2(Q)} + ||A||_{L^2(0,T)})||\omega_k||_{V_1}|\Delta t|^{\frac12}, \tag{2.11}$$

where $C$ is a positive constant independent of $m, k$. From the above inequality, we can deduce that for any fixed positive integer $k$, $c_{km}$ is equicontinuous with respect to $m$ in $[0, T]$. Thus by Ascoli-Arzela theorem, we can extract a subsequence of $\{c_{km}\}$ (still denoted by $\{c_{km}\}$) such that as $m \to \infty$,

$$c_{km} \to c_k \quad \text{uniformly in } [0, T]. \tag{2.12}$$

For any positive integer $r \le m$, (2.10) yields

$$\sum_{k=1}^r c_{km}^2(t) \le C, \quad \forall t \in (0, T), \tag{2.13}$$

where $C$ is a positive constant independent of $m, k, t$. Let $m \to \infty$ in (2.13), then for any positive integer $r$ we have

$$\sum_{k=1}^r c_k^2(t) \le C. \tag{2.14}$$

Let $\tilde u(x,t) = \sum_{k=1}^\infty c_k(t)\omega_k(x)$, (2.14) imply that $\tilde u(x,t) \in L^2(\tilde\Omega_1 \cup \tilde\Omega_2)$ for all $t \in [0, T]$. For any fixed positive integer $k$, from (2.12) it follows that

$$(\tilde u_m(\cdot,t) - \tilde u(\cdot,t), \omega_k) = c_{km} - c_k \to 0, \quad \text{uniformly in } [0, T]. \tag{2.15}$$

Noting $\{\omega_k\}_{k=1}^\infty$ is a complete orthonormal basis in $L^2(\tilde\Omega_1 \cup \tilde\Omega_2)$, we deduce

$$\tilde u_m \to \tilde u \quad \text{weakly in } C([0,T]; L^2(\tilde\Omega_1 \cup \tilde\Omega_2)). \tag{2.16}$$

Thus (2.9) and (2.16) imply that

$$\tilde u_m \to \tilde u \quad \text{weakly in } L^2(0,T; V_1). \tag{2.17}$$

Convergence (2.16) yields

$$u_m(\cdot,0) \to \tilde u(\cdot,0) \quad \text{weakly in } L^2(\tilde\Omega_1 \cup \tilde\Omega_2). \tag{2.18}$$

Consequently $\tilde u(0) = \tilde\varphi_0$.

For any given a sequence of smooth function $\{\psi_k(t)\}_{k=1}^\infty$ defined in [0, T] with $\psi_k(T) = 0$, multiplying the Galerkin equation (2.5) by $\psi_k(t)$ and using integration by parts, we obtain

$$\begin{aligned} &-\int_0^T \int_{\tilde\Omega_1\cup\tilde\Omega_2} \tilde u_m \frac{\partial \psi_k}{\partial t}\omega_k \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\tilde\Omega_1\cup\tilde\Omega_2} \sum_{i,j=1}^N \tilde a_{ij} \frac{\partial \omega_k}{\partial x_i} \frac{\partial \tilde u_m}{\partial x_j} \psi_k \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\tilde\Omega_1\cup\tilde\Omega_2} F\psi_k\omega_k \mathrm{d}x\mathrm{d}t + \int_0^T A\psi\omega_k|_{\tilde\Gamma_1\cup\tilde\Gamma_2}\mathrm{d}t + \int_{\tilde\Omega_1\cup\tilde\Omega_2} \tilde\varphi_{0m}(x)\psi(0)\omega_k \mathrm{d}x. \end{aligned} \tag{2.19}$$

According to (2.17)-(2.18), let $m \to \infty$ in (2.19), it is easy to prove that

$$\begin{aligned} &-\int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \tilde{u} \frac{\partial \psi_k}{\partial t} \omega_k \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial \omega_k}{\partial x_i} \frac{\partial \tilde{u}}{\partial x_j} \psi_k \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F \psi_k \omega_k \mathrm{d}x\mathrm{d}t + \int_0^T A\psi\omega_k|_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2} \mathrm{d}t + \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \tilde{\varphi}_0(x)\psi(0)\omega_k \mathrm{d}x. \end{aligned} \tag{2.20}$$

For any positive integer $r$, let

$$\psi(x,t) = \sum_{k=1}^r \psi_k(t)\omega_k(x). \tag{2.21}$$

Replacing $\psi_k(t)\omega_k(x)$ by the above $\psi(x,t)$ in (2.20), we have

$$\begin{aligned} &-\int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \tilde{u} \frac{\partial \psi}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \tilde{u}}{\partial x_j} \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F \psi \mathrm{d}x\mathrm{d}t + \int_0^T A\psi|_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2} \mathrm{d}t + \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \tilde{\varphi}_0(x)\psi(x,0) \mathrm{d}x. \end{aligned} \tag{2.22}$$

Since the set composed of functions such as (2.21) is dense in the space $U$, then for any $\psi \in U$ (2.22) holds. Thus we obtain $\tilde{u}$ is the weak solution to problem (2.3). Let

$$u = \begin{cases} \tilde{u} & \text{in } \tilde{Q}\,, \\ \tilde{C}(t) & \text{in } \tilde{\Sigma}\,. \end{cases} \tag{2.23}$$

It is easy to verify that $u \in L^2(0,T;\ V)$ and satisfy (2.2). Thus we obtain $u$ is the weak solution to problem (1.1).

