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EXISTENCE OF SOLUTIONS TO FIRST-ORDER SINGULAR AND NONSINGULAR INITIAL VALUE PROBLEMS

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ABSTRACT. Under barrier strip type arguments we investigate the existence of global solutions to the initial value problem $x' = f(t, x, x')$, $x(0) = A$, where the scalar function $f(t, x, p)$ may be singular at $t = 0$.

1. INTRODUCTION

Results presented in Kelevedjiev O'Regan [12] show the solvability of the singular initial-value problem (IVP)

$$x' = f(t, x, x'), \quad x(0) = A, \tag{1.1}$$

where the function f may be unbounded when $t \to 0^-$. In this paper we give existence results for problem (1.1) under less restrictive assumptions which alow f to be unbounded when $t \to 0$; i.e., here f may be unbounded for t tending to 0 from both sides. In fact, we consider the nonsingular problem (1.1) with $f : D_t \times D_x \times D_p \to \mathbb{R}$ continuous on a suitable subset of $D_t \times D_x \times D_p$ containing $(0, A)$ and the singular problem (1.1) with $f(t, x, p)$ discontinuous for $(t, x, p) \in S$ and defined at least for $(t, x, p) \in (D_t \times D_x \times D_p) \setminus S$, where $D_t, D_x, D_p \subseteq \mathbb{R}$ may be bounded, and $S = \{0\} \times \mathsf{X} \times \mathsf{P}$ for some sets $\mathsf{X} \subseteq D_x$ and $\mathsf{P} \subseteq D_p$.

Singular and nonsingular IVPs for the equation $x' = f(t, x)$ have been discussed extensively in the literature; see, for example, [2, 3, 4, 5, 6, 7, 8, 9, 11, 14]. Singular IVPs of the form (1.1) have been received very little attention; we mention only [1, 12].

This paper is divided into three main sections. For the sake of completeness, in Section 2 we state the Topological transversality theorem [10]. In Section 3 we discus the nonsingular problem (1.1). Obtain a new existence result applying the approach [10]. Moreover, we again use the barrier strips technique initiated in [13]. In Section 4 we use the obtained existence result for the nonsingular problem (1.1) to study the solvability of the singular problem (1.1).

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2. Topological preliminaries

Let X be a metric space, and Y be a convex subset of a Banach space E. We say that the homotopy $\{H_\lambda : X \to Y\}$, $0 \le \lambda \le 1$, is compact if the map $H(x,\lambda) : X \times [0,1] \to Y$ given by $H(x,\lambda) \equiv H_\lambda(x)$ for $(x,\lambda) \in X \times [0,1]$ is compact.

Let $U \subset Y$ be open in Y, ∂U be the boundary of U in Y, and $\overline{U} = \partial U \cup U$. The compact map $F : \overline{U} \to Y$ is called admissible if it is fixed point free on ∂U. We denote the set of all such maps by $\mathbf{L}_{\partial U}(\overline{U}, Y)$.

Definition 2.1 ([10, Chapter I, Def. 2.1])**.** The map F in $\mathbf{L}_{\partial U}(\overline{U}, Y)$ is inessential if there is a fixed point free compact map $G : \overline{U} \to Y$ such that $G|\partial U = F|\partial U$. The map F in $\mathbf{L}_{\partial U}(\overline{U}, Y)$ which is not inessential is called essential.

