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REGULARITY RESULT FOR THE PROBLEM OF VIBRATIONS OF A NONLINEAR BEAM

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Abstract. A model for the dynamics of the Gao nonlinear beam, which allows for buckling, is studied. Existence and uniqueness of the local weak solution was established in Andrews *et al.* (2008). In this work the further regularity in time of the weak solution is shown using recent results for evolution problems. Moreover, the weak solution is shown to be global, existing on each finite time interval.

1. Introduction

A model for the vibrations of a nonlinear beam that takes into account the beam's thickness which, however, is one-dimensional, was derived by Gao in [4, 5]. The existence of the unique local weak solution for the problem was established recently in [2]. In this work we show that the weak solution has additional regularity in time, when the problem data is smoother. This allows us to establish that the weak solution is also a global solution existing on each finite time interval.

The motivation for the introduction of various models for beams was to capture more fully the nonlinearities that beams exhibit, in particular buckling, which is associated with a double-well in the energy function of the beam. Among the models derived in [5], the one-dimensional model studied in [2] and here is the simplest. However, it is nonlinear, it allows for buckling, and has interest in and of itself. The literature on the beam includes also [9] and the references therein.

This work is the continuation of [2] where, in addition to the analysis, a finite difference scheme for the beam was introduced based on the Newmark time discretization and Hermite cubic finite elements, and the results of numerical simulations depicted. Here, we prove that if the problem data is more regular, then the solution has additional time regularity. The proof is based on the problem for the time derivative of the solution. We first study the truncated problem, and use results for variational problems for pseudomonotone operators of [6, 7]. Then, we use a continuity argument to show that there exists a unique weak solution such that for some time the truncation is inactive. Using the energy balance and a priori estimates derived from it, we also show that for a sufficiently large truncation

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ceiling, the truncation is not active on each preassigned time interval, and so the solution is global.

The rest of the paper is organized as follows. In Section 2 we present shortly the classical formulation of the model, following [4, 5, 2]. In Section 2 the weak formulation and the statement of the existence and uniqueness result in [2] is given, the weak or abstract formulation of the problem for the time derivative presented and the statements of the existence and uniqueness results for the truncated problem and the full problem stated. Our main regularity result is states in Theorem 3.4. The proof is provided in Section 6, and is based on the results for the truncated problem in Section 5. In Section 4 we present the energy balance equation for the original problem, derive a priori estimates on $\|w_x\|_{L^\infty(0,T;L^\infty(0,1))}$, and based on this estimate we conclude that the solution of the problem is global in time.

2. The Model

The derivation of the model was done in Gao in [4, 5], and a more detailed description can be found in [2]. Here, we just present it with very few comments. The beam's centerline is, in dimensionless variables, $[0, 1]$ and its thickness $2h$, and we denote by $w(x, t)$ the transverse displacement of its central axis, for $0 \le x \le 1$ and $0 \le t \le T$, for $0 < T$.

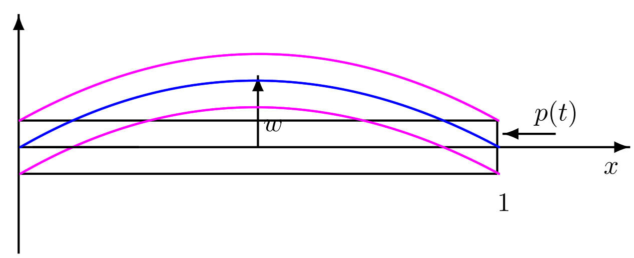

FIGURE 1. The beam; w is the displacement of the central axis.

The beam is subjected to a laterally distributed load $f = f(x, t)$, and a horizontal traction which acts at the end $x = 1$. The simplest finite deformations nonlinear model derived in [4, 5] for the beam consists of the evolution equation

$$w_{tt} + kw_{xxxx} + (\nu p - aw_x^2)w_{xx} = f, \tag{2.1}$$

in $\Omega_T = (0, 1) \times (0, T)$, where all the variables are in dimensionless form, and k, ν, and a are system parameters, assumed to be positive constants. The term with $p = p(t)$ describes the effect of the horizontal force (compression/tension) acting on the right end. When $0 < p$ the beam is being compressed, and when $p < 0$ it is under tension.

We assume that the beam is clamped at both ends and initially the displacement is w_0 and the velocity is v_0. Also, for the sake of generality, we add a viscosity term γw_{txxxx}, with viscosity coefficient $\gamma > 0$, assumed to be small.

The classical formulation of the *dynamic model for a beam with finite deformations* with viscosity, is as follows.

Problem P_{cl}. Find the displacement field $w = w(x,t)$ for $x \in (0,1)$ and $t \in (0,T)$, such that

$$w_{tt} + kw_{xxxx} + \gamma w_{txxxx} - (aw_x^2 - \nu p)w_{xx} = f, \tag{2.2}$$
$$w(0,t) = w_x(0,t) = 0, \tag{2.3}$$
$$w(1,t) = w_x(1,t) = 0, \tag{2.4}$$
$$w(x,0) = w_0(x), \quad w_t(x,0) = v_0(x). \tag{2.5}$$

The system is nonlinear and the existence and uniqueness of the local weak solution to the problem has been established in [2]. We first present the weak formulation of the problem and then show that additional assumptions on the problem data lead to an improved regularity, in time, of the solution.