(2) Proof of uniqueness: If $u_1$ and $u_2$ are two weak solutions to problem (1.1), by Definition 2.1 we get

$$\begin{aligned} &-\int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_i \frac{\partial \psi}{\partial t} dxdt + \int_0^T \int_\Omega \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_i}{\partial x_j} \frac{\partial \psi}{\partial x_i}\, dx\, dt \\ &= \int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F\psi dx\, dt + \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \tilde{\varphi}_0(x)\psi(x,0) dx + \int_0^T A(t)\psi|_{\tilde{\Sigma}} dt, \quad i = 1,2. \end{aligned}$$

Let $u_0 = u_1 - u_2$, then the above equality yields

$$-\int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_0 \frac{\partial \psi}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j} \frac{\partial \psi}{\partial x_i} \mathrm{d}x\mathrm{d}t = 0. \tag{2.24}$$

For a given $0 < h < T$, let

$$u_{0h} = \frac{1}{h} \int_t^{t+h} u_0(x,\tau) \mathrm{d}\tau\,, \quad \varphi = \begin{cases} u_{0h} & 0 \le t < (T-h), \\ 0 & (T-h) \le t \le T\,, \end{cases} \tag{2.25}$$

$$\hat{\varphi} = \begin{cases} 0 & t > (T-h) \\ ,\varphi & 0 < t \le (T-h), \\ 0 & t \le 0, \end{cases} \quad \hat{\varphi}_h = \frac{1}{h} \int_{t-h}^t \hat{\varphi}(x,\tau) \mathrm{d}\tau. \tag{2.26}$$

It is easy to prove that $\hat{\varphi}_h \in U$. Taking $\psi = \hat{\varphi}_h$ in (2.24), we have

$$
\begin{aligned}
&\int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_0 \frac{\partial \hat{\varphi}_h}{\partial t} \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_0 \frac{[\hat{\varphi}(x,t) - \hat{\varphi}(x,t-h)]}{h} \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{h}\Big[\int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_0 \hat{\varphi}(x,t)\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_0 \hat{\varphi}(x,t-h)\mathrm{d}x\mathrm{d}t\Big] \\
&= \frac{1}{h}\int_0^{T-h} \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_0 \hat{\varphi}(x,t)\mathrm{d}x\mathrm{d}t - \int_{-h}^{T-h} \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_0(x,t+h) \hat{\varphi}(x,t)\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{h}\Big[\int_0^{T-h} \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_0 \varphi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^{T-h} \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} u_0(x,t+h) \varphi(x,t)\mathrm{d}x\mathrm{d}t\Big] \\
&= -\int_0^{T-h} \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \frac{[u_0(x,t+h) - u(x,t)]}{h} \varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&= -\int_0^{T-h} \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \frac{\partial u_{0h}}{\partial t} \varphi(x,t)\mathrm{d}x\mathrm{d}t\,.
\end{aligned}
\tag{2.27}
$$

Similarly to the above equality, we can get

$$
\begin{aligned}
&\int_0^T \int_\Omega \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j} \frac{\partial \hat{\varphi}_h}{\partial x_i} \mathrm{d}t\mathrm{d}x \\
&= \frac{1}{h}\int_0^T \int_\Omega \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j} \int_{t-h}^t \frac{\partial \hat{\varphi}(x,\tau)}{\partial x_i} \mathrm{d}\tau\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{h}\int_\Omega \int_0^T \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j} \int_{t-h}^t \frac{\partial \hat{\varphi}(x,\tau)}{\partial x_i} \mathrm{d}\tau\mathrm{d}t\mathrm{d}x \\
&= \frac{1}{h}\int_\Omega \Big[\int_{-h}^0 \frac{\partial \hat{\varphi}}{\partial x_i} \int_0^{\tau+h} \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j} \mathrm{d}t\mathrm{d}\tau + \int_0^{T-h} \frac{\partial \hat{\varphi}}{\partial x_i} \int_\tau^{\tau+h} \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j} \mathrm{d}t\mathrm{d}\tau \\
&\quad + \int_{T-h}^T \frac{\partial \hat{\varphi}}{\partial x_i} \int_\tau^T \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j} \mathrm{d}t\mathrm{d}\tau\Big]\mathrm{d}x \\
&= \frac{1}{h}\int_\Omega \int_0^{T-h} \frac{\partial \varphi(x,\tau)}{\partial x_i} \int_\tau^{\tau+h} \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j} \mathrm{d}t\mathrm{d}\tau\mathrm{d}x \\
&= \int_\Omega \int_0^{T-h} \frac{\partial \varphi(x,t)}{\partial x_i} \Big(\sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j}\Big)_h \mathrm{d}t\mathrm{d}x.
\end{aligned}
\tag{2.28}
$$