Theorem 2.2 ([10, Chapter I, Theorem 2.2])**.** *Let $p \in U$ be arbitrary and $F \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ be the constant map $F(x) = p$ for $x \in \overline{U}$. Then F is essential.*

Proof. Let $G : \overline{U} \to Y$ be a compact map such that $G|\partial U = F|\partial U$. Define the map $H : Y \to Y$ by

$$H(x) = \begin{cases} p & \text{for } x \in Y \setminus \overline{U}, \\ G(x) & \text{for } x \in \overline{U}. \end{cases}$$

Clearly $H : Y \to Y$ is a compact map. By Shauder fixed point theorem, H has a fixed point $x_0 \in Y$; i. e., $H(x_0) = x_0$. By definition of H we have $x_0 \in U$. Thus, $G(x_0) = x_0$ since H equals G on U. So every compact map from $\overline{U}$ into Y which agrees with F on ∂U has a fixed point. That is, F is essential. □

Definition 2.3 ([10, Chapter I, Def. 2.3])**.** The maps $F, G \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ are called homotopic ($F \sim G$) if there is a compact homotopy $H_\lambda : \overline{U} \to Y$, such that H_λ is admissible for each $\lambda \in [0,1]$ and $G = H_0$, $F = H_1$.

Lemma 2.4 ([10, Chapter I, Theorem 2.4])**.** *The map $F \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ is inessential if and only if it is homotopic to a fixed point free map.*

Proof. Let F be inessential and $G : \overline{U} \to Y$ be a compact fixed point free map such that $G|\partial U = F|\partial U$. Then the homotopy $H_\lambda : \overline{U} \to Y$, defined by

$$H_\lambda(x) = \lambda F(x) + (1-\lambda) G(x), \quad \lambda \in [0,1],$$

is compact, admissible and such that $G = H_0$, $F = H_1$.

Now let $H_0 : \overline{U} \to Y$ be a compact fixed point free map, and $H_\lambda : \overline{U} \to Y$ be an admissible homotopy joining H_0 and F. To show that $H_\lambda, \lambda \in [0,1]$, is an inessential map consider the map $H : \overline{U} \times [0,1] \to Y$ such that $H(x,\lambda) \equiv H_\lambda(x)$ for each $x \in \overline{U}$ and $\lambda \in [0,1]$ and define the set $B \subset \overline{U}$ by

$$B = \{x \in \overline{U} : H_\lambda(x) \equiv H(x,\lambda) = x \text{ for some } \lambda \in [0,1]\}.$$

If B is empty, then $H_1 = F$ has no fixed point which means that F is inessential. So we may assume that B is non-empty. In addition B is closed and such that $B \cap \partial U = \emptyset$ since $H_\lambda, \lambda \in [0,1]$, is an admissible map. Now consider the Urysohn function $\theta : \overline{U} \to [0,1]$ with

$$\theta(x) = 1 \text{ for } x \in \partial U \quad \text{and} \quad \theta(x) = 0 \text{ for } x \in B$$

and define the homotopy $H^*_\lambda : \overline{U} \to Y, \lambda \in [0,1]$, by

$$H^*_\lambda = H(x, \theta(x)\lambda) \quad \text{for } (x,\lambda) \in \overline{U} \times [0,1].$$

It easy to see that $H^*_\lambda : \overline{U} \to Y$ is inessential. In particular $H_1 = F$ is inessential, too. The proof is complete. □

Lemma 2.4 leads to the Topological transversality theorem:

Theorem 2.5 ([10, Chapter I, Theorem 2.6]). *Let Y be a convex subset of a Banach space E, and $U \subset Y$ be open. Suppose that*

(i) *$F, G : \overline{U} \to Y$ are compact maps.*
(ii) *$G \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ is essential.*
(iii) *$H_\lambda(x), \lambda \in [0,1]$, is a compact homotopy joining F and G; i.e., $H_0(x) = G(x)$, $H_1(x) = F(x)$.*
(iv) *$H_\lambda(x), \lambda \in [0,1]$, is fixed point free on ∂U.*

Then $H_\lambda, \lambda \in [0,1]$, has a least one fixed point $x_0 \in U$, and in particular there is a $x_0 \in U$ such that $x_0 = F(x_0)$.

