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EXISTENCE OF POSITIVE SOLUTIONS FOR SEMIPOSITONE DYNAMIC SYSTEM ON TIME SCALES

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Abstract. In this paper, we study the following semipositone dynamic system on time scales

$$
\begin{gathered}
-x^{\Delta\Delta}(t) = f(t,y) + p(t), \quad t \in (0,T)_{\mathbb{T}}, \\
-y^{\Delta\Delta}(t) = g(t,x), \quad t \in (0,T)_{\mathbb{T}}, \\
x(0) = x(\sigma^2(T)) = 0, \\
\alpha y(0) - \beta y^{\Delta}(0) = \gamma y(\sigma(T)) + \delta y^{\Delta}(\sigma(T)) = 0.
\end{gathered}
$$

Using fixed point index theory, we show the existence of at least one positive solution. The interesting point is the that nonlinear term is allowed to change sign and may tend to negative infinity.

1. Introduction

The theory of dynamic equations on time scales is undergoing rapid development. This is not only because it can provide a unifying structure for the study of differential equations in the continuous case and the study of finite difference equations in the discrete case, but also because the study of time scales has led to many important applications, e.g., in the study of insect population models, neural networks, biology, heat transfer, stock market, crop harvesting and epidemic models [1, 2, 3, 7, 8].

Let $\mathbb{T}$ be a time scale (arbitrary nonempty closed subset of the real numbers $\mathbb{R}$). For each internal I of $\mathbb{R}$, we denote by $I_{\mathbb{T}} = I \bigcap \mathbb{T}$. In this paper, we are interested in the nonlinear dynamic system on a time scale $\mathbb{T}$,

$$
\begin{gathered}
-x^{\Delta\Delta}(t) = f(t,y) + p(t), \quad t \in (0,T)_{\mathbb{T}}, \\
-y^{\Delta\Delta}(t) = g(t,x), \quad t \in (0,T)_{\mathbb{T}}, \\
x(0) = x(\sigma^2(T)) = 0, \\
\alpha y(0) - \beta y^{\Delta}(0) = \gamma y(\sigma(T)) + \delta y^{\Delta}(\sigma(T)) = 0,
\end{gathered} \tag{1.1}
$$

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where $p : (0,T) \to \mathbb{R}$ is Lebesgue integrable and $f, g \in C((0,T)_{\mathbb{T}} \times \mathbb{R}^+, \mathbb{R}^+)$, $0, T \in \mathbb{T}$, $\alpha, \beta, \gamma, \delta > 0$ are constants such that $\rho = \alpha\delta + \beta\gamma + \alpha\gamma\sigma(T) > 0$.

Recently, there is much attention paid to question of positive solutions of boundary value problems on time scales, see [3, 4, 5, 9, 10, 11] and the references therein. However, to the best of our knowledge, very few works can be found for the problem when the nonlinearity can change sign.

Motivated by the main ideas in [12], the purpose of this paper is to study the existence at least one positive solution for the semipositone differential system (1.1). Different from the previous papers, in this paper the nonlinearity term $p(t)$ is allowed to be negative and may tend to infinity. Our results are new even for the special case of difference equations.

The rest of the paper is organized as follows. In section 2, we provide some lemmas which are useful later. In Section 3, we give the main result of the paper and an example is presented to demonstrate the application of our main results.

2. Preliminaries

Let $\mathbb{X} = \{x | x : [0, \sigma^2(T)]_{\mathbb{T}} \to \mathbb{R}$ is continuous$\}$ be a Banach space endowed with the norm $\|u\| = \max_{t\in[0,\sigma^2(T)]_{\mathbb{T}}} |u(t)|$.