By (2.24), (2.27) and (2.28), we deduce

$$
\int_0^{T-h} \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} \varphi \frac{\partial u_{0h}}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^{T-h} \int_\Omega \Big(\sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u_0}{\partial x_j}\Big)_h \frac{\partial \varphi(x,t)}{\partial x_i} \mathrm{d}x\mathrm{d}t = 0\,.
$$

Taking $\varphi$ defined in (2.25), we get

$$\int_0^{T-h}\int_{\tilde\Omega_1\cup\tilde\Omega_2} u_{0h}\frac{\partial u_{0h}}{\partial t}\mathrm{d}x\mathrm{d}t+\int_0^{T-h}\int_\Omega\Big(\sum_{i,j=1}^N \tilde a_{ij}\frac{\partial u_0}{\partial x_j}\Big)_h\frac{\partial u_{0h}}{\partial x_i}\mathrm{d}x\mathrm{d}t=0\,.$$

Letting $h\to 0$ in in the above equation, we have

$$\frac12\int_{\tilde\Omega_1\cup\tilde\Omega_2} u_0^2\mathrm{d}x\Big|_0^T+\int_0^T\int_\Omega\sum_{i,j=1}^N\tilde a_{ij}\frac{\partial u_0}{\partial x_i}\frac{\partial u_0}{\partial x_j}\mathrm{d}x\mathrm{d}t=0\,.$$

Hence

$$\int_0^T\int_\Omega |Du_0|^2\mathrm{d}x\mathrm{d}t=0.$$

Thus we can prove $u_0=0$ a.e. in $\Omega$. Thus the proof of uniqueness is completed. □

## 3. Parabolic boundary value problem with equivalued interface

To study the limit behavior of solutions to problem (1.1), we need to study the following equivalued interface problem (3.1). Here we give another division of $\Omega$ as shown in Fig.2. $\Omega$ is composed of two non-overlapping subdomains $\Omega_1$ and $\Omega_2$, and $\tilde\Gamma$ is the interface of $\Omega_1$ and $\Omega_2$. Denote $Q_0=(\Omega_1\cup\Omega_2)\times(0,T), \tilde\Sigma_0=\tilde\Gamma\times(0,T)$.

$\Omega_1$ $\Omega_2$ $\widetilde\Gamma$ $\Gamma$

Figure 2. The compostion of $\Omega$ with thin interface $\widetilde\Gamma$

In this section we consider the boundary value problem with equivalued interface:

$$\begin{gathered}
\frac{\partial u}{\partial t}-\sum_{i,j=1}^N\frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j})=F(x,t)\quad\text{in } Q_0,\\
u=0\quad\text{on }\Sigma,\\
u_+=u_-=C(t)\quad\text{on }\tilde\Sigma_0,\\
\int_{\tilde\Gamma}(\frac{\partial u}{\partial n_L})_+\mathrm{d}s=\int_{\tilde\Gamma}(\frac{\partial u}{\partial n_L})_-\mathrm{d}s+A(t)\quad\text{a.e. t }\in(0,T),\\
u(x,0)=\varphi_0(x)\quad\text{in }\Omega,
\end{gathered}\tag{3.1}$$

where the subscripts + and − denote the values on both sides of $\tilde\Gamma$, and the unit normal vector $\vec n=(n_1,\dots,n_N)$ takes the same direction on both sides of $\tilde\Gamma$.

We state the following assumption to $a_{ij}$:

(H1) The functions $a_{ij}$ are piecewise smooth in $Q$, $a_{ij} = a_{ji}$, and there exist two constants $\alpha, \beta > 0$ such that

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leq \beta|\xi|^2, \quad \forall \xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N, \text{ a.e. } (x,t) \in Q.$$

Let

$$V_0 = \{v \in H_0^1(\Omega) : v|_{\tilde{\Gamma}} = \text{constant}\}, \tag{3.2}$$

$$U_0 = \{v \in \mathring{W}_2^{1,1}(Q) : v|_{\tilde{\Sigma}_0} = C(t),\ v(x,T) = 0\}. \tag{3.3}$$

