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INTEGRAL REPRESENTATION OF SOLUTIONS TO BOUNDARY-VALUE PROBLEMS ON THE HALF-LINE FOR LINEAR ODES WITH SINGULARITY OF THE FIRST KIND

YULIA HORISHNA, IGOR PARASYUK, LYUDMYLA PROTSAK

Abstract. We study the existence of solutions to a non-homogeneous system of linear ODEs which has the pole of first order at $x = 0$; these solutions should vanish at infinity and be continuously differentiable on $[0, \infty)$. The resonant case where the corresponding homogeneous problem has nontrivial solutions is of great interest to us. Under the conditions that the homogeneous system is exponentially dichotomic on $[1, \infty)$ and the residue of system's operator at $x = 0$ does not have eigenvalues with real part 1, we construct the so-called generalized Green function. We also establish conditions under which the main non-homogeneous problem can be reduced to the Noetherian problem with nonzero index.

1. Introduction

In the space $\mathbb{R}^n$ endowed with a scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$, we consider the linear singular system

$$y' = \Big(\frac{A}{x} + B(x)\Big)y + \frac{a}{x} + f(x). \tag{1.1}$$

Here A is a linear operator in $\mathrm{Hom}(\mathbb{R}^n)$, a is a constant in $\mathbb{R}^n$, $B(\cdot) : [0, \infty) \to \mathrm{Hom}(\mathbb{R}^n)$ and $f(\cdot) : [0, \infty) \to \mathbb{R}^n$ are continuous bounded mappings for which there exists a constant $M > 0$ such that $\|B(x)\| \le M$ and $\|f(x)\| \le M$ for all $x \in [0, \infty)$. (The norm of a linear operator in $\mathbb{R}^n$ is considered to be concordant with the norm in $\mathbb{R}^n$.)

We seek a solution $y(x)$ of the system (1.1) which satisfies the following two conditions:

$$y(\cdot) \in C^1\big([0, \infty) \to \mathbb{R}^n\big), \quad y(+\infty) = 0. \tag{1.2}$$

The stated problem belongs to the class of singular problems on account of both having a singularity at the point $x = 0$ and unboundedness of the interval where the independent variable is defined. The problems of such a kind often arise when constructing and investigating solutions of various equations of mathematical physics. Majority of papers devoted to study of such problems deal with second and

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higher order equations (see e.g. [1, 2, 3, 4, 8, 9, 10, 13, 14, 16, 18, 17, 19]). Despite the fact that corresponding bibliography amounts to several hundreds of titles, we failed to find a ready-made procedure for establishing existence conditions and integral representation of solutions to the problem (1.1)–(1.2). The necessity of such representation naturally arises when solving the problem about perturbations of solutions to singular non-linear boundary value problems on the semi-axis [21, 22].

While considering the above problem, we did not exclude the so-called resonance case when the corresponding homogeneous problem has non-trivial solutions. In this connection results of papers [5, 6, 7, 11, 20, 23, 25, 24] should be mentioned, which are devoted to the problem of existence of solutions to linear non-homogeneous systems bounded on the entire axis, in particular, extension of Fredholm and Noether theory over such systems. It should be noted that in the papers [2, 4, 13] the authors find quite general sufficient conditions for boundary value problems on finite interval with nonintegrable singularities to have the Fredholm property with index zero.

The present paper is organized as follows. The section 2 contains an auxiliary result about the structure of a fundamental operator of a linear homogeneous system with continuous (however non-analytic) coefficients on the interval $(0, x_0)$ and singular point of the first kind at $x = 0$. In section 3, we describe additional conditions imposed on the linear homogeneous system, and classify its solutions in accordance with their asymptotical behavior when $x \to +0$ and $x \to +\infty$. In section 4, the existence criterion for the solution to a boundary value problem with homogeneous boundary conditions is established and the Green function for this problem is constructed. Finally, in section 5, the main result is stated — the theorem about existence and integral representation of solutions to the problem (1.1)–(1.2).

2. Structure of the fundamental operator of linear systems near a singular point of the first kind

Consider the linear homogeneous system associated with (1.1):

$$y' = \Big(\frac{A}{x} + B(x)\Big)y. \tag{2.1}$$

In the analytical theory of differential equations, the structure of the fundamental operator of the system (2.1) is completely investigated under the assumption that the mapping $B(\cdot)$ is holomorphic in the neighborhood of the singular point $x = 0$ (see e.g. [15]). In the case where $B(\cdot)$ is continuous only, the following proposition which is a simple modification of the result stated in [7, p. 275] holds.

Proposition 2.1. *There exist numbers $x_0 \in (0, \infty)$, $K > 0$ and $r > 0$ such that the fundamental operator of the system* (2.1) *admits the representation in the form*

$$Y(x) = (E + U(x))\, x^A, \quad x \in (0, x_0], \tag{2.2}$$

where $E \in \mathrm{Hom}(\mathbb{R}^n)$ is a unit operator, and the mapping $U(\cdot) \in C^1\big((0, \infty) \to \mathrm{Hom}(\mathbb{R}^n)\big)$ satisfies the estimate

$$\|U(x)\| \le Kx|\ln x|^r, \quad x \in (0, x_0].$$

Proof. The mapping $Y(\cdot) : (0, x_0] \to \mathrm{Hom}(\mathbb{R}^n)$ defined by (2.2) is a fundamental operator of the system (2.1) if $U(x)$ satisfies the equation

$$U' = \frac{1}{x}(AU - UA) + B(x)(E + U), \quad x \in (0, x_0].$$

After the substitution $x = e^{-t}$ we obtain the following equation for the operator $V(t) := U(e^{-t})$:

$$\dot{V} = VA - AV - e^{-t}B(e^{-t})\,(E+V)\,. \tag{2.3}$$

Thus we are to find the solution to this equation which satisfies the inequality

$$\|V(t)\| \le Kt^r e^{-t}, \quad t_0 \in [t_0,\infty)$$

for certain value of $t_0 > 0$.

Equation (2.3) can be identified in $\mathbb{R}^{n^2}$ with the system of the form

$$\dot{v} = \mathcal{A}v + e^{-t}\big(H(t)v + h(t)\big), \tag{2.4}$$

where $\mathcal{A} \in \mathrm{Hom}(\mathbb{R}^{n^2})$ is a constant operator, and the mappings $H(\cdot) \in C\big([t_0,\infty) \to \mathrm{Hom}(\mathbb{R}^{n^2})\big)$ and $h(\cdot) \in C([t_0,\infty) \to \mathbb{R}^{n^2})$ satisfy the inequalities $\|H(t)\| \le M$, $\|h(t)\| \le M$ for $t \in [t_0,\infty)$.

Now the required result can be obtained as an obvious consequence of two lemmas stated below. □

Lemma 2.2. *Let $\mathcal{A} \in \mathrm{Hom}(\mathbb{R}^N))$. Then there exists a mapping $G_{\mathcal{A}}(\cdot) \in C^\infty\big(\mathbb{R} \to \mathrm{Hom}(\mathbb{R}^N)\big)$ such that for any function $\mathfrak{f}(t) \in C([t_0,\infty) \to \mathbb{R}^N)$ satisfying the estimate*

$$\|\mathfrak{f}(t)\| \le M_{\mathfrak{f}}e^{-t}, \quad t \in [t_0,\infty),$$

with some constant $M_{\mathfrak{f}} > 0$, the system

$$\dot{y} = \mathcal{A}y + \mathfrak{f}(t) \tag{2.5}$$

possesses a bounded on the semi-axis $[t_0,\infty)$ solution of the form

$$y(t) = \int_{t_0}^{\infty} G_{\mathcal{A}}(t-s)\mathfrak{f}(s)ds.$$

This solution satisfies the inequality

$$\|y(t)\| \le C_{\mathcal{A}}M_{\mathfrak{f}}e^{-t}\left(1 + (t-t_0)^r\right), \tag{2.6}$$

where $C_{\mathcal{A}}$ is a positive constant depending on $\mathcal{A}$ only, and r is the maximum dimension of Jordan blocks corresponding to eigenvalues with the real part equal to -1 in the normal form matrix of the operator $\mathcal{A}$.