3. Weak formulation and results

We first describe the weak formulation of Problem P_{cl}, the assumptions on the problem data, and state the existence and uniqueness result for local weak solutions. We follow [2]. Then, we describe the problem obtained by differentiating Problem P_{cl} with respect to time.

We denote by $(\cdot,\cdot)$ the inner product in $H = L^2(0,1)$, and let

$$W = H_0^2(0,1) = \{u \in H^2(0,1) : u = u_x = 0 \text{ at } \ x = 0,1\},$$

be a Hilbert space endowed with the inner product $(w,u)_W = \int_0^1 w_{xx}u_{xx}\,dx$, and the associated norm $\|w\|_W^2 = (w,w)_W$ which, in view of the boundary conditions and the Poincare theorem, is equivalent to the usual $H^2(0,1)$ norm on W. The dual of W is denoted by W', and since $W \subseteq H \subseteq W'$, by identifying $H' = H$, it follows that (W,H,W') is a Gelfand triple. Next, let $\mathcal{H} = L^2(0,T;H)$, and $\mathcal{W} = L^2(0,T;W)$ with inner product $(\cdot,\cdot)_{\mathcal{W}}$ and duality pairing $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ between $\mathcal{W}$ and its dual $\mathcal{W}'$, which we write as $\langle\cdot,\cdot\rangle$. Again, we have

$$\mathcal{W} \subseteq \mathcal{H} = \mathcal{H}' \subseteq \mathcal{W}'.$$

Next, proceeding as usual, we obtain the following *variational formulation of the problem of vibrations of a nonlinear beam.*

Problem P_V. Find the displacement field $w : [0,T] \to W$ and the velocity $v = w_t$, such that for a.a. $t \in [0,T]$ and $\psi \in W$,

$$\begin{gathered}\langle v_t(t),\psi\rangle_W + k(w_{xx}(t),\psi_{xx}) + \gamma(v_{xx}(t),\psi_{xx}) \\ +\frac{1}{3}a(w_x^3(t),\psi_x) - \nu p(t)(w_x(t),\psi_x) = (f(t),\psi),\end{gathered} \tag{3.1}$$
$$w(0) = w_0, \quad w_t(0) = v_0. \tag{3.2}$$

We make the following assumptions on the problem data:

$$w_0,\ v_0 \in W, \quad \|w_{0xx}\|_{L^2(0,1)}, \|w_{0x}\|_{L^\infty(0,1)} \le R^*, \tag{3.3}$$
$$p \in C^1([0,T]), \quad |p|, |p'| \le p^*, \tag{3.4}$$
$$f \in \mathcal{H}. \tag{3.5}$$

Here, R^* and p^* are two positive constants.

The main existence and uniqueness result in [2] is the following.

Theorem 3.1. *Assume that* (3.3)–(3.5) *hold. Then there exists* $T^* > 0$ *and a unique solution* w *to Problem* P_V *on the time interval* $[0, T^*)$ *such that*

$$w, v \in L^\infty(0, T^*; W), \quad v' \in L^2(0, T^*; W'). \tag{3.6}$$

To establish additional regularity of w, we study the problem for $v = w'$, where here and below we denote by a prime the (weak) time derivative. We differentiate equation (3.1) with respect to t, set $z = w'' = v'$, and, for $\psi \in W$, we obtain

$$\begin{aligned} &\langle z'(t), \psi\rangle_W + k(v_{xx}(t), \psi_{xx}) + \gamma(z_{xx}(t), \psi_{xx}) + a(w_x^2(t)v_x(t), \psi_x) \\ &-\nu p'(t)(w_x(t), \psi_x) - \nu p(t)(v_x(t), \psi_x) = (f'(t), \psi). \end{aligned} \tag{3.7}$$

To obtain the initial condition for z, we formally set $t = 0$ in (2.2) and obtain condition (3.9) below.

We have the following problem for the triple $\{w, v, z\}$.

Problem P_{Vz}**.** Find the displacement field $w : [0, T] \to W$, the velocity field $v : [0, T] \to W$, and the acceleration $z : [0, T] \to W$ such that for a.a. $t \in [0, T]$ and every $\psi \in W$ the variational equation (3.7) holds, together with

$$w(t) = w_0 + \int_0^t v(\tau)\, d\tau, \quad v(t) = v_0 + \int_0^t z(\tau)\, d\tau, \tag{3.8}$$

$$z(0) = -k w_{0xxxx} - \gamma v_{0xxxx} + a w_{0x}^2 w_{0xx} - \nu p(0) w_{0xx} + f(0). \tag{3.9}$$

Problem (3.7)–(3.9) makes sense only if we assume, in addition to (3.3)–(3.5), that

$$f, f' \in \mathcal{W}', \quad f(0) \in H, \tag{3.10}$$

and to ensure that $z(0) \in L^2(0,1)$ we assume that

$$w_0,\ v_0 \in H^4(0,1), \quad w_0 \in H_0^2(0,1), \quad \|w_{0x}\|_{L^\infty(0,1)} \le R^*. \tag{3.11}$$

We note that the term $a w_{0x}^2 w_{0xx}$ is well defined.