**Definition 3.1.** A measurable function $u \in L^2(0,T;V_0)$ is called a weak solution to problem (3.1), if for any $\psi \in U_0$,

$$\begin{aligned} &-\int_0^T \int_\Omega u \frac{\partial \psi}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial u}{\partial x_j} \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega F u \mathrm{d}x\mathrm{d}t + \int_\Omega \varphi_0(x)\psi(x,0)\mathrm{d}x + \int_0^T A(t)\psi|_{\tilde{\Sigma}_0}\mathrm{d}t. \end{aligned} \tag{3.4}$$

Now we state the main result of this section as follows:

**Theorem 3.2.** *Suppose that $\varphi_0 \in L^2(\Omega), F \in L^2(Q)$, $A \in L^2(0,T)$ and* (H1) *hold. Then there exists a unique weak solution $u \in L^2(0,T;V_0)$ to* (3.1).

The proof of this theorem is similar to Theorem 2.2, so we omit it.

## 4. Limiting behavior of solutions to problem (1.1)

Let $\varepsilon > 0$ be a small parameter and replace $\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}$ by $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \tilde{\Omega}^\varepsilon$, also interface $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ by the interface $\tilde{\Gamma}_1^\varepsilon$ and $\tilde{\Gamma}_2^\varepsilon$, respectively as shown in Figure 3. Let $\tilde{Q}^\varepsilon = (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times (0,T), \tilde{\Sigma}_\varepsilon = (\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon) \times (0,T)$.

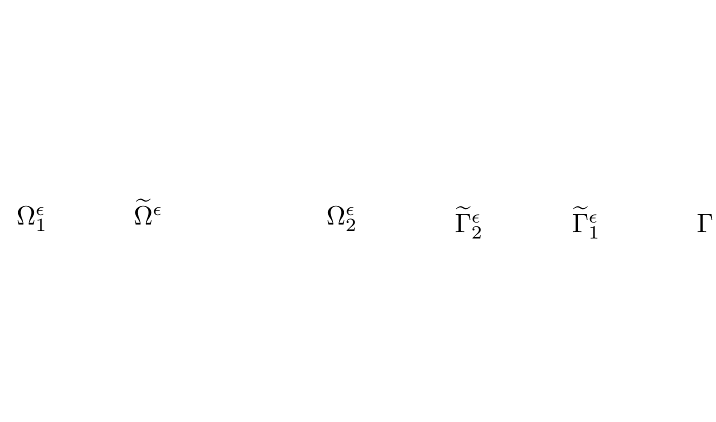


Figure 3. The compostion of $\Omega$ described by parameter $\varepsilon$

Here we will discuss the problem

$$
\begin{gathered}
\frac{\partial u_\varepsilon}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}^\varepsilon(x,t)\frac{\partial u_\varepsilon}{\partial x_j}) = F(x,t) \quad \text{in } \tilde{Q}^\varepsilon \\
u_\varepsilon = 0 \quad \text{on } \Sigma, \\
u_\varepsilon = \tilde{C}_\varepsilon(t) \quad \text{on } \tilde{\Sigma}_\varepsilon, \\
\int_{\tilde{\Gamma}_1^\varepsilon} \frac{\partial u_\varepsilon}{\partial n_{L^\varepsilon}} \mathrm{d}s = \int_{\tilde{\Gamma}_2^\varepsilon} \frac{\partial u_\varepsilon}{\partial n_{L^\varepsilon}} \mathrm{d}s + A(t) \quad \text{a.e. t } \in (0,T), \\
u_\varepsilon(x.0) = \varphi_{0\varepsilon}(x) \quad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon.
\end{gathered}
\tag{4.1}
$$

We state the following assumptions:

(H2) $\tilde{\Gamma} \subset \tilde{\Omega}^\varepsilon$, for all $\varepsilon > 0$; $\tilde{\Omega}^\varepsilon$ shrinks to $\tilde{\Gamma}$, as $\varepsilon \to 0$.

(H3) For any given domain $\tilde{\Omega}$ such that $\tilde{\Gamma} \subset \tilde{\Omega} \subset \Omega$, then for any $\varepsilon > 0$ small enough, we have $\tilde{\Omega}_\varepsilon \subset \tilde{\Omega}$.

(H4) $a_{ij}^\varepsilon$ are piecewise smooth functions in $Q$, $a_{ij}^\varepsilon = a_{ji}^\varepsilon$, and there exist two constants $\alpha, \beta > 0$ independent of $\varepsilon$ such that

$$
\alpha|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}^\varepsilon(x,t)\xi_i\xi_i \le \beta|\xi|^2, \quad \forall\ \xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N,
$$

a.e. $(x,t) \in Q$.