3. Nonsingular problem

Consider the problem

$$x' = f(t, x, x'), \quad x(a) = A, \tag{3.1}$$

where $f : D_t \times D_x \times D_p \to R$, and the sets $D_t, D_x, D_p \subseteq \mathbb{R}$ may be bounded. Assume that:

(R1) There are constants $T > a$, $Q > 0$, L_i, F_i, $i = 1, 2$, and a sufficiently small $\tau > 0$ such that $[a, T] \subseteq D_t$, $L_2 - \tau \geq L_1 \geq \max\{0, A\}$, $F_2 + \tau \leq F_1 \leq \min\{0, A\}$, $[F_2, L_2] \subseteq D_x$, $[h - \tau, H + \tau] \subseteq D_p$ for $h = -Q - L_1$ and $H = Q - F_1$,

$$\begin{gathered} f(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in [a,T] \times [L_1, L_2] \times D_p^+ \quad \text{where } D_p^+ = D_p \cap (0, \infty), \\ f(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in [a,T] \times [F_2, F_1] \times D_p^- \quad \text{where } D_p^- = D_p \cap (-\infty, 0), \\ pf(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in [a,T] \times [F_1 - \tau, L_1 + \tau] \times (D_Q^- \cup D_Q^+), \\ \text{where } D_Q^- = \{p \in D_p : p < -Q\} \text{ and } D_Q^+ = \{p \in D_p : p > Q\}. \end{gathered} \tag{3.2}$$

Remark. The sets D_p^-, D_p^+, D_Q^- and D_Q^+ are not empty because $h - \tau < h = -Q - L_1 < -Q < 0$, $H + \tau > H = Q - F_1 > Q > 0$ and $[h - \tau, H + \tau] \subseteq D_p$.

(R2) $f(t,x,p)$ and $f_p(t,x,p)$ are continuous for $(t,x,p) \in \Omega_\tau = [a,T] \times [F_1 - \tau, L_1 + \tau] \times [h - \tau, H + \tau]$ and for some $\varepsilon > 0$

$$f_p(t,x,p) \leq 1 - \varepsilon \quad \text{for } (t,x,p) \in \Omega_\tau,$$

where T, F_1, L_1, h, H and τ are as in (R1).

Now for $\lambda \in [0,1]$ construct the family of IVPs

$$x' + (1 - \lambda)x = \lambda f(t, x, x' + (1 - \lambda)x), \quad x(a) = A. \tag{3.3}$$

Note that (3.3) with $\lambda = 1$ is problem (1.1), and that when $\lambda = 0$, this problem has a unique solution $x(t) = Ae^{a-t}$, $t \in \mathbb{R}$.

For the proof of the main result of this section we need the following auxiliary result.

Lemma 3.1 ([12, Lemma 3.1]). *Let* (R1) *hold and* $x(t) \in C^1[a,T]$ *be a solution to* (3.3) *with* $\lambda \in [0,1]$. *Then*

$$F_1 \le x(t) \le L_1 \quad \text{and} \quad -Q - L_1 \le x'(t) \le Q - F_1 \quad \text{for } t \in [a,T].$$

We will omit the proof of the above lemma. Note only that (3.2) yields

$$-Q \le x'(t) + (1-\lambda)x(t) \le Q \quad \text{for } \lambda \in [0,1] \text{ and } t \in [a,T], \tag{3.4}$$

which together with the obtained bounds for $x(t)$ gives the bounds for $x'(t)$.

Lemma 3.2. *Let* (R1) *and* (R2) *hold. Then there exists a function* $\Phi(\lambda,t,x)$ *continuous for* $(\lambda,t,x) \in [0,1] \times [a,T] \times [F_1 - \tau, L_1 + \tau]$ *and such that:*

(i) *The family*

$$x' + (1-\lambda)x = \Phi(\lambda,t,x), \quad x(a) = A,$$

and family (3.3) *are equivalent.*

(ii) $\Phi(0,t,x) = 0$ *for* $(t,x) \in [a,T] \times [F_1 - \tau, L_1 + \tau]$.