To deal with the term with $w_x^2 v_x$ we introduce the truncation

$$\Psi_R(r) = \begin{cases} R & \text{for } R \le r, \\ r & \text{for } |r| \le R, \\ -R & \text{for } r \le -R, \end{cases} \tag{3.12}$$

where R is a large positive number, and we replace w_x^2 with $\Psi_R^2(w_x)$. Eventually, we show that when R is sufficiently large, the truncation is inactive.

To proceed with the abstract formulation of the truncated problem, we define the operators $B, K : \mathcal{W} \to \mathcal{W}'$, and $K_{RN} : \mathcal{W} \times \mathcal{W} \to \mathcal{W}'$ by

$$\langle B(w), \psi\rangle = \int_0^T \int_0^1 w_x \psi_x \, dx\, dt, \tag{3.13}$$

$$\langle K(w), \psi\rangle = \int_0^T \int_0^1 w_{xx} \psi_{xx} \, dx\, dt, \tag{3.14}$$

$$\langle K_{NR}(w, v), \psi\rangle = \int_0^T \int_0^1 \Psi_R^2(w_x) v_x \psi_x \, dx\, dt. \tag{3.15}$$

We introduce the function space

$$\mathcal{Y} = \mathcal{W} \times \mathcal{W} \times \mathcal{W},$$

and denote its dual by $\mathcal{Y}'$.

The abstract formulation of the truncated version of Problem P_{Vz} is:

Problem P_{VzR}. Find $(w, v, z) \in \mathcal{Y}$, with $z' \in \mathcal{W}'$ such that:

$$v = w', \quad z = v', \tag{3.16}$$

$$z' + kK(v) + \gamma K(z) + aK_{NR}(w, v) - \nu p' B(w) - \nu p B(v) = f', \tag{3.17}$$

in $\mathcal{W}'$, together with (3.8) and (3.11).

The abstract formulation of Problem P_{Vz} is obtained by reinstating w_x^2 in place of $\Psi_R^2(w_x)$ in K_{NR}.

Next, we rewrite (3.16) in an equivalent form,

$$K(v) = K(w'), \quad K(z) = K(v').$$

The equivalence follows from the boundary conditions. This allows us to show the coercivity of the operator $A_{\rm reg}$.

The operator $A : \mathcal{Y} \to \mathcal{Y}'$, for $y = (w, v, \varphi) \in \mathcal{Y}$, is defined by

$$A(w, v, \varphi) = kK(v) + \gamma K(\varphi) + aK_{NR}(w, v) - \nu p' B(w) - \nu p B(v).$$

The operator $A_{\rm reg} : \mathcal{Y} \to \mathcal{Y}'$ is defined, for $y = (w, v, \varphi) \in \mathcal{Y}$, by

$$A_{\rm reg}(y) = (-K(v), -K(\varphi), A(w, v, \varphi)).$$

We let $G = W \times W \times H$ with dual G', we define $\mathcal{G} = \mathcal{W} \times \mathcal{W} \times \mathcal{H}$, the operator $D : \mathcal{G} \to \mathcal{G}'$ is defined, for $y = (w, v, \varphi) \in \mathcal{G}$, by

$$D(y) = (K(w), K(v), \varphi),$$

and the functional $F : \mathcal{Y} \to \mathbb{R}^3$ as

$$\langle F, y\rangle = (0, 0, \int_0^T \int_0^1 f'\varphi\, dx\, dt).$$

Problem P_{VzR} can now be written in the following abstract form.

Problem P_{AR}. Find $y = (w, v, z) \in \mathcal{Y}$ such that

$$(Dy)' + A_{\rm reg}(y) = F, \quad \text{in } \mathcal{Y}',$$
$$Dy(0) = Dy_0, \quad \text{in } G',$$

where w and v are given in (3.11), $y_0 = (w_0, v_0, z(0))$, and $z(0)$ in (3.9).

Theorem 3.2. *Assume that* (3.3)–(3.5)*,* (3.10) *and* (3.11) *hold. Then Problem $P_{AR} = P_{VzR}$ has a unique weak solution.*

The proof of the theorem is given in the next section. The main step to the main result of this paper is the following.

Theorem 3.3. *Assume that* (3.3)–(3.5)*,* (3.10) *and* (3.11) *hold. Then there exists $0 < T^*$ such that Problem P_{Vz} has a unique weak solution on the time interval $[0, T^*)$.*

The proof is provided in Section 6, and is based on the observation that if R is sufficiently large and w_{0x} is bounded in L^∞, then, by the continuity of the solution, the L^∞ norm of w_x is bounded by R on a time interval $[0, T^*)$, hence the truncation $\Psi_R(w_x)$ is inactive, and on this time interval the problem with or without truncation has the same solution.

Then, the argument presented in the next section, which provides an a priori estimate based on an energy balance, shows that the unique weak solution is actually global in time, and exists on each time interval $[0, T]$.