(H5) For any given domain $\tilde{\Omega}$ such that $\tilde{\Gamma} \subset \tilde{\Omega} \subset \Omega$, then

$$
a_{ij}^\varepsilon(x,t) \to a_{ij}(x,t) \quad \text{strongly in } L^\infty(0,T;L^\infty(\Omega\backslash\tilde{\Omega})).
$$

Set

$$
\begin{gathered}
V_\varepsilon = \{v \in H_0^1(\Omega) : v|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon} = \text{constant}\}, \\
U_\varepsilon = \{v \in \mathring{W}_2^{1,1}(Q) : v(x,T) = 0,\ v|_{\tilde{\Sigma}_\varepsilon} = C(t)\},
\end{gathered}
$$

where $C(t)$ is arbitrary function of $t$.

**Definition 4.1.** A measurable function $u_\varepsilon \in L^2(0,T;V_\varepsilon)$ is a weak solution to problem (4.1), if for any $\psi \in U_\varepsilon$,

$$
\begin{aligned}
&- \int_0^T \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} u_\varepsilon \frac{\partial \psi}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=1}^{N} a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \psi}{\partial x_i} \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} F\psi \mathrm{d}x\mathrm{d}t + \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} \varphi_{0\varepsilon}(x)\psi(x,0)\mathrm{d}x + \int_0^T A(t)\psi|_{\tilde{\Sigma}_\varepsilon} \mathrm{d}t.
\end{aligned}
\tag{4.2}
$$

**Remark 4.2.** For every fixed $\varepsilon > 0$, if (H4) and $\varphi_{0\varepsilon} \in L^2(\Omega)$, $F \in L^2(Q)$, $A \in L^2(0,T)$ hold, we can similarly prove that (4.1) admits a unique weak solution $u_\varepsilon \in L^2(0,T;V_\varepsilon)$ in the sense of Definition 4.1.

To prove the main result in this section, we need the following Lemma.

**Lemma 4.3.** *Under hypothesis* (H2)–(H3)*, for any given $\psi \in U_0$, there exist $\psi_\varepsilon \in U_\varepsilon$ such that as $\varepsilon \to 0$,*

$$
\psi_\varepsilon \to \psi \quad \textit{strongly in } U_0,
\tag{4.3}
$$

*where $U_0$ can be seen in* (3.3).

*Proof.* For convenience, we assume the origin is the interior point of $\Omega_2$ (see Figure 2).

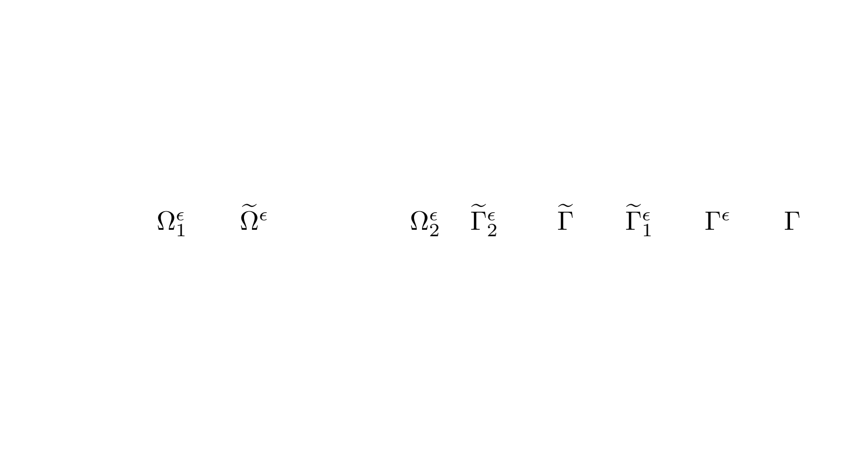


FIGURE 4. Scaling down or up the compostion of $\Omega$ in Figure 2

For fixed $\varepsilon > 0$ small enough, let $\Omega_2^\varepsilon = \{x(1-\varepsilon) | x \in \Omega_2\}$, $\Omega_1' = \{\frac{x}{1-\varepsilon} | x \in \Omega_2\}$, $\Omega_1^\varepsilon = \Omega \backslash \Omega_1'$, $\tilde{\Omega}^\varepsilon = \Omega_1' \backslash \Omega_2^\varepsilon$. Defining $\Gamma^\varepsilon = \{x(1-\varepsilon) | x \in \Gamma\}$ and assuming $\tilde{\Gamma}_1^\varepsilon$, $\tilde{\Gamma}_2^\varepsilon$ are the interfaces of $\tilde{\Omega}^\varepsilon$ with $\Omega_1^\varepsilon$ and $\Omega_2^\varepsilon$, we can write $\Gamma^\varepsilon \times (0,T) = \Sigma_\varepsilon$ (see Figure 4). Let