Define $P = \{x \in \mathbb{X} : x(t) \geq 0, t \in [0, \sigma^2(T)]_{\mathbb{T}}\}$ and $K = \{x \in P : x(t) \geq q(t)\|x\|, t \in [0, \sigma^2(T)]\}$, where $q(t) = \frac{t(\sigma^2(T)-t)}{(\sigma^2(T))^2}$, it is easy to see that P and K are cones of $\mathbb{X}$ and $K \subset P$. To obtain solutions of the system (1.1), we first denote the Green's functions of the following boundary value problems:

$$\begin{gathered} -x^{\Delta\Delta}(t) = 0, \quad t \in (0,T)_{\mathbb{T}}, \\ x(0) = x(\sigma^2(T)) = 0, \end{gathered}$$

and

$$\begin{gathered} -y^{\Delta\Delta}(t) = 0, \quad t \in (0,T)_{\mathbb{T}}, \\ \alpha y(0) - \beta y^{\Delta}(0) = \gamma y(\sigma(T)) + \delta y^{\Delta}(\sigma(T)) = 0, \end{gathered}$$

by $G(t,s)$ and $H(t,s)$ respectively. From [4, 5], we know that

$$G(t,s) = \frac{1}{\sigma^2(T)} \begin{cases} t(\sigma^2(T) - \sigma(s)), & t \leq s, \\ \sigma(s)(\sigma^2(T) - t), & t \geq \sigma(s); \end{cases} \tag{2.1}$$

$$H(t,s) = \frac{1}{\rho} \begin{cases} (\beta + \alpha t)(\gamma(\sigma(T) - \sigma(s)) + \delta), & t \leq s, \\ (\beta + \alpha\sigma(s))(\gamma(\sigma(T) - t) + \delta), & t \geq \sigma(s), \end{cases} \tag{2.2}$$

Obviously,

$$0 \leq G(t,s) \leq \frac{t(\sigma^2(T) - t)}{\sigma^2(T)}, \quad 0 \leq H(t,s) \leq H(\sigma(s), s). \tag{2.3}$$

For the sake of convenience, we state the following hypotheses:

(C1) $f : (0,T)_{\mathbb{T}} \times [0, +\infty) \to [0, +\infty)$ is continuous and for any $t \in (0,T)_{\mathbb{T}}$, $y \in [0, +\infty)$, $f(t,y)$ is nondecreasing on y and satisfying $f(t,y) \leq p^*(t)h(y)$, where $p^* : (0,T)_{\mathbb{T}} \to [0, +\infty)$ and $h : (0, +\infty) \to [0, +\infty)$ are continuous, $\lim_{y\to+\infty} \frac{f(t,y)}{y} = +\infty$ uniformly for t on any close subinterval of $(0,T)_{\mathbb{T}}$.

(C2) $g:(0,T)_{\mathbb{T}}\times[0,+\infty)\to[0,+\infty)$ is continuous, there exist constants $\lambda_1\geq\lambda_2\geq 1$ such that for any $t\in[0,T]_{\mathbb{T}}, x\in[0,+\infty)$,
$$c^{\lambda_1}g(t,x)\leq g(t,cx)\leq c^{\lambda_2}g(t,x),\quad \forall 0\leq c\leq 1, \tag{2.4}$$
with $0<\int_0^{\sigma(T)}H(\sigma(t),t)g(t,1)\Delta t<+\infty$.

(C3) $p:(0,T)_{\mathbb{T}}\to(-\infty,+\infty)$ is Lebesgue integrable such that $\int_0^{\sigma(T)}p_-(t)\Delta t>0$ and
$$0<\int_0^{\sigma(T)}G(\sigma(t),t)[p^*(t)+p_+(t)]\Delta t<\frac{\sigma^2(T)\int_0^{\sigma(t)}p_-(t)\Delta t}{2(\max_{0\leq\tau\leq R}h(\tau)+1)},$$
where $p_+(t)=\max\{p(t),0\}, p_-(t)=\max\{0,-p(t)\}$, and
$$R=\Big(\int_0^{\sigma(T)}P_-(t)\Delta t+1\Big)^{\lambda_1}\int_0^{\sigma(T)}H(\sigma(t),t)g(t,1)\Delta t.$$

From $(C2)$, as in [10, Remark2.2, Lemma2.2] we obtain the following result.