If, in addition, $\mathfrak{f}(t) = o(e^{-t})$ as $t \to \infty$, then the solution $y(t)$ has the property $y(t) = o\big(e^{-t}t^r\big)$ as $t \to \infty$.

Proof. We give the proof of the first part of the Proposition for the case where $r \ge 1$. Note that there exist three projectors $\mathcal{P}_i : \mathbb{R}^N \to \mathbb{R}^N$, $i = \overline{1,3}$, such that $\mathcal{P}_i\mathcal{P}_k = 0, i \ne k$, $\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 = E$, and for some constants $K_{\mathcal{A}} > 0$, $\gamma_1 > -1$, $\gamma_2 < -1$ the following inequalities hold

$$\|e^{\mathcal{A}\tau}\mathcal{P}_1\| \le K_{\mathcal{A}}e^{\gamma_1\tau}, \quad \tau \le 0,$$
$$\|e^{\mathcal{A}\tau}\mathcal{P}_2\| \le K_{\mathcal{A}}\left(1 + \tau^{r-1}\right)e^{-\tau}, \quad \tau \ge 0,$$
$$\|e^{\mathcal{A}\tau}\mathcal{P}_3\| \le K_{\mathcal{A}}e^{\gamma_2\tau}, \quad \tau \ge 0.$$

Now we define the function

$$G_{\mathcal{A}}(\tau) = \begin{cases} -e^{\mathcal{A}\tau}\mathcal{P}_1, & \tau \le 0, \\ e^{\mathcal{A}\tau}(\mathcal{P}_2 + \mathcal{P}_3), & \tau > 0. \end{cases}$$

The function

$$\begin{aligned} y(t) &:= \int_{t_0}^{\infty} G_{\mathcal{A}}(t-s)\mathfrak{f}(s)ds \\ &\equiv \int_{t_0}^{t} e^{\mathcal{A}(t-s)}\mathcal{P}_2\mathfrak{f}(s)ds + \int_{t_0}^{t} e^{\mathcal{A}(t-s)}\mathcal{P}_3\mathfrak{f}(s)ds - \int_{t}^{\infty} e^{\mathcal{A}(t-s)}\mathcal{P}_1\mathfrak{f}(s)ds \end{aligned}$$

is well defined and there exists a constant $C_{\mathcal{A}} > 0$ dependent on the operator $\mathcal{A}$ only such that

$$\begin{aligned} \|y(t)\| &\le K_{\mathcal{A}} M_{\mathfrak{f}} \Big(\int_{t_0}^{t} \big(1 + (t-s)^{r-1}\big) e^{-(t-s)} e^{-s} ds \\ &\quad + \int_{t_0}^{t} e^{\gamma_2 (t-s)} e^{-s} ds + \int_{t}^{\infty} e^{\gamma_1 (t-s)} e^{-s} ds \Big) \\ &\le K_{\mathcal{A}} M_{\mathfrak{f}} e^{-t} \Big((t-t_0) + \frac{(t-t_0)^r}{r} + \frac{1 - e^{-(|\gamma_2|-1)(t-t_0)}}{|\gamma_2| - 1} + \frac{1}{\gamma_1 + 1} \Big) \\ &\le C_{\mathcal{A}} M_{\mathfrak{f}} e^{-t} \big(1 + (t-t_0)^r\big). \end{aligned}$$

Therefore, for $y(t)$ the inequality (2.6) holds. One can easily make sure by the direct check that this function is in fact the solution to the system (2.5).

Now let $\mathfrak{f}(t) = o(e^{-t})$ as $t \to \infty$. Then for an arbitrary $\epsilon > 0$ one can choose $T(\epsilon) > t_0$ in such a way that $\|\mathfrak{f}(t)\| \le \epsilon e^{-t}$ for $t \ge T(\epsilon)$. Represent the solution $y(t)$ in the form

$$y(t) = \int_{t_0}^{T(\epsilon)} G_{\mathcal{A}}(t-s)\mathfrak{f}(s)\,ds + \int_{T(\epsilon)}^{\infty} G_{\mathcal{A}}(t-s)\mathfrak{f}(s)\,ds.$$

In accordance with what has been proved above, the norm of the second addend does not exceed $C_{\mathcal{A}} \epsilon e^{-t}(1 + (t - T(\epsilon))^r)$ for any $t \ge T(\epsilon)$. For the first addend, when $t \ge T(\epsilon)$ we have:

$$\int_{t_0}^{T(\epsilon)} G_{\mathcal{A}}(t-s)\mathfrak{f}(s)\,ds = \int_{t_0}^{T(\epsilon)} e^{\mathcal{A}(t-s)}\mathcal{P}_2\mathfrak{f}(s)ds + \int_{t_0}^{T(\epsilon)} e^{\mathcal{A}(t-s)}\mathcal{P}_3\mathfrak{f}(s)ds.$$

If $r = 0$, then $\mathcal{P}_2 = 0$, and

$$\Big\| \int_{t_0}^{T(\epsilon)} G_{\mathcal{A}}(t-s)\mathfrak{f}(s)\,ds \Big\| = O\big(e^{\gamma_2 t}\big) = o\big(e^{-t}\big), \quad t \to \infty.$$

If $r > 0$, then

$$\Big\| \int_{t_0}^{T(\epsilon)} G_{\mathcal{A}}(t-s)\mathfrak{f}(s)\,ds \Big\| = O\big(e^{-t}\big) = o\big(e^{-t} t^r\big), \quad t \to \infty.$$

□

Lemma 2.3. *Assume that $H(\cdot) \in C\big([t_0,\infty) \to \mathrm{Hom}(\mathbb{R}^N)\big)$, $h(\cdot) \in C\big([t_0,\infty) \to \mathbb{R}^N)\big)$, and that there exist constants $M > 0$, $m > 0$ such that $\|H(t)\| \le M$, $\|h(t)\| \le m$ for any $t \ge t_0$. Let $C_{\mathcal{A}}$ and r be the numbers defined in Lemma 2.2. If the inequalities*

$$t_0 > r, \quad q := 2C_{\mathcal{A}} M t_0^r e^{-t_0} < 1, \tag{2.7}$$

hold true, then the system (2.4) has a solution $v(t)$ such that

$$\|v(t)\| \le \frac{2C_{\mathcal{A}} m}{1-q} t^r e^{-t}, \quad t \ge t_0.$$

If, in addition, $h(t) \to 0$ as $t \to \infty$, then $v(t) = o(t^r e^{-t})$ as $t \to \infty$.

Proof. In view of the Lemma 2.2, we are going to find the solution to the system (2.4) satisfying the integral equation

$$v(t) = \int_{t_0}^{\infty} G_{\mathcal{A}}(t-s)e^{-s}\big(H(s)v(s) + h(s)\big)ds. \tag{2.8}$$

Denote

$$\mathcal{G}[v(\cdot)](t) := \int_{t_0}^{\infty} G_{\mathcal{A}}(t-s)e^{-s}\big(H(s)v(s) + h(s)\big)ds$$

and define the space of functions

$$\mathcal{M}_{t_0,C} := \{v(t) \in C([t_0,\infty) \to \mathbb{R}^N) : \|v(t)\| \le Ct^r e^{-t},\, t \ge t_0\}.$$

Let us show that if (2.7) holds, then it is possible to choose the constant $C > 0$ in such a way that $\mathcal{G} : \mathcal{M}_{t_0,C} \to \mathcal{M}_{t_0,C}$ and this mapping is a contraction in the uniform metric.