The main result of this paper shows that the solution of Problem P_V has additional regularity in time.

Theorem 3.4. *Assume that* (3.3)–(3.5), (3.10)–(3.11) *hold. Then, for each* $0 < T$, *Problem* P_V, (3.1) *and* (3.2), *has a unique weak solution on the time interval* $[0, T]$, *and the solution satisfies*

$$w, v, v' \in L^\infty(0, T; W), \quad v'' \in L^2(0, T, W'). \tag{3.18}$$

We conclude that the acceleration v' is a function and only its time derivative may be a distribution, and moreover,

$$w \in C^1([0, T]); W), \quad v \in C([0, T]; W), \quad v' \in C([0, T]; H). \tag{3.19}$$

In the next section we show that the solution is global and exists on each finite time interval $[0, T]$.

4. Energy estimate and global solution

In this section we use an energy balance to obtain an a priori estimate that allows us to conclude that the solution of Problem P_V is global.

Proceeding formally, the following energy balance was obtained in [2] for Problem P_V on $[0, T^*)$:

$$\begin{aligned} E(t) &\equiv \frac{1}{2}\|v(t)\|_H^2 + \frac{k}{2}\|w_{xx}(t)\|_H^2 + \frac{a}{12}\|w_x^2(t)\|_H^2 - \frac{1}{2}\nu p(t)\|w_x(t)\|_H^2 \\ &= E(0) - \gamma\int_0^t \|v_{xx}(\tau)\|_H^2\, d\tau - \frac{1}{2}\nu\int_0^t p'(\tau)\|w_x(\tau)\|_H^2\, d\tau \\ &\quad + \int_0^t (f(\tau), v(\tau))\, d\tau. \end{aligned} \tag{4.1}$$

The initial energy is

$$E(0) = \frac{1}{2}\|v_0\|_H^2 + \frac{k}{2}\|w_{0xx}\|_H^2 + \frac{a}{12}\|w_{0x}^2\|_H^2 - \frac{1}{2}\nu p(0)\|w_{0x}\|_H^2.$$

The first integral on the right-hand side in (4.1) is the viscous dissipation, the second one is related to the work done by p, while the third one is the work of the applied force f.

It follows from the assumptions (3.3)–(3.5) on the problem data that $|E(0)| = E_0 < \infty$, and also that the energy balance is actual, as the regularity of the solution, (3.18) and (3.19), justifies the manipulations that lead to (4.1).

Rearranging the terms, using the fact that $\|w_x(t)\|_H^2 \le \|w_{xx}(t)\|_H^2$, and manipulations that include the Cauchy inequality, we obtain for $0 \le t < T^*$,

$$\begin{aligned} &\frac{1}{2}\|v(t)\|_H^2 + \frac{k}{2}\|w_{xx}(t)\|_H^2 + \gamma\int_0^t \|v_{xx}(\tau)\|_H^2\, d\tau \\ &\le E_0 + \frac{1}{2}\int_0^t \|f(\tau)\|_H^2\, d\tau + \frac{1}{2}\nu p^*\|w_x(t)\|_H^2 - \frac{a}{12}\|w_x^2(t)\|_H^2 \\ &\quad + C\int_0^t \left(\|v(\tau)\|_H^2 + \|w_{xx}(\tau)\|_H^2\right)\, d\tau. \end{aligned}$$

Here, $C = C(k, \nu, p^*)$ is a positive constant. Next, we study $J = J(t)$ which is given by

$$J(t) \equiv \frac{1}{2}\nu p^* \|w_x(t)\|_H^2 - \frac{a}{12}\|w_x^2(t)\|_H^2 = \frac{a}{12}\int_0^1 \left(\frac{6\nu p^*}{a} - w_x^2(x,t)\right) w_x^2(x,t)\,dx.$$

Let $\chi_+(x,t)$ be the subset of $[0,1]$ where $w_x^2(x,t) \le 6\nu p^*/a$ and let $\chi_-(x,t)$ be the complement where $w_x^2(x,t) > 6\nu p^*/a$. Then

$$J(t) \le \frac{a}{12}\int_{\chi_+(x,t)} \left(\frac{6\nu p^*}{a} - w_x^2(x,t)\right) w_x^2(x,t)\,dx \le 3\nu^2(p^*)^2 \equiv c_{*\nu}.$$

Using now the Gronwall inequality yields

$$\|v(t)\|_H^2 + \|w_{xx}(t)\|_H^2 \le \exp(CT^*)\left((E_0 + c_{*\nu})T^* + \|f\|^2_{L^2(0,T^*,H)}\right), \tag{4.2}$$

for all $0 \le t < T^*$. Thus, $\|w_{xx}(t)\|_H \le C^*$, where C^* depends only on the data and on T^*. Finally, we note that the Hölder inequality yields

$$|w_x(x,t)| \le \|w_x(t)\|_{L^\infty(0,1)} \le \int_0^1 |w_{xx}(r,t)|\,dr \le \|w_{xx}(t)\|_{L^2(0,1)} \le C^*,$$

for $0 \le t \le T^*$. But this means that if we choose the truncation ceiling R such that $C^* << R$, then by a continuity argument the truncation Ψ_R (used in [2]) is inactive on $0 \le t \le T^* + \varepsilon$, where ε depends only on C^*, and therefore, the solution exists on any finite time interval $[0,T]$.