Remark 2.1. If $(C2)$ is satisfied, then for $t\in[0,T]_{\mathbb{T}}$, $g(t,x)$ is increasing on x and for $(t,x)\in[0,T]_{\mathbb{T}}\times[0,+\infty)$, $c\in[1,+\infty), \lambda_1\geq\lambda_2>1$,
$$c^{\lambda_2}g(t,x)\leq g(t,cx)\leq c^{\lambda_1}g(t,x),\quad \lim_{x\to+\infty}\min_{t\in[0,T]_{\mathbb{T}}}\frac{g(t,x)}{x}=+\infty \tag{2.5}$$

For convenience, we define a functions
$$[x(t)]^*=\begin{cases}x(t), & x(t)\geq 0,\\ 0, & x(t)<0,\end{cases}$$
and
$$\omega(t)=\int_0^{\sigma(T)}G(t,s)p_-(s)\Delta s,\quad t\in[0,\sigma^2(T)]_{\mathbb{T}}.$$
By the definition of the function $\omega(t)$ and $G(t,s)\geq 0$, we have
$$0\leq\omega(t)=\int_0^{\sigma(T)}G(t,s)p_-(s)\Delta s\leq\frac{t(\sigma^2(t)-t)}{\sigma^2(T)}\int_0^{\sigma(T)}p_-(s)\Delta s<\infty,$$
thus $\omega(t)\in P$, $-\omega^{\Delta\Delta}(t)=p_-(t)$ and $\omega(0)=\omega(\sigma^2(T))=0$.

Next, we consider the approximate system
$$\begin{gathered}-x^{\Delta\Delta}(t)=f(t,y)+p_+(t),\quad t\in(0,T)_{\mathbb{T}},\\ -y^{\Delta\Delta}(t)=g(t,[x(t)-\omega(t)]^*),\quad t\in(0,T)_{\mathbb{T}},\\ x(0)=x(\sigma^2(T))=0,\\ \alpha y(0)-\beta y^{\Delta}(0)=\gamma y(\sigma(T))+\delta y^{\Delta}(\sigma(T))=0.\end{gathered}\tag{2.6}$$

It is well known that (x,y) is a solution of system (2.6) if and only if (x,y) is the solution of the nonlinear integral equation system
$$\begin{gathered}x(t)=\int_0^{\sigma(T)}G(t,s)[f(s,y(s))+p_+(s)]\Delta s,\quad t\in[0,\sigma^2(T)]_{\mathbb{T}},\\ y(t)=\int_0^{\sigma(T)}H(t,s)g(s,[x(s)-\omega(s)]^*)\Delta s.\end{gathered}\tag{2.7}$$

Obviously, the integral system (2.7) is equivalent to the nonlinear integral equation

$$x(t)=\int_0^{\sigma(T)}G(t,s)\Big[f\Big(s,\int_0^{\sigma(T)}H(s,\tau)g(\tau,[x(\tau)-\omega(\tau)]^*)\Delta\tau\Big)+p_+(s)\Big]\Delta s,\quad (2.8)$$

$t\in[0,\sigma^2(T)]_{\mathbb{T}}$.

Now, we define the operator $F:P\to\mathbb{X}$ by

$$(Fx)(t)=\int_0^{\sigma(T)}G(t,s)\Big[f\Big(s,\int_0^{\sigma(T)}H(s,\tau)g(\tau,[x(\tau)-\omega(\tau)]^*)\Delta\tau\Big)+p_+(s)\Big]\Delta s. \tag{2.9}$$

Then the existence of solutions to system (2.7) is equivalent to the existence of solution for nonlinear integral equation (2.8). Therefore, if x is a fixed point of the operator F in $\mathbb{X}$, then the system (2.7) has one solution (u,v) which can be written as

$$\begin{gathered}u(t)=x(t),\\ v(t)=\int_0^{\sigma(T)}H(t,s)g(s,[x(s)-\omega(s)]^*)\Delta s,\quad t\in[0,\sigma^2(T)]_{\mathbb{T}}.\end{gathered}\tag{2.10}$$