Proof. (i) Consider the function

$$G(\lambda,t,x,p) = \lambda f(t,x,p) - p \quad \text{for } (\lambda,t,x,p) \in [0,1] \times \Omega_\tau.$$

Since $h - \tau < -Q$ and $H + \tau > Q$, (3.2) implies

$$f(t,x,h-\tau) \ge 0, \quad f(t,x,H+\tau) \le 0 \quad \text{for } (t,x) \in [a,T] \times [F_1 - \tau, L_1 + \tau],$$

which together with the definition of the function G yields

$$G(\lambda,t,x,h-\tau)\, G(\lambda,t,x,H+\tau) < 0, \quad (\lambda,t,x) \in [0,1]\times[a,T]\times[F_1-\tau, L_1+\tau]. \tag{3.5}$$

In addition, $G(\lambda,t,x,p)$ and

$$G_p(\lambda,t,x,p) = \lambda f_p(t,x,p) - 1 \tag{3.6}$$

are continuous for $(\lambda,t,x,p) \in [0,1] \times \Omega_\tau$ because $f(t,x,p)$ and $f_p(t,x,p)$ are continuous for $(t,x,p) \in \Omega_\tau$. Besides, from $f_p(t,x,p) \le 1 - \varepsilon$ for $(t,x,p) \in \Omega_\tau$ we have

$$G_p(\lambda,t,x,p) \le \lambda(1-\varepsilon) - 1 \le \max\{-\varepsilon, -1\} \quad \text{for } (\lambda,t,x,p) \in [0,1] \times \Omega_\tau. \tag{3.7}$$

Using (3.5), (3.6) and (3.7) we conclude that the equation

$$G(\lambda,t,x,p) = 0, \quad (\lambda,t,x,p) \in [0,1] \times \Omega_\tau$$

defines a unique function $\Phi(\lambda,t,x)$ continuous for $(\lambda,t,x) \in [0,1] \times [a,T] \times [F_1 - \tau, L_1 + \tau]$ and such that

$$G(\lambda,t,x,\Phi(\lambda,t,x)) = 0 \quad \text{for } (\lambda,t,x) \in [0,1] \times [a,T] \times [F_1 - \tau, L_1 + \tau];$$

i.e., $p = \Phi(\lambda,t,x)$ for $(\lambda,t,x) \in [0,1] \times [a,T] \times [F_1 - \tau, L_1 + \tau]$.

Now write the differential equation (3.3) as

$$\lambda f(t,x,x' + (1-\lambda)x) - (x' + (1-\lambda)x) = 0$$

and use that for $\lambda \in [0,1]$ and $t \in [a,T]$,

$$x(t) \in [F_1, L_1] \subset [F_1 - \tau, L_1 + \tau],$$

by lemma 3.1, and

$$x'(t) + (1-\lambda)x(t) \in [-Q, Q] \subset [h - \tau, H + \tau],$$

according to (3.4), to conclude that the first part of the assertion is true.

(ii) It follows immediately from $G(0,t,x,0)=0$ for $(t,x)\in \times[a,T]\times[F_1-\tau,L_1+\tau]$. □

We will only sketch the proof of the following result since it is similar to the proof of [12, Theorem 2.3].

Theorem 3.3. *Let* (R1) *and* (R2) *hold. Then the nonsingular IVP* (1.1) *has at least one solution in* $C^1[a,T]$.