$$\psi_\varepsilon = \psi_\varepsilon^+ - \psi_\varepsilon^-,$$

where

$$\psi_\varepsilon^+ = \begin{cases} \big(\psi((1-\varepsilon)x,t) - \underset{\sup_\varepsilon}{\psi}{}^{+}(x,t)\big)^+, & (x,t) \in \Omega_1^\varepsilon \times (0,T), \\ \big(\psi(x,t)|_{\tilde{\Gamma}\times(0,T)} - \sup_{\Sigma_\varepsilon} \psi^+(x,t)\big)^+, & (x,t) \in (\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon) \times (0,T), \\ \big(\psi(\frac{x}{1-\varepsilon},t) - \sup_{\Sigma_\varepsilon} \psi^+(x,t)\big)^+, & (x,t) \in \Omega_2^\varepsilon \times (0,T)\,, \end{cases}$$

and

$$\psi_\varepsilon^- = \begin{cases} \big(\psi((1-\varepsilon)x,t) - \inf_{\Sigma_\varepsilon} \psi^-(x,t)\big)^-, & (x,t) \in \Omega_1^\varepsilon \times (0,T), \\ \big(\psi(x,t)|_{\tilde{\Gamma}\times(0,T)} - \inf_{\Sigma_\varepsilon} \psi^-(x,t)\big)^-, & (x,t) \in (\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon) \times (0,T), \\ \big(\psi(\frac{x}{1-\varepsilon},t) - \inf_{\Sigma_\varepsilon} \psi^-(x,t)\big)^-, & (x,t) \in \Omega_2^\varepsilon \times (0,T)\,. \end{cases}$$

Obviously, $\psi_\varepsilon^+ \in U_\varepsilon$, $\psi_\varepsilon^- \in U_\varepsilon$, so we have $\psi_\varepsilon \in U_\varepsilon$. It is easy to prove that $\psi_\varepsilon^+$ and $\psi_\varepsilon^-$ converge strongly to $\psi^+$ and $\psi^-$ in $U_0$ respectively. We omit the details. □

Now we give the limit behavior of solutions to (4.1) as follows.

**Theorem 4.4.** *Suppose that* (H1)-(H5) *and* $F \in L^2(Q)$, $A \in L^2(0,T)$ *hold, if as* $\varepsilon \to 0$,

$$\varphi_{0\varepsilon} \to \varphi_0 \quad \text{weakly in } L^2(\Omega), \tag{4.4}$$

*then for every weak solution* $u_\varepsilon$ *to* (4.1) *we have*

$$u_\varepsilon \to u \quad \text{weakly in } L^2(0,T;V_0), \tag{4.5}$$

*where* $u$ *is the weak solution to problem* (1.1) *and definition of* $V_0$ *can be seen in* (3.2).

*Proof.* Let $u_\varepsilon$ be the solution to problem (4.1). For a given $0 < h < T$, let $u_0$ and $u_{0h}$ be replaced by $u_\varepsilon$ and $u_{\varepsilon h}$ in (2.27) respectively. Taking $\psi = \hat{\varphi}_h$ (see (2.26)) in (4.2), we have

$$\begin{aligned}
&-\int_0^T \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} u_\varepsilon \frac{\partial \hat{\varphi}_h}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \hat{\varphi}_h}{\partial x_i} \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} F \hat{\varphi}_h \mathrm{d}x\mathrm{d}t + \int_0^T A(t) \hat{\varphi}_h(x,t)|_{\tilde{\Sigma}_\varepsilon} \mathrm{d}t.
\end{aligned}$$

Similar to (2.27) and (2.28), it is easy to prove that

$$\begin{gathered}
-\int_0^T \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} u_\varepsilon \frac{\partial \hat{\varphi}_h}{\partial t} \mathrm{d}x\mathrm{d}t = \int_0^{T-h} \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} \frac{\partial u_{\varepsilon h}}{\partial t} u_{\varepsilon h} \mathrm{d}x\mathrm{d}t, \\
\int_0^T \int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \hat{\varphi}_h}{\partial x_i} \mathrm{d}x\mathrm{d}t = \int_0^{T-h} \int_\Omega \sum_{i,j=1}^N \frac{\partial u_{\varepsilon h}}{\partial x_i} \big(a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\big)_h \mathrm{d}x\mathrm{d}t, \\
\int_0^T \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} F \hat{\varphi}_h \mathrm{d}x\mathrm{d}t = \int_0^{T-h} \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} u_{\varepsilon h} F_h \mathrm{d}x\mathrm{d}t, \\
\int_0^T A(t) \hat{\varphi}_h|_{\tilde{\Sigma}_\varepsilon} \mathrm{d}t = \frac{1}{|\tilde{\Gamma}|} \int_0^T \int_{\tilde{\Gamma}} A(t) \hat{\varphi}_h \mathrm{d}s\mathrm{d}t = \frac{1}{|\tilde{\Gamma}|} \int_0^{T-h} \int_{\tilde{\Gamma}} (A(t))_h u_{\varepsilon h} \mathrm{d}s\mathrm{d}t.
\end{gathered}$$