Lemma 2.2. *If (u,v) with $u(t)\geq\omega(t)$ for $t\in[0,\sigma^2(T)]_{\mathbb{T}}$ is a positive solution of system* (2.7)*, then $(u-\omega,v)$ is a positive solution of semipositone dynamical system* (1.1)*.*

Proof. Suppose that (u,v) is a positive solution of (2.7) with $u(t)\geq\omega(t)$ for $t\in[0,\sigma^2(T)]_{\mathbb{T}}$, then from (2.7) and the definition of $[u(t)]^*$, we have

$$\begin{gathered}-u^{\Delta\Delta}(t)=f(t,v(t))+p_+(t),\quad t\in(0,T)_{\mathbb{T}},\\ -v^{\Delta\Delta}(t)=g(t,[u(t)-\omega(t)]^*),\quad t\in(0,T)_{\mathbb{T}},\\ u(0)=u(\sigma^2(T))=0,\\ \alpha v(0)-\beta v^{\Delta}(0)=\gamma v(\sigma(T))+\delta v^{\Delta}(\sigma(T))=0.\end{gathered}\tag{2.11}$$

Set $u_1=u-\omega$, then $u_1^{\Delta\Delta}=u^{\Delta\Delta}-\omega^{\Delta\Delta}$ which implies

$$\begin{gathered}-u_1^{\Delta\Delta}(t)=f(t,v(t))+p_+(t)-p_-(t),\quad t\in(0,T)_{\mathbb{T}},\\ -v^{\Delta\Delta}(t)=g(t,u_1(t)),\quad t\in(0,T)_{\mathbb{T}},\\ u_1(0)=u_1(\sigma^2(T))=0,\\ \alpha v(0)-\beta v^{\Delta}(0)=\gamma v(\sigma(T))+\delta v^{\Delta}(\sigma(T))=0,.\end{gathered}\tag{2.12}$$

From (C3) we know $p_+(t)-p_-(t)=p(t)$, then (u_1,v) is a positive solution of (1.1). □

Lemma 2.3. *If* (C1)–(C3) *are satisfied, then $F:K\to K$ is completely continuous.*

Proof. For any $x\in K$, let $u(t)=(Fx)(t)$. By the definition of the operator F, we have $u(t)\geq 0$, and $u(0)=u(\sigma^2(T))=0$. Hence, there exists a $t_0\in[0,\sigma^2(T)]_{\mathbb{T}}$ such that $\|u\| = u(t_0)$. Since

$$\frac{G(t,s)}{G(t_0,s)} = \begin{cases} \frac{t}{t_0}, & t, t_0 \le s, \\ \frac{t(\sigma^2(T)-\sigma(s))}{\sigma(s)(\sigma^2(T)-t_0)}, & t \le s < t_0, \\ \frac{\sigma(s)(\sigma^2(T)-t)}{t_0(\sigma^2(T)-\sigma(s))}, & t_0 \le s < t, \\ \frac{\sigma^2(T)-t}{\sigma^2(T)-t_0}, & t, t_0 \ge s, \end{cases} \tag{2.13}$$

we have

$$\frac{G(t,s)}{G(t_0,s)} \ge \frac{t(\sigma^2(T)-t)}{(\sigma^2(T))^2} = q(t).$$

Thus

$$\begin{aligned} &(Fx)(t) \\ &= \int_0^{\sigma(T)} G(t,s)\Big[f\Big(s, \int_0^{\sigma(T)} H(s,\tau)g(\tau,[x(\tau)-\omega(\tau)]^*)\Delta\tau\Big) + p_+(s)\Big]\Delta s \\ &\ge q(t)\int_0^{\sigma(T)} G(t_0,s)\Big[f\Big(s, \int_0^{\sigma(T)} H(s,\tau)g(\tau,[x(\tau)-\omega(\tau)]^*)\Delta\tau\Big) + p_+(s)\Big]\Delta s \\ &\ge q(t)(Fx)(t_0) = q(t)\|Fx\|, \end{aligned}$$

and $F(K) \subseteq K$. By standard argument, we can easily obtain that the operator $F : K \to K$ is completely continuous. The proof is complete. □

For our arguments, the following fixed point index theory [6] is crucial.