Then Lemma 2.2 implies

$$\begin{aligned}\|\mathcal{G}[v(\cdot)](t)\| &\le C_{\mathcal{A}}\big(M \sup_{t\ge t_0}\big(Ct^r e^{-t}\big) + m\big)e^{-t}\big(1 + (t-t_0)^r\big)\\ &\le 2C_{\mathcal{A}}\big(MCt_0^r e^{-t_0} + m\big)t^r e^{-t}, \quad t_0 > r,\end{aligned}$$

for any function $v(t) \in \mathcal{M}_{t_0,C}$. Besides, when $t_0 > r$, for any $v(t)$, $u(t) \in \mathcal{M}_{t_0,C}$ we obtain:

$$\begin{aligned}\|\mathcal{G}[v(\cdot) - u(\cdot)](t)\| &\le C_{\mathcal{A}}Me^{-t}\big(1 + (t-t_0)^r\big) \sup_{t\ge t_0} \|v(t) - u(t)\|\\ &\le 2C_{\mathcal{A}}Mt_0^r e^{-t_0} \sup_{t\ge t_0} \|v(t) - u(t)\| = q \sup_{t\ge t_0} \|v(t) - u(t)\|.\end{aligned}$$

Since $q < 1$, it is clear that $\mathcal{G}$ is a contraction mapping on $\mathcal{M}_{t_0,C}$, once the following inequality holds

$$2C_{\mathcal{A}}(MCt_0^r e^{-t_0} + m) \le C.$$

Hence, by setting

$$C := \frac{2C_{\mathcal{A}}m}{1-q}$$

we guarantee the existence of a unique solution $v(t) \in \mathcal{M}_{t_0,C}$ to the equation (2.8).

Now, suppose in addition that $h(t) \to 0$ as $t \to \infty$. Since the solution $v(t)$ can be represented in the form

$$v(t) = \int_{t_0}^{\infty} G_{\mathcal{A}}(t-s)\mathfrak{f}(s)ds$$

where $\mathfrak{f}(t) = e^{-t}(H(t)v(t) + h(t)) = o(e^{-t})$, $t \to \infty$, then in accordance with the Lemma 2.2 we obtain: $v(t) = o(t^r e^{-t})$ as $t \to \infty$. □

3. Additional conditions for the linear homogeneous system

Hereafter we assume that for the linear homogeneous system (2.1) the conditions (A), (B) described below hold. These conditions concern the local properties of the system in neighborhoods of the points $x = 0$ and $x = +\infty$.

(A) The characteristic polynomial of the operator A has no roots with real part equal to 1;

(B) the system (2.1) is exponentially dichotomic on the semi-axis $[x_0, \infty)$ for some (and therefore, for any) positive x_0.

Let $y(x, y_0)$ be a solution to the system (2.1) satisfying the initial condition $y(x_0, y_0) = y_0$. For the sake of generality we assume that the characteristic polynomial of the operator A has roots with real parts both less and greater than 1 and the system (2.1) has both bounded and unbounded solutions on the half-line $[x_0, \infty)$.

Under the conditions (A) and (B) there exist subspaces $\mathbb{V}_+$ and $\mathbb{U}_-$ with the following properties:

(1) There exists $\alpha > 0$ such that for any subspace $\mathbb{V}_-$ which is a direct supplement of $\mathbb{V}_+$ to $\mathbb{R}^n$ one can choose a constant $c_0 > 0$ in such a way that

$$\|y(x, y_0)\| \le c_0 \big(\frac{x}{s}\big)^{1+\alpha} \|y(s, y_0)\|, \quad 0 < x \le s \le x_0, \quad \text{if} y_0 \in \mathbb{V}_+; \tag{3.1}$$

$$\|y(x, y_0)\| \le c_0 \big(\frac{x}{s}\big)^{1-\alpha} \|y(s, y_0)\|, \quad 0 < s \le x \le x_0, \quad \text{if} y_0 \in \mathbb{V}_-. \tag{3.2}$$

(This property results from the Proposition 2.1 and the condition (A).)

(2) There exists a constant $\gamma > 0$ such that for any subspace $\mathbb{U}_+$ which is a direct supplement of $\mathbb{U}_-$ to $\mathbb{R}^n$ one can choose a constant $c_* > 0$ in such a way that

$$\|y(x, y_0)\| \le c_* e^{-\gamma(x-s)} \|y(s, y_0)\|, \quad x_0 \le s \le x, \quad \text{if} y_0 \in \mathbb{U}_- \tag{3.3}$$

$$\|y(x, y_0)\| \le c_* e^{\gamma(x-s)} \|y(s, y_0)\|, \quad x_0 \le x \le s, \quad \text{if} y_0 \in \mathbb{U}_+. \tag{3.4}$$

(See [7, Remark 3.4 p. 235])

If the subspace $\ker A$ is non-trivial, then there exists a subspace $\mathbb{V}^0_-$ isomorphic to the subspace $\ker A$ and having the next property:

(3) For each $y_* \in \mathbb{V}^0_-$ there exists a unique vector $\zeta \in \ker A$ such that

$$y(x, y_*) = \big(E + \Theta(x)\big)\zeta, \quad x \to +0, \tag{3.5}$$

where $\Theta(\cdot) \in C^1\big([0, x_0] \to \operatorname{Hom}(\mathbb{R}^n)\big)$ and $\Theta(x) = x(E-A)^{-1}B(0) + o(x),\ x \to +0$. At the same time $\mathbb{V}^0_- \cap \mathbb{V}_+ = \{0\}$ and the subspace $\mathbb{V}_+ \oplus \mathbb{V}^0_-$ coincides with the subspace of initial values (for $x = x_0$) of continuously differentiable on $[0, \infty)$ solutions to the system (2.1). (See the corollary from the Proposition 4.1 which is stated in section 5.)

Now the space $\mathbb{R}^n$ can be represented as the direct sum of six subspaces $\mathbb{L}_1, \dots, \mathbb{L}_6$ defined in the following way:

(1) $\mathbb{L}_1 := \mathbb{U}_- \cap \mathbb{V}_+$;
(2) $\mathbb{L}_2$ is a direct supplement of the subspace $\mathbb{L}_1$ to $\mathbb{U}_- \cap (\mathbb{V}_+ \oplus \mathbb{V}^0_-)$, so that
$$\mathbb{L}_1 \oplus \mathbb{L}_2 = \mathbb{U}_- \cap (\mathbb{V}_+ \oplus \mathbb{V}^0_-);$$
(3) $\mathbb{L}_3$ is a direct supplement of the subspace $\mathbb{U}_- \cap (\mathbb{V}_+ \oplus \mathbb{V}^0_-)$ to $\mathbb{U}_-$, so that
$$\mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3 = \mathbb{U}_-;$$
(4) $\mathbb{L}_4$ is a direct supplement of the subspace $\mathbb{L}_1 = \mathbb{U}_- \cap \mathbb{V}_+$ to $\mathbb{V}_+$, so that
$$\mathbb{V}_+ = \mathbb{L}_1 \oplus \mathbb{L}_4;$$
(5) $\mathbb{L}_5$ is a direct supplement of the subspace $(\mathbb{U}_- \cap (\mathbb{V}_+ \oplus \mathbb{V}^0_-)) \oplus \mathbb{L}_4$ to $\mathbb{V}_+ \oplus \mathbb{V}^0_-$, so that
$$\mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_4 \oplus \mathbb{L}_5 = \mathbb{V}_+ \oplus \mathbb{V}^0_-,$$

and taking into account the equalities $(\mathbb{L}_1 \oplus \mathbb{L}_4) \cap \mathbb{V}_-^0 = \{0\}$ and $\dim \mathbb{L}_2 + \dim \mathbb{L}_5 = \dim \mathbb{V}_-^0$ we choose $\mathbb{L}_5 \subset \mathbb{V}_-^0$;

(6) $\mathbb{L}_6$ is a direct supplement of the subspace $\mathbb{L}_1 \oplus \dots \oplus \mathbb{L}_5 = \mathbb{U}_- \oplus \mathbb{L}_4 \oplus \mathbb{L}_5$ to $\mathbb{R}^n$.