Proof. Consider the family of IVPs

$$x'+(1-\lambda)x=\Phi(\lambda,t,x),\ x(a)=A, \tag{3.8}$$

where Φ is the function from Lemma 3.2, define the maps

$$j: C^1_I[a,T]\to C[a,T] \quad \text{by} \quad jx=x,$$
$$V_\lambda: C^1_I[a,T]\to C[a,T] \quad \text{by} \quad V_\lambda x=x'+(1-\lambda)x, \lambda\in[0,1],$$
$$\Phi_\lambda: C[a,T]\to C[a,T] \quad \text{by} \quad (\Phi_\lambda x)(t)=\Phi(\lambda,t,x(t)),\ t\in[a,T],\ \lambda\in[0,1],$$

where $C^1_I[a,T]=\{x(t)\in C^1[a,T]: x(a)=A\}$, and introduce the set

$$U=\big\{x\in C^1_I[a,T]: F_1-\tau<x<L_1+\tau,\ h-\tau<x'<H+\tau\big\}.$$

Next, define the compact homotopy

$$H:\overline{U}\times[0,1]\to C^1_I[a,T] \quad \text{by} \quad H(x,\lambda)\equiv H_\lambda(x)\equiv V_\lambda^{-1}\Phi_\lambda j(x).$$

By Lemma 3.1, the $C^1[a,T]$-solutions to the family (3.3) do not belong to ∂U. This means, according to (i) of Lemma 3.2, that the family (3.8) has no solutions in ∂U. Consequently, the homotopy is admissible because its fixed points are solutions to (3.8). Besides, from (ii) of Lemma 3.2 it follows $(\Phi_0 x)(t)=0$ for each $x\in U$. Then for each $x\in U$ we have

$$H_0(x)=V_0^{-1}\Phi_0 j(x)=V_0^{-1}(0)=Ae^{a-t}$$

where Ae^{a-t} is the unique solution to the problem

$$x'+x=0, \quad x(a)=A.$$

According to Theorem 2.2 the constant map $H_0=Ae^{a-t}$ is essential. Then, by Theorem 2.5, H_1 has a fixed point in U. This means that problem (3.8) with $\lambda=1$ has at least one solution $x(t)\in C^1[a,T]$. Finally, use Lemma 3.2 to see that $x(t)$ is also a solution to problem (3.3) with $\lambda=1$ which coincides with problem (1.1). □

The following result is known, but we state it for completeness. We will need it in Section 4.

Lemma 3.4. *Suppose that there are constants* m_i, M_i, $i=0,1$, *such that:*

(i) $f(t,x,p)$ *is continuously differentiable for* $(t,x,p)\in[a,T]\times[m_0,M_0]\times[m_1,M_1]$.
(ii) $1-f_p(t,x,p)\neq 0$ *for* $(t,x,p)\in[a,T]\times[m_0,M_0]\times[m_1,M_1]$.
(iii) $x(t)\in C^1[a,T]$ *is a solution to the IVP* (1.1) *satisfying the bounds*

$$m_0\le x(t)\le M_0,\quad m_1\le x'(t)\le M_1 \quad \text{for } t\in[a,T].$$

Then $x''(t)$ *exists and is continuous on* $[a,T]$ *and*

$$x''(t)=\frac{f_t(t,x(t),x'(t))+x'(t)f_x(t,x(t),x'(t))}{1-f_p(t,x(t),x'(t))}$$

for $t\in[a,T]$.

Proof. In view of (i) and (iii) for $t,\ t+h \in [a,T]$ we can work out the identity

$$
\begin{aligned}
&f(t,x,x') - f(t,x,x') + f(t_h,x,x') - f(t_h,x,x') + f(t_h,x_h,x') \\
&- f(t_h,x_h,x') + f(t_h,x_h,x'_h) - f(t_h,x_h,x'_h) + x' - x' + x'_h - x'_h = 0,
\end{aligned}
$$

where $t_h = t+h$, $x_h = x(t+h)$ and $x'_h = x'(t+h)$. Using that $x(t)$ is a solution to (1.1) we obtain

$$
\begin{aligned}
&f(t_h,x,x') - f(t,x,x') + f(t_h,x_h,x') - f(t_h,x,x') \\
&+ f(t_h,x_h,x'_h) - f(t_h,x_h,x') + x' - x'_h = 0
\end{aligned}
$$