5. Proof of Theorem 3.2

The proof is based on the existence results for evolution problems established in Kuttler [6] and [7]. To use these results we need to show that the operator $A_{\text{reg}} : \mathcal{Y} \to \mathcal{Y}'$ is bounded, coercive, and pseudomonotone, that is, for $y = (w, v, \varphi) \in \mathcal{Y}$, it satisfies the following conditions:

(1) $\|A_{\text{reg}}(y)\|_{\mathcal{Y}'} \le C(1 + \|y\|_{\mathcal{Y}})$.

(2) $\displaystyle\lim_{\|y\|_{\mathcal{Y}}\to\infty} \frac{\langle \lambda Dy + A_{\text{reg}}(y),\, y\rangle}{\|y\|_{\mathcal{Y}}} = \infty$, for λ sufficiently large.

(3) $y \to A_{\text{reg}}(y)$ is a pseudomonotone map from $\mathcal{Y}$ to $\mathcal{Y}'$.

These conditions guarantee the existence of a weak solution to the truncated problem.

Actually, to obtain the coercivity (2) one has to use an exponential shift in which the new dependent variable $\widetilde{y}$ is given by $\widetilde{y} = ye^{-\lambda t}$. Then in all the linear terms we find that $\lambda D + A_{\text{reg}}$ replaces A_{reg} and the nonlinear term K_{NR} changes with an exponential $e^{\lambda t}$. Since the problem is only considered on a finite interval, to save on notation and simplify the presentation, we ignore this minor technical detail and note that it suffices to show that (1)–(3) hold for $y = (w, v, \varphi) \in \mathcal{Y}$.

These steps are summarized in the following lemmas. The proofs parallel the ones presented in [2], so we only present the modifications which deal with the different nonlinear term. Indeed, the operators B and K are linear, so we need only to study K_{NR}.

Below, we let C denote a positive constant which depends on the problem data and whose value may change from place to place.

Lemma 5.1. *The operator K_{NR} is bounded.*

Proof. Since $|\Psi_R^2(w_{\varphi\, x})| \le R^2$, by using the Hölder inequality we obtain

$$\begin{aligned}
|\langle K_{NR}(w,v),\psi\rangle| &\le \int_0^T\int_0^1 |\Psi_R^2(w_x)||v_x||\psi_x|\,dx\,dt\\
&\le R^2\int_0^T\int_0^1 |v_x||\psi_x|\,dx\,dt\\
&\le R^2\|v\|_{L^2(0,T;H^1)}\|\psi\|_{L^2(0,T;H^1)}\\
&\le CR^2\|v\|_{\mathcal{W}}\|\psi\|_{\mathcal{W}}.
\end{aligned}$$

Therefore,

$$\|K_{NR}(w,v)\|_{\mathcal{W}'} \le CR^2\|v\|_{\mathcal{W}} \le CR^2\|y\|_{\mathcal{Y}}.$$

The rest of the estimates leading to the boundedness of A are straightforward (see [2, Lemma 3.3]) and from these we obtain the boundedness of $A_{\rm reg}$. □

Lemma 5.2. *The operator $\lambda D + A_{\rm reg}$ is coercive for all λ sufficiently large.*

Proof. We have

$$\langle \lambda Dy + A_{\rm reg}(y), y\rangle = \langle \lambda Dy, y\rangle + \langle A_{\rm reg}(y), y\rangle.$$

Then,

$$\begin{aligned}
\langle \lambda Dy, y\rangle = (\lambda Dy, y) &= \lambda(\|w_{xx}\|_{\mathcal{H}}^2 + \|v_{xx}\|_{\mathcal{H}}^2 + \|\varphi\|_{\mathcal{H}}^2)\\
&= \lambda(\|w\|_{\mathcal{W}}^2 + \|v\|_{\mathcal{W}}^2 + \|\varphi\|_{\mathcal{H}}^2).
\end{aligned}$$

Here, we used $\|w_{xx}\|_{\mathcal{H}}$ as the norm on $\mathcal{W}$. Next,

$$\begin{aligned}
\langle A_{\rm reg}(y), y\rangle &\ge -|\langle Kv, w\rangle| - |\langle K\varphi, v\rangle| + \langle A(w,v,\varphi),\varphi\rangle|\\
&\ge -\|v\|_{\mathcal{W}}\|w\|_{\mathcal{W}} - \|\varphi\|_{\mathcal{W}}\|v\|_{\mathcal{W}} + \langle A(w,v,\varphi),\varphi\rangle.
\end{aligned}$$

Using the Cauchy inequality with ϵ leads to

$$\|v\|_{\mathcal{W}}\|w\|_{\mathcal{W}} + \|\varphi\|_{\mathcal{W}}\|v\|_{\mathcal{W}} \ge -C(\|v\|_{\mathcal{W}}^2 + \|w\|_{\mathcal{W}}^2) - \frac{1}{4}\gamma\|\varphi\|_{\mathcal{W}}^2.$$