If the two subspaces $\mathbb{U}_+$ and $\mathbb{V}_-$, which are direct supplements of the subspaces $\mathbb{U}_-$ and $\mathbb{V}_+$ respectively, are defined by the equalities

$$\mathbb{U}_+ := \mathbb{L}_4 \oplus \mathbb{L}_5 \oplus \mathbb{L}_6, \quad \mathbb{V}_- := \mathbb{L}_2 \oplus \mathbb{L}_3 \oplus \mathbb{L}_5 \oplus \mathbb{L}_6,$$

then above assumptions allow us to distinguish six types of solutions to (2.1). Namely: if $y_0 \in \mathbb{L}_1$, then the solution $y(x, y_0)$ satisfies the inequalities (3.1) and (3.3); the solution for which $y_0 \in \mathbb{L}_2$ fulfills the inequality (3.3), and there exists unique $y_* \in \mathbb{V}_-^0$ such that

$$\|y(x, y_0) - y(x, y_*)\| = o(x), \quad x \to 0;$$

the solution for which $y_0 \in \mathbb{L}_3$ satisfies the inequalities (3.2) and (3.3), besides, for this solution the derivative $y'(+0; y_0)$ does not exist; for the solution with $y_0 \in \mathbb{L}_4$ the inequalities (3.1) and (3.4) hold true; the solution having initial value from $\mathbb{L}_5$ fulfills the inequality (3.4) and there is a unique $\zeta \in \ker A$ for which (3.5) is valid; finally, if $y_0 \in \mathbb{L}_6$, then the solution $y(x, y_0)$ satisfies inequalities (3.2) and (3.4), and for such a solution the derivative $y'(+0; y_0)$ does not exist.

Let $E = P_1 + \dots + P_6$ be the decomposition of the unit operator into the sum of mutually disjunctive projectors generated by the decomposition $\mathbb{R}^n = \mathbb{L}_1 \oplus \dots \oplus \mathbb{L}_6$. Define the operators:

$$Q_+ := P_1 + P_4, \quad Q_- := P_2 + P_3 + P_5 + P_6,$$
$$P_- := P_1 + P_2 + P_3, \quad P_+ := P_4 + P_5 + P_6.$$

It is clear that the projectors Q_+, Q_- correspond to the decomposition $\mathbb{R}^n = \mathbb{V}_+ \oplus \mathbb{V}_-$, while P_-, P_+ correspond to the decomposition $\mathbb{R}^n = \mathbb{U}_- \oplus \mathbb{U}_+$, and there exist constants $C_0 > 0$ and $C_* > 0$ such that for the normalized at the point x_0 evolution operator $Y(x; x_0)$ of the system (2.1) the following estimates are valid:

$$\|Y(x; x_0) Q_+ Y^{-1}(s; x_0)\| \le C_0 \Big(\frac{x}{s}\Big)^{1+\alpha}, \quad 0 < x \le s \le x_0, \tag{3.6}$$

$$\|Y(x; x_0) Q_- Y^{-1}(s; x_0)\| \le C_0 \Big(\frac{x}{s}\Big)^{1-\alpha}, \quad 0 < s \le x \le x_0, \tag{3.7}$$

and

$$\|Y(x; x_0) P_- Y^{-1}(s; x_0)\| \le C_* e^{-\gamma(x-s)}, \quad x_0 \le s \le x, \tag{3.8}$$

$$\|Y(x; x_0) P_+ Y^{-1}(s; x_0)\| \le C_* e^{-\gamma(s-x)}, \quad x_0 \le x \le s. \tag{3.9}$$

4. Generalized Green function for boundary-value problems with homogeneous boundary conditions

Consider the boundary-value problem

$$y' = \Big(\frac{A}{x} + B(x)\Big) y + g(x), \tag{4.1}$$

$$y(\cdot) \in C^1([0, \infty) \to \mathbb{R}^n), \quad y(+0) = 0, \quad y(+\infty) = 0, \tag{4.2}$$

in the case of function $g(\cdot) \in C([0, \infty) \to \mathbb{R}^n)$ vanishing at infinity: $g(x) \to 0$ when $x \to +\infty$. Let $m := \sup_{x \in [0, \infty)} \|g(x)\|$.

First, we prove that any element of $\ker A$ can be brought into correspondence with at least one solution which is continuously differentiable over $[0,\infty)$.

Proposition 4.1. *Under condition* (A)*, for any $\zeta \in \ker A$ there exists a solution to the system* (4.1) *of the form*

$$y_\zeta(x) = \zeta + \zeta_1 x + o(x), \quad x \to +0, \tag{4.3}$$

where $\zeta_1 := (E-A)^{-1}\big(B(0)\zeta + g(0)\big)$. Conversely, every continuously differentiable on $[0,\infty)$ solution to the system (4.1) *can be represented in the form* (4.3)*.*

Proof. The change of dependent variable $y = \zeta + \zeta_1 x + z$ in (4.1) leads to the system

$$z' = \Big(\frac{A}{x} + B(x)\Big)z + \tilde g(x)$$

where $\tilde g(x) = (B(x) - B(0))\zeta + g(x) - g(0) + xB(x)\zeta_1 = o(1)$, $x \to +0$. After the substitution $x = e^{-t}$ we obtain the system

$$\dot z = -\big(A + e^{-t}B\big(e^{-t}\big)\big)z - e^{-t}\tilde g\big(e^{-t}\big). \tag{4.4}$$

The value $t_0 > 0$ can be chosen sufficiently large, so that the conditions of Lemma 2 hold true for this system. In accordance with this Lemma and taking into account that the characteristic polynomial of the operator $-A$ has no roots with the real part equal to -1, there exists the solution $\tilde z(t)$ to the system (4.4) satisfying the equality

$$\tilde z(t) = -\int_{t_0}^{\infty} G_{-A}(t-s)e^{-s}\big(B\big(e^{-s}\big)\tilde z(s) + \tilde g(e^{-s})\big)\,ds,$$

and, having the property $\tilde z(t) = o(e^{-t})$ as $t \to \infty$. But in such a case the function $z(x) := \tilde z(-\ln x) = o(x)$ as $x \to 0$, generates the required solution $y(x) = \zeta + \zeta_1 x + z(x)$ of the system (4.1). The second part of this proposition is obvious. ◻

Corollary 4.2. *There exists a mapping $\Theta(\cdot) \in C^1\big([0,x_0] \to \operatorname{Hom}(\mathbb{R}^n)\big)$ of the form $\Theta(x) = x(E-A)^{-1}B(0) + o(x)$, $x \to +0$, such that for any $\zeta \in \ker A$ the function*

$$y_\zeta(x) = \big(E + \Theta(x)\big)\zeta$$

is a solution to the homogeneous system (2.1) *corresponding to the vector ζ.*

Proposition 4.3. *The family of functions defined as*

$$\begin{aligned}\bar y_v(x) = Y(x;x_0)v &+ \int_0^x Y(x;x_0)Q_- Y^{-1}(s;x_0)g(s)ds \\ &+ \int_{x_0}^x Y(x;x_0)Q_+ Y^{-1}(s;x_0)g(s)ds,\end{aligned} \tag{4.5}$$

where v is an arbitrary vector in $\mathbb{V}_+ \oplus \mathbb{V}_-^0$, determines all solutions to the system (4.1) *of the class $C^1([0,\infty) \to \mathbb{R}^n)$. Each of such solutions satisfies the condition $\bar y_v(+0) = 0$ if and only if $v \in \mathbb{L}_1 \oplus \mathbb{L}_4 = \mathbb{V}_+$.*

Proof. In view of the estimates (3.6), (3.7), for any $x \in [0, x_0)$ the integrals in the formula (4.5) satisfy

$$\begin{aligned}\Big\| \int_0^x Y(x;x_0)Q_- Y^{-1}(s;x_0)g(s)ds\Big\| &\le mC_0 x^{1-\alpha}\int_0^x s^{\alpha-1}ds = mC_0\frac{x}{\alpha};\\ \Big\|\int_x^{x_0} Y(x;x_0)Q_+ Y^{-1}(s;x_0)g(s)ds\Big\| &\le mC_0 x^{1+\alpha}\int_x^{x_0} s^{-1-\alpha}ds\\ &\le mC_0\frac{x^{1+\alpha}(x^{-\alpha}-x_0^{-\alpha})}{\alpha}\\ &\le \frac{mC_0 x}{\alpha}.\end{aligned}$$