Lemma 2.4. *Let $\mathbb{X}$ be a Banach space and K be a cone in $\mathbb{X}$. Assume that Ω is a bounded open subset of $\mathbb{X}$ with $\theta \in \Omega$ and let $\Phi : K \cap \overline{\Omega} \to K$ be a completely continuous operator.*

(i) *If $\Phi z \ne \lambda z$ for all $z \in K \bigcap \partial\Omega$, $\lambda \ge 1$, then $i(\Phi, K \bigcap \Omega, K) = 1$;*
(ii) *if $\Phi z \not\le z$ for all $z \in K \bigcap \partial\Omega$, then $i(\Phi, K \bigcap \Omega, K) = 0$.*

3. Main Results

Before presenting the main result, we give two lemmas which are important in establishing the existence of one positive solutions to the problem (1.1).

Lemma 3.1. *Assume* (C1)–(C3) *hold, if we let $r = \sigma^2(T)\int_0^{\sigma(T)} p_-(t)\Delta t$, $\Omega_r = \{x \in \mathbb{X} : \|x\| < r\}$, then $i(F, K \cap \Omega_r, K) = 1$.*

Proof. Suppose that there exist $\lambda_0 \ge 1$ and $x_0 \in K \cap \partial\Omega_r$ such that $Fx_0 = \lambda_0 x_0$. By $0 \le G(t,s) \le t(\sigma^2(T)-t)/\sigma^2(T)$, we have

$$0 < \omega(t) = \int_0^{\sigma(T)} G(t,s)p_-(s)\Delta s \le \frac{t(\sigma^2(T)-t)}{\sigma^2(T)}\int_0^{\sigma(T)} p_-(s)\Delta s < +\infty, \tag{3.1}$$

for $t \in [0,\sigma^2(T)]_{\mathbb{T}}$, which together with $x_0(t) \ge q(t)\|x_0\| = rq(t)$, $t \in [0,\sigma^2(T)]_{\mathbb{T}}$, yield

$$x_0(t) - \omega(t) \ge rq(t) - \frac{t(\sigma^2(T)-t)}{\sigma^2(T)}\int_0^{\sigma(T)} p_-(s)\Delta s = 0, \quad t \in [0,\sigma^2(T)]_{\mathbb{T}}. \tag{3.2}$$

Thus, from $x_0 = \frac{1}{\lambda_0} F x_0$ and (3.2), we obtain

$$\begin{gathered} -x_0^{\Delta\Delta}(t) = \frac{1}{\lambda_0}\Big[f\Big(t, \int_0^{\sigma(T)} H(t,\tau)g(\tau, x_0(\tau)-\omega(\tau))\Delta\tau\Big) + p_+(t)\Big], \quad t \in [0,T]_{\mathbb{T}}, \\ x_0(0) = x_0(\sigma^2(T)) = 0, \end{gathered} \tag{3.3}$$

which shows that there exists a $t_0 \in [0, \sigma^2(T)]_{\mathbb{T}}$ such that $x_0(t_0) = \|x_0\| = r$, $x_0^{\Delta}(t_0) \le 0$. Then

$$0 \le x_0(t) - \omega(t) \le x_0(t) \le \|x_0\| = r < r+1. \tag{3.4}$$

Let $t \in [0, t_0]_{\mathbb{T}}$, integrating the equation in (3.3) from t to t_0, we have