Also,

$$\begin{aligned}
\langle A(w,v,\varphi),\varphi\rangle &= k(v_{xx},\varphi_{xx})_{\mathcal{H}} + \gamma\|\varphi\|_{\mathcal{W}}^2 - \nu(p'w_x,\varphi_x)_{\mathcal{H}}\\
&\quad - \nu(pv_x,\varphi_x)_{\mathcal{H}} + a(\Psi_R^2(w_x)v_x,\varphi_x)_{\mathcal{H}}.
\end{aligned}$$

Therefore, using the Cauchy inequality with ϵ, again, yields

$$\begin{aligned}
\langle A(w,v,\varphi),\varphi\rangle &\ge \gamma\|\varphi\|_{\mathcal{W}}^2 - k\|v_{xx}\|_{\mathcal{H}}\|\varphi_{xx}\|_{\mathcal{H}} - \nu p^*\|w_x\|_{\mathcal{H}}\|\varphi_x\|_{\mathcal{H}}\\
&\quad - \nu p^*\|v_x\|_{\mathcal{H}}\|\varphi_x\|_{\mathcal{H}} - aR^2\|v_x\|_{\mathcal{H}}\|\varphi_x\|_{\mathcal{H}}\\
&\ge \frac{1}{2}\gamma\|\varphi\|_{\mathcal{W}}^2 - C\left(\|w\|_{\mathcal{W}}^2 + \|v\|_{\mathcal{W}}^2\right).
\end{aligned}$$

Collecting these estimates and rearranging the constants we obtain

$$\langle A_{\rm reg}(y), y\rangle \ge \frac{1}{4}\gamma\|\varphi\|_{\mathcal{W}}^2 - C\left(\|w\|_{\mathcal{W}}^2 + \|v\|_{\mathcal{W}}^2\right).$$

Thus,

$$\begin{aligned}
&\langle \lambda Dy + A_{\rm reg}(y), y\rangle\\
&\ge \lambda(\|w\|_{\mathcal{W}}^2 + \|v\|_{\mathcal{W}}^2) + (\lambda\|\varphi\|_{\mathcal{H}}^2 + \frac{1}{4}\gamma\|\varphi\|_{\mathcal{W}}^2) - C\left(\|w\|_{\mathcal{W}}^2 + \|v\|_{\mathcal{W}}^2\right).
\end{aligned}$$

It follows that when λ is sufficiently large,

$$\lim_{\|y\|_{\mathcal{Y}}\to\infty} \frac{\langle(\lambda Dy + A_{\rm reg})(y), y\rangle}{\|y\|_{\mathcal{Y}}} = \infty.$$

Hence, the operator $\lambda D + A_{\rm reg}$ is coercive. □

We prove, next, that the operator $A_{\rm reg}$ is pseudomonotone on a smaller space, which is sufficient for our purposes. Let

$$\mathcal{Z} = \{y \in \mathcal{Y};\ y' \in \mathcal{Y}'\}$$

be a Banach space with the product norm $\|y\|_{\mathcal{Z}} = \|y\|_{\mathcal{Y}} + \|y'\|_{\mathcal{Y}'}$. Then, we consider the operator as $A_{\rm reg} : \mathcal{Z} \to \mathcal{Z}'$. We have the following result.

Lemma 5.3. *The operator $A_{\rm reg} : \mathcal{Z} \to \mathcal{Z}'$ is pseudomonotone.*

Proof. We have

$$A(w, v, \varphi) = kK(v) + \gamma K(\varphi) + aK_{NR}(w, v) - \nu p' B(w) - \nu p B(v).$$

Since $\mathcal{Z}$ is a reflexive Banach space, it suffices to show that

$$K_{NR} : \mathcal{Z} \to \mathcal{Z}'$$

is a weak to norm continuous, where K_{NR} is trivially restricted to to $\mathcal{Z}$. We observe that the sum of the other operators in $A_{\rm reg}$ is linear, bounded and monotone, thus pseudomonotone.

Let $\{y_m\}$ be a sequence which converges weakly in $\mathcal{Z}$ to y. Then, the sequence $\{y_m = (w_m, v_m, \varphi_m)\}$ converges weakly in $\mathcal{Y}$ to $y = (w, v, \varphi)$.

Since $\{y_m\}$ is bounded in $\mathcal{Z}$, it follows from the theorems of Simon [10] or Aubin [1], that each one of the sequences $\{\varphi_m\}, \{v_m\}$ and $\{w_m\}$ is relatively compact in $L^2(0, T; H^1(0, 1))$. Thus, there exist subsequences $\{\varphi_{m_j}\}, \{v_{m_j}\}$ and $\{w_{m_j}\}$ which converge strongly in $L^2(0, T; H^1(0, 1))$ and almost everywhere in $[0, 1] \times [0, T]$ to φ, v and w, respectively. Then,

$$\Psi_R^2(w_{m_j x}) v_{m_j x} \to \Psi_R^2(w_x) v_x,$$

strongly in $\mathcal{H} = L^2(0, T; L^2(0, 1)$ and a.e. in $[0, 1] \times [0, T]$. Thus, for each $\psi \in \mathcal{Y}$, we have