and apply the mean value theorem to get

$$
\begin{aligned}
&\big(1 - f_p(t_h,x_h,x' + \theta_p(x'_h - x'))\big)(x'_h - x') \\
&= f_t(t+\theta_t h,x,x')h + f_x(t_h,x+\theta_x(x_h - x),x')(x_h - x),
\end{aligned}
$$

for some $\theta_t,\theta_x,\theta_p \in (0,1)$. Dividing by $\big(1 - f_p(t_h,x_h,x'+\theta_p(x'_h - x'))\big)h$, (ii) allows us to obtain

$$
\begin{aligned}
&\lim_{h\to 0} \frac{x'(t+h) - x'(t)}{h} \\
&= \lim_{h\to 0} \Big(f_t(t+\theta_t h, x(t), x'(t)) \\
&\quad + f_x\big(t+h, x(t) + \theta_x(x(t+h) - x(t)), x'(t)\big) \frac{x(t+h) - x(t)}{h} \Big) \\
&\quad \div \Big(1 - f_p(t+h, x(t+h), x' + \theta_p(x'(t+h) - x'(t))) \Big),
\end{aligned}
$$

from where the lemma follows. □

4. Singular problem

Consider problem (1.1) for

$f(t,x,p)$ is discontinuous for $(t,x,p) \in S$ and is defined at least for $(t,x,p) \in (D_t \times D_x \times D_p) \setminus S$, where $D_t, D_x, D_p \subseteq R$, $S = \{0\} \times \mathsf{X} \times \mathsf{P}$, $\mathsf{X} \subseteq D_x$ and $\mathsf{P} \subseteq D_p$. (4.1)

which allows f to be unbounded at $t = 0$.

In this section we assume the following:

(S1) There exist constants $T,\ Q > 0,\ L_i, F_i,\ i = 1,2$, and a sufficiently small $\tau > 0$ such that $(0,T] \subseteq D_t$, $L_2 - \tau \geq L_1 \geq \max\{0,A\}$, $F_2 + \tau \leq F_1 \leq \min\{0,A\}$, $[F_2,L_2] \subseteq D_x$, $[h-\tau, H+\tau] \subseteq D_p$ for $h = -Q - L_1$ and $H = Q - F_1$,

$$
\begin{gathered}
f(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in (0,T] \times [L_1,L_2] \times D_p^+, \\
f(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in (0,T] \times [F_2,F_1] \times D_p^-, \\
pf(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in (0,T] \times [F_1 - \tau, L_1 + \tau] \times (D_Q^- \cup D_Q^+),
\end{gathered}
$$

where the sets $D_p^-, D_p^+, D_Q^-, D_Q^+$ are as in (R1).

(S2) $f(t,x,p)$ and $f_p(t,x,p)$ are continuous for (t,x,p) in $(0,T] \times [F_1 - \tau, L_1 + \tau] \times [h-\tau, H+\tau]$, and for some $\varepsilon > 0$,

$$
f_p(t,x,p) \leq 1 - \varepsilon \quad \text{for } (t,x,p) \in (0,T] \times [F_1 - \tau, L_1 + \tau] \times [h-\tau, H+\tau], \tag{4.2}
$$

where the constants T, F_1, L_1, h, H, τ are as in (S1).

(S3) $f_t(t,x,p)$ and $f_x(t,x,p)$ are continuous for $(t,x,p) \in (0,T]\times[F_1,L_1]\times[h,H]$, where T, F_1, L_1, h, H, τ are as in (S1).

Note, in [12] the condition (4.2) has the form

$$f_p(t,x,p) \le -K_p < 0 \quad \text{for } (t,x,p) \in (0,T] \times [F_1-\tau, L_1+\tau] \times [h-\tau, H+\tau]$$

where K_p is a positive constant. Besides, in contrast to [12], here we do not need the assumption

$$\Big|\frac{f_t(t,x,p)+pf_x(t,x,p)}{1-f_p(t,x,p)}\Big| \le M, \quad (t,x,p) \in (0,T] \times [F_1,L_1] \times [h,H],$$

for some constant M.