$$\begin{aligned} x_0^{\Delta}(t) - x_0^{\Delta}(t_0) &= \int_t^{t_0} -x_0^{\Delta\Delta}(s)\Delta s \\ &= \int_t^{t_0} \frac{1}{\lambda_0}\Big[f\Big(s, \int_0^{\sigma(T)} H(s,\tau)g(\tau, x_0(\tau)-\omega(\tau))\Delta\tau\Big) + p_+(s)\Big]\Delta s \\ &\le \int_t^{t_0} \Big[f\Big(s, \int_0^{\sigma(T)} H(s,\tau)g(\tau, x_0(\tau)-\omega(\tau))\Delta\tau\Big) + p_+(s)\Big]\Delta s \\ &\le \int_t^{t_0} \Big[p^*(s)h\Big(\int_0^{\sigma(T)} H(s,\tau)g(\tau, x_0(\tau)-\omega(\tau))\Delta\tau\Big) + p_+(s)\Big]\Delta s. \end{aligned}$$

Since $g(t,x)$ is increasing with respect to x,

$$\begin{aligned} \int_0^{\sigma(T)} H(s,\tau)g(\tau, x_0(\tau)-\omega(\tau))\Delta\tau &\le \int_0^{\sigma(T)} H(\sigma(\tau),\tau)g(\tau, x_0(\tau)-\omega(\tau))\Delta\tau \\ &\le \int_0^{\sigma(T)} H(\sigma(\tau),\tau)g(\tau, r+1)\Delta\tau \\ &\le (r+1)^{\lambda_1}\int_0^{\sigma(T)} H(\sigma(\tau),\tau)g(\tau,1)\Delta\tau. \end{aligned}$$

So

$$x_0^{\Delta}(t) \le \big(\max_{0\le\tau\le R} h(\tau)+1\big)\int_t^{t_0}(p^*(s)+p_+(s))\Delta s, \tag{3.5}$$

where $R = (r+1)^{\lambda_1}\int_0^{\sigma(T)} H(\sigma(\tau),\tau)g(\tau,1)\Delta\tau$. Integrating (3.5) from 0 to t_0, we have

$$\begin{aligned} r &= \int_0^{t_0} x_0^{\Delta}(s)\Delta s \le [\max_{0\le\tau\le R} h(\tau)+1]\int_0^{t_0}\int_s^{t_0}(p^*(\tau)+p_+(\tau))\Delta\tau\Delta s \\ &\le [\max_{0\le\tau\le R} h(\tau)+1]\sigma^2(T)\int_0^{t_0}(p^*(\tau)+p_+(\tau))\Delta\tau \\ &\le \frac{[\max_{0\le\tau\le R} h(\tau)+1]\sigma^2(T)}{\sigma^2(T)-t_0}\int_0^{t_0}\frac{\sigma(\tau)(\sigma^2(T)-\sigma(\tau))}{\sigma^2(T)}(p^*(\tau)+p_+(\tau))\Delta\tau. \\ &\le \frac{[\max_{0\le\tau\le R} h(\tau)+1]\sigma^2(T)}{\sigma^2(T)-t_0}\int_0^{\sigma(T)}\frac{\sigma(\tau)(\sigma^2(T)-\sigma(\tau))}{\sigma^2(T)}(p^*(\tau)+p_+(\tau))\Delta\tau. \end{aligned}$$

Consequently,

$$\int_0^{\sigma(T)} G(\sigma(\tau),\tau)(p^*(\tau)+p_+(\tau))\Delta\tau \ge \frac{r(\sigma^2(T)-t_0)}{[\max_{0\le\tau\le R} h(\tau)+1]\sigma^2(T)}. \tag{3.6}$$

Integrate (3.5) from t_0 to t, we have the following inequality in the same way

$$\int_0^{\sigma(T)} G(\sigma(\tau),\tau)(p^*(\tau)+p_+(\tau))\Delta\tau \geq \frac{rt_0}{[\max_{0\leq\tau\leq R} h(\tau)+1]\sigma^2(T)}\,. \tag{3.7}$$