$$\begin{aligned} &|\langle K_{NR}(w_{m_j}, v_{m_j}) - K_{NR}(w, v), \psi\rangle| \\ &\le C\|\Psi_R^2(w_{m_j x}) v_{m_j x} - \Psi_R^2(w_x) v_x\|_{\mathcal{H}} \|\psi\|_{\mathcal{Y}}, \end{aligned}$$

and, as $m_j \to \infty$,

$$\|K_{NR}(w_{m_j}, v_{m_j}) - K_{NR}(w, v)\|_{\mathcal{Y}'} \le C\|\Psi_R^2(w_{m_j x}) v_{m_j x} - \Psi_R^2(w_x) v_x\|_{\mathcal{H}} \to 0.$$

Thus, K_{NR} is weak to norm continuous and, since the sum of the other terms is linear, bounded and monotone, we conclude that $A_{\rm reg}$ is pseudomonotone. □

Since the conditions (1), (2) and (3) are satisfied, the abstract existence theorems in [6, 7] provide a solution to Problem P_{VzR} for each sufficiently large R. This completes the proof of the existence part of Theorem 3.2. The uniqueness is shown by using the Gronwall inequality. However, we skip it here since it will be used below in a similar manner.

6. Proof of Theorem 3.3

We prove the theorem by showing the following result, and then prove the uniqueness of the solution.

Proposition 6.1. *Let R be sufficiently large. Then, there exists $T^* > 0$ such that the solution (w_R, v_R, z_R) of Problem P_{VzR} satisfies $\Psi_R(w_x) = w_x$ a.e. on $[0,1] \times [0,T^*)$.*

Proof. It follows from Theorem 3.2 that $w = w_R, v = w'_R \in L^2(0,T : W)$, then [8, Lemma 1.2] asserts that $w \in C([0,T]; W)$. Thus, the mappings $w_x : [0,T] \to H_0^1(0,1)$, and $w_{xx} : [0,T] \to L^2(0,1)$ are continuous. Now, $w_x(x,t) = \int_0^x w_{xx}(r,t)\,dr$, and since $w_x(t) \in H_0^1(0,1)$, the Hölder inequality yields

$$|w_x(x,t)| \le \|w_x(t)\|_{L^\infty(0,1)} \le \int_0^1 |w_{xx}(r,t)|\,dr \le \|w_{xx}(t)\|_{L^2(0,1)},$$

for $0 \le t \le T$. Let $h(t) = \|w_{xx}(t)\|_{L^2(0,1)}$ then $h : [0,T] \to \mathbb{R}$ is continuous on a compact set, so it is bounded. Since $h(0) = \|w_{0xx}\|_{L^2(0,1)} \le R^* < R$, there exists $T^* \le T$ such that $h(t) \le R$ for all $t \in [0,T^*)$. It follows that

$$\|w_x(t)\|_{L^\infty(0,1)} \le R, \tag{6.1}$$

and, therefore, the truncation is inactive on the time interval $[0,T^*)$, i.e., $\Psi_R^2(w_x) = w_x^2$, and so the solution $(w_R, v_R, z_R) = (w, v, z)$ is a solution of the Problem P_{Vz} on $[0,T^*)$. □

This completes the proof of the existence of a local solution of Problem P_{Vz}.

We note in passing that it follows from the proof that if $R_1 < R_2$ then the associated times satisfy $T_1^* < T_2^*$.

We next prove the uniqueness of the solution.

Proposition 6.2. *The solution $(w,v,z) \in \mathcal{Y}$ of Problem P_{Vz} on $[0,T^*)$ is unique.*

Proof. Let $y_i = (w_i, v_i, z_i)$, for $i = 1,2$, be two solutions of the problem. We subtract (3.7) for y_1 from (3.7) for y_2, use $\psi = z_2(t) - z_1(t)$ as a test function, and for a.a. $t \in (0,T)$, obtain

$$\begin{aligned}
&\langle z_2'(t) - z_1'(t), z_2(t) - z_1(t)\rangle_W + k(v_{2xx}(t) - v_{1xx}(t), z_{2xx}(t) - z_{1xx}(t))\\
&+ \gamma\|z_{2xx}(t) - z_{1xx}(t)\|_H^2 + \nu p'(t)(w_{2xx}(t) - w_{1xx}(t), z_2(t) - z_1(t))\\
&+ \nu p(t)(v_{2xx}(t) - v_{1xx}(t),\ z_2(t) - z_1(t))\\
&= -a(w_{2x}^2(t)v_{2x}(t) - w_{1x}^2(t)v_{1x}(t), z_{2x}(t) - z_{1x}(t)),
\end{aligned}$$

where we used the facts that in view of the boundary conditions

$$(v_{2x}(t) - v_{1x}(t),\ z_{2x}(t) - z_{1x}(t)) = -(v_{2xx}(t) - v_{1xx}(t), z_2(t) - z_1(t)),$$

and similarly for the expression with w. We may write it as

$$\begin{aligned}
&\frac{d}{dt}\|z(t)\|_H^2 + k\frac{d}{dt}\|v_{xx}(t)\|_H^2 + 2\gamma\|z_{xx}(t)\|_H^2 + 2\nu p(t)(v_{xx}(t), z(t))\\
&- 2\nu p'(t)(w_{xx}(t), z(t))\\
&= -2a(w_{2x}^2(t)(v_{2x}(t) - v_{1x}(t)), z_x(t)) - 2a((w_{2x}^2(t) - w_{1x}^2(t))v_{1x}(t), z_x(t)),
\end{aligned}$$

where $z(t) = z_2(t) - z_1(t)$, $v = v_2(t) - v_1(t)$ and $w = w_2(t) - w_1(t)$.