By a direct check, one can easily verify that each function of the set (4.5) is a solution to the system (4.1). From the definition of $\mathbb{V}_+$, $\mathbb{V}_-^0$, properties of the spaces $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_4, \mathbb{L}_5$ (see p. 7) it follows that for any $v \in \mathbb{V}_+ \oplus \mathbb{V}_-^0$ there exists a limit $\lim_{x\to+0} Y(x;x_0)v =: \zeta(v) \in \ker A$, and $Y(x;x_0)v = \zeta(v) + O(x)$, $x \to 0$. Therefore, $\bar{y}_v(x) = \zeta(v) + O(x)$, $x \to 0$, and the difference $\bar{y}_v(x) - y_{\zeta(v)}(x)$, where $y_{\zeta(v)}(x)$ is the solution from the Proposition 4.1, is a solution to the system (2.1). Moreover, $\|\bar{y}_v(x) - y_{\zeta(v)}(x)\| = O(x)$, $x \to 0$. This implies that $\|\bar{y}_v(x) - y_{\zeta(v)}(x)\| = o(x)$, $x \to 0$, and thus, $\bar{y}_v(x_0) - y_{\zeta(v)}(x_0) \in \mathbb{L}_1 \oplus \mathbb{L}_4$. Taking into account the Proposition 4.3, we can conclude that $\bar{y}_v(x) \in C^1([0, x_0] \to \mathbb{R}^n)$.

Since each non-trivial solution to the system (2.1) with the initial condition $y_0 \in \mathbb{L}_2 \oplus \mathbb{L}_5$ has a non-zero limit when $x \to +0$, the equality $\bar{y}_v(+0) = 0$ is equivalent to $v \in \mathbb{L}_1 \oplus \mathbb{L}_4$. □

It is well known (see e.g. [7]) that all solutions to the system (4.1) which are bounded on the semi-axis $[x_0, \infty)$ form a family

$$\begin{aligned}\hat{y}_u(x) = Y(x;x_0)u &+ \int_{x_0}^x Y(x;x_0)P_- Y^{-1}(s;x_0)g(s)ds\\ &- \int_x^\infty Y(x;x_0)P_+ Y^{-1}(s;x_0)g(s)ds,\end{aligned}$$

where u is an arbitrary vector from $\mathbb{U}_-$.

It is also known that the following proposition holds.

Proposition 4.4. *If $g(x) \to 0$ as $\to \infty$, then $\hat{y}_u(x) \to 0$ as $x \to \infty$.*

Proof. For the sake of completeness we sketch the proof. For an arbitrary $\epsilon > 0$ let choose the value $x(\epsilon) > x_0$ in such a way that $\|g(x)\| < \epsilon$ for any $x > x(\epsilon)$. Then for $x > x(\epsilon)$ we have

$$\begin{aligned}\hat{y}_u(x) = Y(x;x_0)u &+ \int_{x_0}^{x(\epsilon)} Y(x;x_0)P_- Y^{-1}(s;x_0)g(s)\,ds\\ &+ \int_{x(\epsilon)}^x Y(x;x_0)P_- Y^{-1}(s;x_0)g(s)\,ds + \int_x^\infty Y(x;x_0)P_+ Y^{-1}(s;x_0)g(s)\,ds.\end{aligned}$$

The first addend in this expression tends to zero when $x \to \infty$, norm of each of the last two addends does not exceed $\epsilon K/\gamma$, and for the second addend it holds

$$\Big\|\int_{x_0}^{x(\epsilon)} Y(x;x_0)P_- Y^{-1}(s;x_0)g(s)\,ds\Big\| = O(e^{-\gamma x}), \quad x \to \infty.$$

□

Now to find all solutions to (4.1) which satisfy the conditions (4.2) we bind parameters $v \in \mathbb{L}_1 \oplus \mathbb{L}_4$ and $u \in \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3$ by means of the equality $\bar{y}_v(x_0) = \hat{y}_u(x_0)$, which can be rewritten in the form

$$P_- u - Q_+ v = \int_0^{x_0} Q_- Y^{-1}(s; x_0) g(s)\, ds + \int_{x_0}^{\infty} P_+ Y^{-1}(s; x_0) g(s)\, ds,$$

or, equivalently,

$$\begin{aligned}(P_1 + P_2 + P_3)u - (P_1 + P_4)v = &\int_0^{x_0} (P_2 + P_3 + P_5 + P_6) Y^{-1}(s; x_0) g(s)\, ds \\ &+ \int_{x_0}^{\infty} (P_4 + P_5 + P_6) Y^{-1}(s; x_0) g(s)\, ds.\end{aligned}$$

From this it follows that

$$P_1 u = P_1 v, \quad P_2 u = \int_0^{x_0} P_2 Y^{-1}(s; x_0) g(s)\, ds,$$
$$P_3 u = \int_0^{x_0} P_3 Y^{-1}(s; x_0) g(s)\, ds, \quad P_4 v = -\int_{x_0}^{\infty} P_4 Y^{-1}(s; x_0) g(s)\, ds,$$

and the function $g(x)$ must satisfy the additional condition

$$\int_0^{\infty} (P_5 + P_6) Y^{-1}(s; x_0) g(s)\, ds = 0. \tag{4.6}$$

Therefore, if the condition (4.6) holds, the solutions to the problem (4.1)–(4.2) can be given by the formula

$$\begin{aligned}y = Y(x; x_0)\Big(&P_1 v + \int_{x_0}^{x} P_1 Y^{-1}(s; x_0) g(s)\, ds \\ &+ \int_0^{x} (P_2 + P_3) Y^{-1}(s; x_0) g(s)\, ds - \int_x^{\infty} (P_4 + P_5 + P_6) Y^{-1}(s; x_0) g(s)\, ds\Big).\end{aligned} \tag{4.7}$$

This formula can also be rewritten as

$$\begin{aligned}y = Y(x; x_0)\Big(&P_1 v - \int_{x_0}^{\infty} P_4 Y^{-1}(s; x_0) g(s)\, ds \\ &+ \int_{x_0}^{x} Q_+ Y^{-1}(s; x_0) g(s)\, ds + \int_0^{x} Q_- Y^{-1}(s; x_0) g(s)\, ds\Big).\end{aligned}$$

Having defined the sets

$$D := \{(x, s) : 0 < x < s < x_0\} \cup \{(x, s) : x_0 \le s \le x\},$$
$$D_+ := \{(x, s) : 0 < s \le x\}, \quad D_- := \{(x, s) : 0 < x < s\},$$

and the functions

$$G_1(x, s) := \begin{cases} Y(x; x_0) P_1 Y^{-1}(s; x_0), & (x, s) \in D \cap D_+, \\ -Y(x; x_0) P_1 Y^{-1}(s; x_0), & (x, s) \in D \cap D_-, \\ 0, & (x, s) \in (D_+ \cup D_-) \setminus D, \end{cases}$$

$$G_2(x, s) := \begin{cases} Y(x; x_0)(P_2 + P_3) Y^{-1}(s; x_0), & (x, s) \in D_+, \\ -Y(x; x_0)(P_4 + P_5 + P_6) Y^{-1}(s; x_0), & (x, s) \in D_-, \end{cases}$$

$$G(x, s) := G_1(x, s) + G_2(x, s),$$

and taking into account (4.7), we get the following result.