Thus we can write the above expression as

$$\begin{aligned}
&\int_0^{T-h} \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} \frac{\partial u_{\varepsilon h}}{\partial t} u_{\varepsilon h} \mathrm{d}x\mathrm{d}t + \int_0^{T-h} \int_\Omega \frac{\partial u_{\varepsilon h}}{\partial x_i} \big(a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\big)_h \mathrm{d}x\mathrm{d}t \\
&= \int_0^{T-h} \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} u_{\varepsilon h} F_h \mathrm{d}x\mathrm{d}t + \frac{1}{|\tilde{\Gamma}|} \int_0^{T-h} \int_{\tilde{\Gamma}} (A(t))_h u_{\varepsilon h} \mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Let $h \to 0$ in the above expression, we have

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} u_\varepsilon^2 \mathrm{d}x \Big|_0^T + \int_0^T \int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} u_\varepsilon F \mathrm{d}x\mathrm{d}t + \frac{1}{|\tilde{\Gamma}|} \int_0^T \int_{\tilde{\Gamma}} A u_\varepsilon \mathrm{d}s\mathrm{d}t.
\end{aligned} \tag{4.6}$$

Condition (H4), Hölder's inequality, the trace theorem, and Poincáre's inequality yield

$$\begin{aligned}
&\alpha \|Du_\varepsilon\|_{L^2(Q)}^2 \\
&\le \|u_\varepsilon\|_{L^2(Q)} \|F\|_{L^2(Q)} + \frac{1}{|\tilde{\Gamma}|^{1/2}} \int_0^T |A| \Big( \int_{\tilde{\Gamma}} u_\varepsilon^2 \mathrm{d}s \Big)^{1/2} \mathrm{d}t \\
&\le \|Du_\varepsilon\|_{L^2(Q)} \|F\|_{L^2(Q)} + \frac{C}{|\tilde{\Gamma}|^{1/2}} \int_0^T |A| \Big( \int_{\Omega_2} |u_\varepsilon|^2 + |Du_\varepsilon|^2 \mathrm{d}x \Big)^{1/2} \mathrm{d}t \\
&\le \|Du_\varepsilon\|_{L^2(Q)} \|F\|_{L^2(Q)} + C \|A\|_{L^2(0,T)} \|Du_\varepsilon\|_{L^2(Q)} \\
&\le (C\|A\|_{L^2(0,T)} + \|F\|_{L^2(Q)}) \|Du_\varepsilon\|_{L^2(Q)}\,,
\end{aligned}$$

where $C$ is a positive constant independent of $\varepsilon$. By Young's inequality we obtain

$$\|Du_\varepsilon\|_{L^2(Q)} \le C. \tag{4.7}$$

Hence we get

$$\|u_\varepsilon\|_{L^2(0,T;V_0)} \le C, \tag{4.8}$$

where $C$ a positive constant independent of $\varepsilon$.

From the above inequality, we can extract a subsequence of $\{u_\varepsilon\}$ (still denoted by $\{u_\varepsilon\}$) such that

$$u_\varepsilon \to u \quad \text{weakly in } L^2(0,T;V_0). \tag{4.9}$$

By Lemma 4.3, for any given $\psi \in U_0$, there exists $\psi_\varepsilon \in U_\varepsilon$ such that

$$\psi_\varepsilon \to \psi \quad \text{strongly in } U_0, \text{ as } \varepsilon \to 0. \tag{4.10}$$

Fixed $\varepsilon_0 > 0$ and for any $0 < \varepsilon < \varepsilon_0$, we have $\tilde{\Omega}^\varepsilon \subset \tilde{\Omega}^{\varepsilon_0}$ and $\psi_{\varepsilon_0} \in U_\varepsilon$, taking $\psi = \psi_{\varepsilon_0}$ in (4.2), we have

$$\begin{aligned} &-\int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} u_\varepsilon \frac{\partial\psi_{\varepsilon_0}}{\partial t}\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial \psi_{\varepsilon_0}}{\partial x_i}\mathrm{d}x\mathrm{d}t \\ &= \int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} F\psi_{\varepsilon_0}\mathrm{d}x\mathrm{d}t + \int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} \varphi_{0\varepsilon}\psi_{\varepsilon_0}(x,0)\mathrm{d}x\mathrm{d}t + \int_0^T A(t)\psi_{\varepsilon_0}|_{\tilde{\Sigma}_{\varepsilon_0}}\mathrm{d}t. \end{aligned} \tag{4.11}$$