Integrating over $0 \le \tau \le t$ and using the Hölder inequality, (3.4) and the boundedness of w_x and v_x, yields

$$\begin{aligned} &\|z(t)\|_H^2 + k\|v_{xx}(t)\|_H^2 + 2\gamma \int_0^t \|z_{xx}(\tau)\|_H^2\, d\tau \\ &\le 2\nu p^* \int_0^t \|w_{xx}(\tau)\|_H \|z(\tau)\|_H\, d\tau + 2\nu p^* \int_0^t \|v_{xx}(\tau)\|_H \|z(\tau)\|_H\, d\tau \\ &\quad + C\int_0^t \|v_x(\tau)\|_H \|z_x(\tau)\|_H\, d\tau + C\int_0^t \|w_x(\tau)\|_H\ \|z_x(\tau)\|_H\, d\tau, \end{aligned} \tag{6.2}$$

where C is a positive number which is independent of z, v or w, and we used the fact that $w(0) = v(0) = 0$, and also (6.1). Using the Cauchy's inequality with ϵ on the right-hand side leads to

$$\begin{aligned} &\le C\int_0^t \left(\|w_{xx}(\tau)\|_H^2 + \|v_{xx}(\tau)\|_H^2 + \|z(\tau)\|_H^2\right)\, d\tau \\ &+ \frac{C}{4\epsilon}\int_0^t \left(\|w_x(\tau)\|_H^2 + \|v_x(\tau)\|_H^2\right)\, d\tau + \epsilon C\int_0^t \left(\|z_x(\tau)\|_H^2\right)\, d\tau, \end{aligned}$$

then, using the estimates on w_x, w_{xx}, v_x, and z_x in terms of v_{xx} and z_{xx}, we find that the above expression is less than or equal to

$$C_\epsilon \int_0^t \left(\|z(\tau)\|_H^2 + \|v_{xx}(\tau)\|_H^2\right)\, d\tau + \epsilon C\int_0^t \|z_{xx}(\tau)\|_H^2\, d\tau.$$

Now, we choose ϵ sufficiently small, say $\epsilon = \gamma/C$, and obtain

$$\|z(t)\|_H^2 + k\|v_{xx}(t)\|_H^2 \le C\int_0^t (\|z(\tau)\|_H^2 + \|v_{xx}(\tau)\|_H^2)\, d\tau. \tag{6.3}$$

It follows from the Gronwall inequality that $\|z(t)\|_H^2 = 0$ and $\|v_{xx}(t)\|_H^2 = 0$. Since $\|v(t)\|_W \le C\|v_{xx}(t)\|_H$ and $\|w_{xx}(t)\|_H \le C\|v_{xx}(t)\|_H$, we conclude that the solution is unique. □

This concludes the proof of Theorem 3.3, and then Theorem 3.4 follows from it and the estimate in Section 4.

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References

[1] J. P. Aubin, *Un théorème de Compacité*, CRAS **256** (1963), 5042–5045.

[2] K. T. Andrews, Y. Dumont M'B. F. M'Bengue and M. Shillor, *Analysis and simulations of a nonlinear dynamic beam*, in preparation.

[3] K. T. Andrews and M. Shillor, *Thermomechanical behaviour of a damageable beam in contact with two stops*, Applicable Analysis **85**(6-7) (2006), 845–865.

[4] D. Y. Gao, *Nonlinear Elastic Beam Theory with Application in Contact Problems and Variational Approaches*, Mechanics Research Communications **23**(1) (1996), 11–17.

[5] D. Y. Gao, *Finite deformation beam models and triality theory in dynamical post-buckling analysis*, Intl. J. Non-Linear Mechanics **35** (2000), 103–131.

[6] K. L. Kuttler, *Time dependent implicit evolution equations*, Nonlin. Anal. **10**(5) (1986), 447–463.

[7] K. L. Kuttler and M. Shillor, *Set-valued pseudomonotone maps and degenerate evolution equations*, Comm. Contemp. Math. **1**(1)(1999), 87–123.

[8] J.-L. Lions, "Quelques Methods de Resolution des Problemes aux Limites Non Lineaires," Dunod, Paris, 1969.

[9] D. L. Russel and L. W. White, *A nonlinear elastic beam system with inelastic contact constraints*, Appl. Math. Optim. **46** (2002), 291–312.

[10] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura. Appl. **146** (1987), 65–96.

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