Proposition 4.5. *There exists a solution to the boundary-value problem* (4.1)–(4.2) *if and only if the condition* (4.6) *holds, and in this case all solutions to the problem are defined by the formula*

$$y = Y(x;x_0)v + \int_0^\infty G(x,s)g(s)\,ds, \quad \forall v \in \mathbb{L}_1.$$

Now we are going to interpret the condition (4.6) in terms of solutions to the adjoint (with respect to the scalar product $\langle\cdot,\cdot\rangle$) homogeneous system

$$\eta' = -\Big(\frac{A^*}{x} + B^*(x)\Big)\eta. \tag{4.8}$$

Let $\eta(x,\eta_0)$ denote the solution to this system satisfying the initial condition $\eta(x_0,\eta_0) = \eta_0$. In what follows, without loss of generality we assume that the scalar product in $\mathbb{R}^n$ is determined in such a way that $P_j^* = P_j,\ j = 1,\dots,6$.

Let $L_1([0,\infty) \to \mathbb{R}^n)$ be the space of functions $f(\cdot) : [0,\infty) \to \mathbb{R}^n$ for which $\int_0^\infty \|f(x)\|\,dx < \infty$.

Proposition 4.6. *The solution $\eta(x,\eta_0)$ belongs to $L_1\big([0,\infty) \to \mathbb{R}^n\big)$ if and only if $\eta_0 \in \mathbb{L}_5 \oplus \mathbb{L}_6$.*

Proof. As is well known, $\big[Y^{-1}(x;x_0)\big]^*$ is a fundamental operator of the adjoint system normalized at the point x_0, and

$$\langle \eta(x,\eta_0), y(x,y_0)\rangle \equiv \langle \eta_0, y_0\rangle.$$

Let $y_0 := (P_1 + \dots + P_4)\eta_0 \neq 0$. If in addition we suppose that $Q_+y_0 \neq 0$, then in view of (3.6)

$$\|Q_+y_0\|^2 \le \|y(x,Q_+y_0)\|\|\eta(x,\eta_0)\| \le c_0(x/x_0)^{1+\alpha}\|Q_+y_0\|\|\eta(x,\eta_0)\|$$

for all $x \in (0,x_0]$, and thus $\|\eta(x,\eta_0)\| \ge \|Q_+y_0\|(x/x_0)^{-1-\alpha}/c_0$ when $x \in (0,x_0]$. This implies that $\eta(x,\eta_0) \notin L_1([0,\infty) \to \mathbb{R}^n)$.

If $Q_+y_0 = 0$ then $y_0 = (P_2 + P_3)y_0 = P_-y_0$. Hence, in view of (3.8),

$$\|P_-y_0\|^2 \le \|y(x,P_-y_0)\|\|\eta(x,\eta_0)\| \le c_*e^{-\gamma(x-x_0)}\|P_-y_0\|\|\eta(x,\eta_0)\|$$

for all $x \ge x_0$. This also implies that $\eta(x,\eta_0) \notin L_1([0,\infty) \to \mathbb{R}^n)$.

On the other hand, if $y_0 = 0$, then $\eta_0 = (P_5 + P_6)\eta_0$. Now from the inequalities (3.7) and (3.9) it follows that

$$\|\eta(x,\eta_0)\| \le \big\|\big[Y^{-1}(x;x_0)\big]^*(P_5+P_6)\big\|\|\eta_0\| \le C_0\Big(\frac{x_0}{x}\Big)^{1-\alpha}\|\eta_0\|,$$

when $x \in (0,x_0]$, and

$$\|\eta(x,\eta_0)\| \le \big\|[Y^{-1}(x;x_0)]^*(P_5+P_6)\big\|\|\eta_0\| \le C_*e^{-\gamma(x-x_0)}\|\eta_0\|$$

when $x \ge x_0$. Hence, $\eta(x,\eta_0)$ belongs to $L_1\big([0,\infty) \to \mathbb{R}^n\big)$. □

The above proposition leads to the following result.

Proposition 4.7. *Condition* (4.6) *holds if and only if the function $g(x)$ is orthogonal (in the sense of the scalar product $\langle\cdot,\cdot\rangle_{L_2} := \int_0^\infty \langle\cdot,\cdot\rangle\,dx$) to each solution of the adjoint system* (4.8) *belonging to the space $L_1([0,\infty) \to \mathbb{R}^n)$.*

Now let us show that problem (4.1)–(4.2) has a generalized Green function $\mathfrak{G}(x,s)$ defined by the following properties:

1. For any $s>0$ and $x\in[0,\infty)\setminus\{s\}$, it holds

$$\mathfrak{L}\mathfrak{G}(x,s)=-F(x;x_0)\Pi Y^{-1}(s;x_0),$$

where $\mathfrak{L}:=\frac{d}{dx}-\big(\frac{A}{x}+B(x)\big)$, $\Pi:=P_5+P_6$, and $F(\cdot,x_0)\in C\big([0,\infty)\to \mathrm{Hom}(\mathbb{R}^n)\big)$ is a bounded mapping with the "biorthonormality" property with respect to the space of solutions of the adjoint system which belong to $L_1([0,\infty)\to\mathbb{R}^n)$:

$$\int_0^\infty \Pi Y^{-1}(x,x_0)F(x,x_0)\,dx=\Pi.$$

For example, we may set

$$F(x;x_0):=\frac{\kappa^{1+\beta}x^\beta}{\Gamma(1+\beta)e^{\kappa x}}Y(x;x_0)\Pi,$$

where κ is an arbitrary number greater than γ, and $\beta>0$ is an arbitrary number with the property that all real parts of eigenvalues of the matrix A exceed $-\beta$. Obviously, $F(+0;x_0)=F(+\infty;x_0)=0$.

2. For any $x>0$, the unit jump property is valid: $\mathfrak{G}(x+0,x)-\mathfrak{G}(x-0,x)=E$.

3. The condition of orthogonality to the space of solutions to the corresponding homogeneous boundary value problem is fulfilled:

$$\int_0^\infty P_1Y^*(x;x_0)\mathfrak{G}(x,s)\,dx=0.$$

4. For any $s>0$, the boundary conditions $\mathfrak{G}(+0,s)=\mathfrak{G}(+\infty,s)=0$ are satisfied.

5. For any $g(\cdot)\in C([0,\infty)\to\mathbb{R}^n)$ satisfying (4.6), the boundedness condition holds:

$$\sup_{x\in[0,\infty)}\int_0^\infty \|\mathfrak{G}(x,s)g(s)\|\,ds<\infty$$

Observe that the operator equation $\mathfrak{L}Y=-F(x;x_0)$ has a particular solution

$$Y=N(x;x_0):=-\int_0^\infty G(x,t)F(t;x_0)\,dt,$$

which can be represented in the form

$$N(x;x_0)=Y(x;x_0)\Big(\Pi-\int_0^x\frac{\kappa^{1+\beta}t^\beta}{\Gamma(1+\beta)e^{\kappa t}}\,dt\Pi\Big).$$

(Note that generally $N(x;x_0)$ is unbounded on $(0,x_0)$, but it vanishes at infinity.)

It is easily seen that the conditions 1–4 hold for the operator

$$\mathfrak{G}(x,s):=G(x,s)+Y(x;x_0)P_1M(s;x_0)+N(x;x_0)\Pi Y^{-1}(s;x_0), \tag{4.9}$$

once we set

$$\begin{aligned}M(s;x_0):=-\Big[\int_0^\infty P_1Y^*(x;x_0)Y(x;x_0)P_1\,dx\Big|_{\mathbb{L}_1}\Big]^{-1}\\ \times\int_0^\infty P_1Y^*(x;x_0)\big(G(x,s)+N(x;x_0)\Pi Y^{-1}(s;x_0)\big)\,dx.\end{aligned}$$

To show that the condition 5 is fulfilled it remains only to verify that $M(s;x_0)$ is absolutely integrable on $[0,\infty)$. This property can be easily obtained from the next six estimates for the function

$$J(s,x;x_0) := \big\|P_1Y^*(x;x_0)(G(x,s)+N(x;x_0)\Pi Y^{-1}(s;x_0))\big\|$$

which are based on inequalities (3.6)–(3.9).