By (4.4), (4.9) and the absolute continuity of Lebegue integral, it is easy to prove that

$$\int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} u_\varepsilon \frac{\partial\psi_{\varepsilon_0}}{\partial t}\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega u\frac{\partial\psi_{\varepsilon_0}}{\partial t}\mathrm{d}x\mathrm{d}t, \tag{4.12}$$

$$\int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} F\psi_{\varepsilon_0}\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega F\psi_{\varepsilon_0}\mathrm{d}x\mathrm{d}t, \tag{4.13}$$

$$\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} \varphi_{0\varepsilon}(x)\psi_{\varepsilon_0}(x,0)\mathrm{d}x \to \int_\Omega \varphi_0(x)\psi_{\varepsilon_0}(x,0)\mathrm{d}x. \tag{4.14}$$

Next we prove that

$$\int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\psi_{\varepsilon_0}}{\partial x_i}\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial\psi_{\varepsilon_0}}{\partial x_i}\mathrm{d}x\mathrm{d}t. \tag{4.15}$$

For any given $\tilde{\Omega}$ such that $\tilde{\Gamma} \subset \tilde{\Omega} \subset \Omega$, we have

$$\begin{aligned} &\int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\psi_{\varepsilon_0}}{\partial x_i}\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial\psi_{\varepsilon_0}}{\partial x_i}\mathrm{d}x\mathrm{d}t \\ &= \int_0^T\int_{\Omega\setminus\tilde{\Omega}} \sum_{i,j=1}^N (a_{ij}^\varepsilon - a_{ij})\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\psi_{\varepsilon_0}}{\partial x_i}\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Omega\setminus\tilde{\Omega}} \sum_{i,j=1}^N a_{ij}\frac{\partial\psi_{\varepsilon_0}}{\partial x_i}\frac{\partial(u_\varepsilon - u)}{\partial x_j}\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^T\int_{\tilde{\Omega}} \sum_{i,j=1}^N a_{ij}^\varepsilon\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\psi_{\varepsilon_0}}{\partial x_i}\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\tilde{\Omega}} \sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial\psi_{\varepsilon_0}}{\partial x_i}\mathrm{d}x\mathrm{d}t \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned} \tag{4.16}$$

For any given $\delta > 0$, by (H1), (H4), (4.9) and the absolute continuity of Lebegue integral, we can take $\tilde{\Omega}$ so small that

$$|\mathrm{III}| + |\mathrm{IV}| < \frac{\delta}{2}. \tag{4.17}$$

Following the above $\tilde{\Omega}$ is chosen, by (H5) and noting (4.9) and (4.10), then there exists $0 < \varepsilon_1 < \varepsilon_0$ such that for any $\varepsilon$ with $0 < \varepsilon < \varepsilon_1$,

$$|\mathrm{I}| + |\mathrm{II}| < \frac{\delta}{2}. \tag{4.18}$$

From (4.16)–(4.18) we get (4.15). Let $\varepsilon \to 0$ in (4.11), (4.12)–(4.15) yield

$$\begin{aligned} &-\int_0^T \int_\Omega u \frac{\partial \psi_{\varepsilon_0}}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \psi_{\varepsilon_0}}{\partial x_i} \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega F \psi_{\varepsilon_0} \mathrm{d}x\mathrm{d}t + \int_\Omega \varphi_0 \psi_{\varepsilon_0}(x,0)\mathrm{d}x + \int_0^T A(t) \psi_{\varepsilon_0}|_{\tilde{\Sigma}_{\varepsilon_0}} \mathrm{d}t \end{aligned} \tag{4.19}$$

Using (4.10), we get

$$\psi_{\varepsilon_0} \to \psi \quad \text{strongly in } C([0,T]; L^2(\Omega)), \text{ as } \varepsilon_0 \to 0. \tag{4.20}$$

Hence as $\varepsilon_0 \to 0$, we also have

$$\psi_{\varepsilon_0}(x,0) \to \psi(x,0) \quad \text{strongly in } L^2(\Omega). \tag{4.21}$$

By (4.10) and trace theorem, we can deduce

$$\psi_{\varepsilon_0} \to \psi \quad \text{strongly in } L^2(\tilde{\Sigma}_0), \text{ as } \varepsilon_0 \to 0. \tag{4.22}$$

Hence

$$\psi_{\varepsilon_0}|_{\tilde{\Sigma}_{\varepsilon_0}} = \psi_{\varepsilon_0}|_{\tilde{\Sigma}_0} \to \psi|_{\tilde{\Sigma}_0} \quad \text{strongly in } L^2(0,T). \tag{4.23}$$

Let $\varepsilon_0 \to 0$ in (4.19), by (4.10), (4.21) and (4.23) we can deduce $u$ satisfies (3.4). By the uniqueness of weak solution to problem (3.1), (4.9) holds for the whole sequence $\{u_\varepsilon\}$. This completes the proof. □

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