Now we are ready to prove the main result of this paper. It guarantees solutions to the problem (1.1) in the case (4.1).

Theorem 4.1. *Let* (S1), (S2), (S3) *hold. Then the singular initial-value problem* (1.1) *has at least one solution in* $C[0,T] \cap C^1(0,T]$.

Proof. For $n \in N_T = \{n \in \mathbb{N} : n^{-1} < T\}$ consider the family of IVP's

$$x' = f(t,x,x'), \quad x(n^{-1}) = A. \tag{4.3}$$

It satisfies (R1) and (R2) with $a = n^{-1}$ for each $n \in N_T$. By Theorem 3.3, (4.3) has a solution $x_n(t) \in C^1[n^{-1},T]$ for each $n \in N_T$; i.e., the sequence $\{x_n\}$, $n \in N_T$, of $C^1[n^{-1},T]$-solutions to (4.3) exists.

Now, we take a sequence $\{\theta_n\}$, $n \in \mathbb{N}$, such that $\theta_n \in (0,T)$, $\theta_{n+1} < \theta_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} \theta_n = 0$.

It is clear, $\{x_n\} \subset C^1[\theta_1,T]$ for $n \in N_1 = \{n \in N_T : n^{-1} < \theta_1\}$. In addition, by Lemma 3.1, we have the bounds

$$F_1 \le x_n(t) \le L_1, \quad h \le x_n'(t) \le H \quad \text{for } t \in [\theta_1,T],$$

independent of n. On the other hand, $f(t,x,p)$ is continuously differentiable for $(t,x,p) \in [\theta_1,T] \times [F_1,L_1] \times [h,H]$ and

$$1 - f_p(t,x,p) \ge \varepsilon > 0 \quad \text{for } (t,x,p) \in [\theta_1,T] \times [F_1,L_1] \times [h,H].$$

The hypotheses of Lemma 3.4 are satisfied. Consequently, $x_n''(t)$ exists for each $n \in N_1$ and is continuous on $[\theta_1,T]$ and

$$x_n''(t) = \frac{f_t(t,x_n(t),x_n'(t)) + x_n'(t) f_x(t,x_n(t),x_n'(t))}{1 - f_p(t,x_n(t),x_n'(t))} \quad \text{for } t \in [\theta_1,T],\ n \in N_1.$$

The a priori bounds for $x_n(t)$ and $x_n'(t)$ on $[\theta_1,T]$ alow us to conclude that there is a constant C_1, independent of n, such that

$$|x_n''(t)| \le C_1, \quad \in [\theta_1,T],\ n \in N_1.$$

Applying the Arzela-Ascoli theorem we extract a subsequence $\{x_{n_1}\}$, $n_1 \in N_1$, such that the sequences $\{x_{n_1}^{(i)}\}$, $i = 0,1$, are uniformly convergent on $[\theta_1,T]$ and if

$$\lim_{n_1\to\infty} x_{n_1}(t) = x_{\theta_1}(t), \quad \text{then } x_{\theta_1}(t) \in C^1[\theta_1,T] \quad \text{and} \quad \lim_{n_1\to\infty} x_{n_1}'(t) = x_{\theta_1}'(t).$$