(1) Let $x<s<x_0$. In this case $G(x,s)=-Y(x;x_0)(P_1+P_4+\Pi)Y^{-1}(s;x_0)$, and therefore there exits a constant $C_1(x_0)>0$ such that

$$\begin{aligned}
J(s;x,x_0) &\le \|Y(x;x_0)P_1\|\big(\|Y(x;x_0)(P_1+P_4)Y^{-1}(s;x_0)\| \\
&\quad + (\kappa x)^{1+\beta}\|Y(x;x_0)\Pi Y^{-1}(s;x_0)\|\big) \\
&\le C_1(x_0)x^{1+\alpha}\big((x/s)^{1+\alpha}+xs^{\alpha-1}\big) \\
&\le C_1(x_0)x^{1+\alpha}(1+s^{\alpha}).
\end{aligned}$$

(2) Let $s\le x<x_0$. Now $G(x,s)=Y(x;x_0)(P_2+P_3)Y^{-1}(s;x_0)$, and there exists a constant $C_2(x_0)>0$ such that

$$\begin{aligned}
&J(s;x;x_0) \\
&\le \|Y(x;x_0)P_1\|\big(\|Y(x;x_0)Q_-Y^{-1}(s;x_0)\| + (\kappa x)^{1+\beta}\|Y(x;x_0)\Pi Y^{-1}(s;x_0)\|\big) \\
&\le C_2(x_0)x^2s^{\alpha-1}.
\end{aligned}$$

(3) Let $s<x_0\le x$. Now $G(x,s)=Y(x;x_0)(P_2+P_3)Y^{-1}(s;x_0)$, hence,

$$\begin{aligned}
&J(s;x;x_0) \\
&\le \|Y(x;x_0)P_1\|\big(\|Y(x;x_0)(P_2+P_3)Y^{-1}(s;x_0)\| + \|N(x;x_0)\Pi Y^{-1}(s;x_0)\|\big) \\
&\le C_0C_*e^{-\gamma(x-x_0)}\big(\frac{x_0}{s}\big)^{1-\alpha}\big(C_*+\sup_{x\in[x_0,\infty)}\|N(x;x_0)\|\big) \\
&\le C_3(x_0)e^{-\gamma x}s^{\alpha-1}
\end{aligned}$$

for some constant $C_3(x_0)>0$.

(4) Let $x<x_0\le s$. Since $G(x,s)=-Y(x;x_0)(P_4+\Pi)Y^{-1}(s;x_0)$, it follows that there exists a constant $C_4(x_0)>0$ such that

$$\begin{aligned}
&J(s;x;x_0) \\
&\le \|Y(x;x_0)P_1\|\big(\|Y(x;x_0)P_4Y^{-1}(s;x_0)\| + (\kappa x)^{1+\beta}\|Y(x;x_0)\Pi Y^{-1}(s;x_0)\|\big) \\
&\le C_4(x_0)x^{1+\alpha}e^{-\gamma s}.
\end{aligned}$$

(5) Let $x_0\le x<s$. In this case, we also have $G(x,s)=-Y(x;x_0)(P_4+\Pi)Y^{-1}(s;x_0)$. Hence,

$$\begin{aligned}
J(s;x;x_0) &\le \|Y(x;x_0)P_1\|\big(\|Y(x;x_0)P_4Y^{-1}(s;x_0)\| + \|Y(x;x_0)\Pi Y^{-1}(s;x_0)\|\big) \\
&\le 2C_*^2e^{-\gamma(x-x_0)}e^{-\gamma(s-x)} = C_5(x_0)e^{-\gamma s}
\end{aligned}$$

where $C_5(x_0):=2C_*^2e^{\gamma x_0}$.

(6) Finally, let $x_0\le s\le x$. Now $G(x,s)=Y(x;x_0)(P_1+P_2+P_3)Y^{-1}(s;x_0)$ and there exists a constant $C_6(x_0)>0$ such that

$$\begin{aligned}
&J(s;x;x_0) \\
&\le \|Y(x;x_0)P_1\|\big(\|Y(x;x_0)(P_1+P_2+P_3)Y^{-1}(s;x_0)\| + \|N(x;x_0)\Pi Y^{-1}(s;x_0)\|\big) \\
&\le C_6(x_0)\big(e^{-\gamma(2x-s)}+e^{-\gamma(x+s)}\big).
\end{aligned}$$

The above arguments prove the following theorem.

Theorem 4.8. *There exists a solution to the boundary-value problem* (4.1)–(4.2) *if and only if the function $g(x)$ is orthogonal (in terms of the scalar product $\langle\cdot,\cdot\rangle_{L_2} := \int_0^\infty \langle\cdot,\cdot\rangle\,dx$) to all solutions to the adjoint system* (4.8) *which belong to $L_1([0,\infty)\to\mathbb{R}^n)$. If the orthogonality condition holds, then the problem* (4.1)–(4.2) *has the family of the solutions which can be represented as the sum of two mutually orthogonal components*

$$y = Y(x;x_0)v + \int_0^\infty \mathfrak{G}(x,s)g(s)\,ds$$

where $v \in \mathbb{L}_1$ is an arbitrary vector and $\mathfrak{G}(x,s)$ is the generalized Green function defined by (4.9).

5. The main theorem

Let us turn back to the main problem of finding solutions to the system (1.1) which possess the properties (1.2). It is clear that a continuously differentiable on $[0,\infty)$ solution $y(x)$ to the system (1.1), provided that it exists, must satisfy the equality $Ay(+0)+a=0$. Thus we require the following condition to hold

(C) $a \in \operatorname{im} A$.

The orthogonal decomposition $\mathbb{R}^n = \operatorname{im} A^* \oplus \ker A$ together with the condition (C) imply the existence of a unique $\eta \in \operatorname{im} A^*$ for which $A\eta + a = 0$.

Hence, it is natural to formulate the *main boundary value problem* in the following way: *Find all $\zeta \in \ker A$ for which the boundary-value problem for the system* (1.1) *with the boundary conditions*

$$y(+0) = \eta + \zeta, \quad y(\infty) = 0,$$

is solvable in the class $C^1\big([0,\infty)\to\mathbb{R}^n\big)$, and construct an integral represent-ation of corresponding solutions. This problem is solved by the following theorem.

Theorem 5.1. *Let the system* (1.1) *satisfies the conditions* (A)–(C) *and $f(x)\to 0$ as $x\to+\infty$. Then the main boundary-value problem is solvable if and only if*

$$\int_0^\infty P_6 Y^{-1}(x;x_0)\big[f(x)+B(x)\eta\big]\,dx = 0. \tag{5.1}$$

Provided that (5.1) holds, the main boundary-value problem has the family of solutions defined by the formulae

$$y = Y(x;x_0)(v_1+v_2) + \eta + \int_0^\infty G(x,s)\big(f(s)+B(s)\eta\big)\,ds, \tag{5.2}$$

$$\zeta = (E+U(x_0))^{-1}v_2 + (E+\Theta(x_0))^{-1}w \tag{5.3}$$

where $v_1 \in \mathbb{L}_1$, $v_2 \in \mathbb{L}_2$ are arbitrary vectors, and

$$w := -\int_0^\infty P_5 Y^{-1}(x;x_0)\big[f(x)+B(x)\eta\big]\,dx. \tag{5.4}$$

There exist positive constants $K_1(\alpha,\gamma,x_0)$, $K_2(\alpha,\gamma,x_0)$ such that

$$\int_0^\infty \big\|G(x,s)\big(f(s)+B(s)\eta\big)\big\|\,ds \le K_1(\alpha,\gamma,x_0)\sup_{x\in[0,\infty)}\|f(s)+B(s)\eta\|,$$

$$\|w\| \le K_2(\alpha,\gamma,x_0)\sup_{x\in[0,\infty)}\|f(s)+B(s)\eta\|.$$

Proof. We seek the solution to the problem (1.1)–(1.2) in the form

$$y = \eta + \varphi(x) + Y(x;x_0)v + e^{-\kappa x}Y(x;x_0)w + y_0(x) \tag{5.5}$$

where $\kappa > \gamma$ is an arbitrary number, $v \in \mathbb{L}_1 \oplus \mathbb{L}_2$ is an arbitrary constant vector, $w \in \mathbb{L}_5$ is a constant vector which is to be determined, $\varphi(\cdot) \in C^1([0,\infty) \to \mathbb{R}^n)$ is an arbitrary function with the properties

$$\varphi(0) = 0, \quad \lim_{x\to\infty} \varphi(x) = -\eta, \quad \lim_{x\to\infty} \varphi'(x) = 0,$$

and $y_0(x)$ is a solution to the problem (4.1)–(4.2) with

$$g(x) := f(x) + B(x)\eta - \mathfrak{L}\varphi(x) + \kappa e^{-\kappa x}Y(x;x_0)w \quad \text{when} x > 0.$$

Observe that there exists $\lim_{x\to+0} g(x)$. In virtue of Theorem 4.8, the existence of the solution $y_0(x)$ is guaranteed by the orthogonality conditions, which can be given in the form

$$\begin{gathered}\int_0^\infty \langle [Y^{-1}(s;x_0)]^* P_5 b, f(s) + B(s)\eta - \mathfrak{L}\varphi(s)\rangle\, ds + w = 0,\\ \int_0^\infty \langle [Y^{-1}(s;x_0)]^* P_6 b, f(s) + B(s)\eta - \mathfrak{L}\varphi(s)\rangle\, ds = 0 \quad \forall b \in \mathbb{R}^n.\end{gathered}$$

Since $\langle [Y^{-1}(s;x_0)]^* P_j b, \varphi(s)\rangle\big|_{s=0}^{\infty} = 0$, $j = 5, 6$, and $\mathfrak{L}^*[Y^{-1}(s;x_0)]^* = 0$, these conditions are equivalent to (5.4), (5.1). The orthogonality conditions also imply

$$\begin{aligned} y_0(x) &:= \int_0^\infty \mathfrak{G}(x,s)g(s)\, ds\\ &= \int_0^\infty G(x,s)g(s)\, ds + Y(x;x_0)P_1 \int_0^\infty M(s;x_0)g(s)\, ds.\end{aligned}$$

The second addend is inessential owing to the presence of an arbitrary vector $v \in \mathbb{L}_1 \oplus \mathbb{L}_2$ in the formula (5.5).

Next, it is not hard to show that

$$\begin{gathered}\int_0^\infty G(x,s)\mathfrak{L}\varphi(s)\, ds = \varphi(x) - Y(x,x_0)P_1\varphi(x_0),\\ \int_0^\infty G(x,s)\kappa e^{-\kappa s}Y(s;x_0)w\, ds = -e^{-\kappa x}Y(x;x_0)w.\end{gathered}$$

Taking into account these equalities, one can rewrite the formula (5.5) in the form (5.2). Finally, in view of (2.2), (3.5), (5.5) and the equality $y_0(+0) = 0$ we obtain (5.3).

Now observe that from the definition of $\mathbb{L}_5$ it follows that the constant $C_7(x_0) := \max_{x\in[0,x_0]} \|Y(x;x_0)P_5\|$ is well defined. Let $\bar{g}(s) := f(s) + B(s)\eta$. Making use of (5.1) and of estimates similar to those which were obtained for the function $J(s,x,x_0)$ in previous section, we have:

(1) if $0 \le x \le x_0$, then

$$
\begin{aligned}
&\int_0^\infty \big\|G(x,s)\bar g(s)\big\|\,ds\\
&\le \sup_{s\in[0,\infty)}\|\bar g(s)\|\Big(\int_0^x \|Y(x;x_0)(P_2+P_3)Y^{-1}(s;x_0)\|\,ds\\
&\quad+\int_x^{x_0}\|Y(x;x_0)(P_1+P_4+P_5)Y^{-1}(s;x_0)\|\,ds\\
&\quad+\int_{x_0}^\infty\|Y(x;x_0)(P_4+P_5)Y^{-1}(s;x_0)\|\,ds\Big)\\
&\le \sup_{s\in[0,\infty)}\|\bar g(s)\|\Big(C_0\int_0^x (x/s)^{1-\alpha}\,ds + C_0\int_x^{x_0}\big((x/s)^{1+\alpha}+C_7(x_0)(x_0/s)^{1-\alpha}\big)\,ds\\
&\quad+C_*\int_{x_0}^\infty\big(C_0\cdot(x/x_0)^{1+\alpha}+C_7(x_0)\big)e^{-\gamma(s-x_0)}\,ds\Big)\\
&\le K_1(\alpha,\gamma,x_0)\sup_{s\in[0,\infty)}\|\bar g(s)\|;
\end{aligned}
$$

(2) if $0 < x_0 < x$, then

$$
\begin{aligned}
\int_0^\infty \big\|G(x,s)\bar g(s)\big\|\,ds &\le \sup_{s\in[0,\infty)}\|\bar g(s)\|\Big(\int_0^{x_0}\|Y(x;x_0)(P_2+P_3)Y^{-1}(s;x_0)\|\,ds\\
&\quad+\int_{x_0}^x\|Y(x;x_0)(P_1+P_2+P_3)Y^{-1}(s;x_0)\|\,ds\\
&\quad+\int_x^\infty\|Y(x;x_0)(P_4+P_5)Y^{-1}(s;x_0)\|\,ds\Big)\\
&\le \sup_{s\in[0,\infty)}\|\bar g(s)\|\Big(C_0C_*\int_0^{x_0}(x_0/s)^{1-\alpha}e^{-\gamma(x-x_0)}\,ds\\
&\quad+C_*\int_{x_0}^x e^{-\gamma(x-s)}\,ds + C_*\int_x^\infty e^{-\gamma(s-x)}\,ds\Big)\\
&\le K_1(\alpha,\gamma,x_0)\sup_{s\in[0,\infty)}\|\bar g(s)\|.
\end{aligned}
$$

Finally, the inequality for $\|w\|$ is easily obtained with the help of estimates from the proof of Proposition 4.6. □

Conclusions. The results obtained can be interpreted in terms of linear equations in Banach spaces in a following way. Let Y be the Banach space of continuous mappings $y(\cdot) : [0,\infty) \to \mathbb{R}^n$ such that $\lim_{x\to+\infty} y(x) = 0$, and $X \subset Y$ be the Banach space of mappings satisfying $y(0) = 0$ (these spaces are endowed with usual supremum norm). Consider the closed linear operator $\mathcal{L} : X \to Y$ defined on the dense domain $D(\mathcal{L}) = \{y(\cdot) \in X \cap C^1([0,\infty) \to \mathbb{R}^n) : \lim_{x\to+\infty} y'(x) = 0\}$ by $\mathcal{L}y(x) := y'(x) - Ay(x)/x - B(x)y(x)$. From Proposition 4.5 it follows that the range $R(\mathcal{L})$ is a closed subspace of Banach space Y. Hence, the operator $\mathcal{L}$ is normally solvable, moreover, it is both n-normal with $n(\mathcal{L}) = \dim\ker\mathcal{L} = \dim\mathbb{L}_1$ and d-normal with $d(\mathcal{L}) = \operatorname{codim} R(\mathcal{L}) = \dim(\mathbb{L}_5 + \mathbb{L}_6)$. This means that we have established conditions under which the operator $\mathcal{L}$ is a Noetherian operator with index $n(\mathcal{L}) - d(\mathcal{L})$.

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Yulia Horishna
National Taras Shevchenko University of Kyiv, Faculty of Mechanics and Mathematics, Volodymyrs'ka 64, Kyiv, 01033, Ukraine
E-mail address: yuliya_g@ukr.net

Igor Parasyuk
National Taras Shevchenko University of Kyiv, Faculty of Mechanics and Mathematics, Volodymyrs'ka 64, Kyiv, 01033, Ukraine
E-mail address: pio@mail.univ.kiev.ua

Lyudmyla Protsak
National Pedagogical Dragomanov University, Pirogova 9, Kyiv, 01601, Ukraine
E-mail address: protsak_l_v@ukr.net