It is clear that $x_{\theta_1}(t)$ is a solution to the differential equation $x' = f(t,x,x')$ on $t \in [\theta_1,T]$. Besides, integrating from n_1^{-1} to t, $t \in (n_1^{-1},T]$, the inequalities $h \le x_{n_1}'(t) \le H$ we get

$$ht - hn_1^{-1} + A \le x_{n_1}(t) \le Ht - Hn_1^{-1} + A \quad \text{for } t \in [n_1^{-1},T],\ n_1 \in N_1,$$

which yields

$$ht + A \le x_{\theta_1}(t) \le Ht + A \quad \text{for } t \in [\theta_1, T].$$

Now we consider the sequence $\{x_{n_1}\}$ for $n_1 \in N_2 = \{n \in N_T : n^{-1} < \theta_2\}$. In a similar way we extract a subsequence $\{x_{n_2}\}$, $n_2 \in N_2$, converges uniformly on $[\theta_2, T]$ to a function $x_{\theta_2}(t)$ which is a $C^1[\theta_2, T]$-solution to the differential equation $x' = f(t, x, x')$ on $[\theta_2, T]$,

$$ht + A \le x_{\theta_2}(t) \le Ht + A \quad \text{for } t \in [\theta_2, T]$$

and $x_{\theta_2}(t) = x_{\theta_1}(t)$ for $t \in [\theta_1, T]$.

Continuing this process, for $\theta_i \to 0$, we establish a function $x(t) \in C^1(0, T]$ which is a solution to the differential equation $x' = f(t, x, x')$ on $(0, T]$,

$$ht + A \le x(t) \le Ht + A \quad \text{for } t \in (0, T] \tag{4.4}$$

and $x(t) \equiv x_{\theta_i}(t)$ for $t \in [\theta_i, T]$, $i \in \mathbb{N}$. Also (4.4) gives $x(0) = A$ and $x(t) \in C[0, T]$. Consequently, $x(t)$ is a $C[0, T] \cap C^1(0, T]$-solution to the singular IVP (1.1). $\square$

Example. Consider the initial-value problem

$$(0.5 - x - \sqrt[3]{x'})e^{1/t} - 2x' = 0, \quad x(0) = 1.$$

Write this equation as

$$x' = (0.5 - x - \sqrt[3]{x'})e^{1/t} - x'$$

and fix $T > 0$. Then

$$f(t, x, p) = (0.5 - x - \sqrt[3]{p})e^{1/t} - p < 0 \quad \text{for } (0, T] \times [2, 4] \times (0, \infty),$$
$$f(t, x, p) = (0.5 - x - \sqrt[3]{p})e^{1/t} - p > 0 \quad \text{for } (0, T] \times [-3, -1] \times (-\infty, 0).$$

In addition, we have

$$f(t, x, p) = (0.5 - x - \sqrt[3]{p})e^{1/t} - p > 0 \quad \text{for } (0, T] \times [-1.5, 2.5] \times (-\infty, -10),$$
$$f(t, x, p) = (0.5 - x - \sqrt[3]{p})e^{1/t} - p < 0 \quad \text{for } (0, T] \times [-1.5, 2.5] \times (10, \infty).$$

Consequently, (S1) holds for $Q = 10$, $F_2 = -3$, $F_1 = -1$, $L_1 = 2$, $L_2 = 4$ and $\tau = 0.5$. Moreover, $h = -Q - L_1 = -12$ and $H = Q - F_1 = 11$. Condition (S2) also holds because

$$f(t, x, p) \quad \text{and} \quad f_p(t, x, p) = -\frac{e^{1/t}}{3\sqrt[3]{p^2}} - 1$$

are continuous for $(t, x, p) \in (0, T] \times [-1.5, 2.5] \times [-12.5, 11.5]$ and

$$f_p(t, x, p) \le -1 \quad \text{for } (t, x, p) \in (0, T] \times [-1.5, 2.5] \times [-12.5, 11.5].$$

Finally, $f_t(t, x, p) = -t^{-2}(0.5 - x - \sqrt[3]{p})e^{1/t}$ and $f_x(t, x, p) = -e^{1/t}$ are continuous for $(t, x, p) \in (0, T] \times [-1, 2] \times [-12, 11]$ which means (S3) holds.

According to Theorem 4.1, the problem under consideration has at least one solution in $C[0, T] \cap C^1(0, T]$.

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