Combining with (3.6) and (3.7),

$$\begin{aligned} 2\int_0^{\sigma(T)} G(\sigma(\tau),\tau)(p^*(\tau)+p_+(\tau))\Delta\tau &\geq \frac{r\sigma^2(T)}{[\max_{0\leq\tau\leq R} h(\tau)+1]\sigma^2(T)} \\ &= \frac{\sigma^2(T)}{\max_{0\leq\tau\leq R} h(\tau)+1}\int_0^{\sigma(T)} p_-(\tau)\Delta\tau. \end{aligned}$$

As a result,

$$\int_0^{\sigma(T)} G(\sigma(\tau),\tau)(p^*(\tau)+p_+(\tau))\Delta\tau \geq \frac{\sigma^2(T)}{2[\max_{0\leq\tau\leq R} h(\tau)+1]}\int_0^{\sigma(T)} p_-(\tau)\Delta\tau, \tag{3.8}$$

which is a contradiction with (C3). So applying Lemma 2.4, $i(F, K\cap\Omega_r, K) = 1$. $\square$

Lemma 3.2. *Assume* (C1)–(C3). *Then there exist* $R^* > r = \sigma^2(T)\int_0^{\sigma(T)} p_-(t)\Delta s$, *such that* $i(F, K\cap\Omega_{R^*}, K) = 0$, *where* $\Omega_{R^*} = \{x | x \in \mathbb{X}, \|x\| < R^*\}$.

Proof. Choose constants $0 < \alpha^* \leq \beta^*$ and L, such that $[\alpha^*, \beta^*]_{\mathbb{T}} \subseteq [0, T]_{\mathbb{T}}$ and

$$L > \frac{2\sigma^2(T)}{\alpha^*(\sigma^2(T)-\beta^*)}\Big(\min_{t\in[0,\sigma^2(T)]_{\mathbb{T}}}\int_{\alpha^*}^{\beta^*} G(t,s)\Delta s\Big)^{-1}$$

By (C1), there exists $R_1^* > 2r$ such that $f(t,y) > Ly$, for $t \in [\alpha^*, \beta^*]_{\mathbb{T}}$, $y \geq R_1^*$. Since $\lim_{x\to+\infty} \min g(t,x)/x = +\infty$, there is $R_2^* > R_1^*$, $t \in [\alpha^*, \beta^*]_{\mathbb{T}}$. When $x > R_2^*$, from Remark 2.1 we have

$$\frac{g(t,x)}{x} \geq \min_{t\in[\alpha^*,\beta^*]_{\mathbb{T}}} \frac{g(t,x)}{x} \geq \frac{1}{\min_{\alpha\leq s\leq\beta}\int_{\alpha^*}^{\beta^*} H(s,\tau)\Delta\tau}. \tag{3.9}$$

So that

$$g(t,x) \geq \frac{x}{\min_{\alpha^*\leq s\leq\beta^*}\int_{\alpha^*}^{\beta^*} H(s,\tau)\Delta\tau} \quad \text{for } t \in [\alpha,\beta]_{\mathbb{T}},\ x \geq R_2^*.$$

Let $R^* = \frac{2R_2^*(\sigma^2(T))^2}{\alpha^*(\sigma^2(T)-\beta^*)}$, then $R^* > R_2^* > R_1^* > 2r$, we assert that $Fx \not\leq x$, $x \in K\cap\partial\Omega_{R^*}$.

Suppose on the contrary that there exists $x_1 \in K \cap \partial\Omega_{R^*}$ such that $Fx_1 \le x_1$. Then for $t \in [\alpha, \beta]_{\mathbb{T}}$, we have

$$\begin{aligned}
x_1(t) - \omega(t) &\ge x_1(t) - \frac{t(\sigma^2(T) - t)}{\sigma^2(T)} \int_0^{\sigma(T)} P_-(\tau)\Delta\tau \\
&= x_1(t) - rq(t) \\
&\ge x_1(t) - \frac{x_1(t)}{\|x_1(t)\|} r \\
&> \frac{1}{2} x_1(t) \\
&\ge \frac{1}{2} q(t)\|x_1\| \\
&= \frac{t(\sigma^2(T) - t)R^*}{2(\sigma^2(T))^2} \\
&\ge \frac{R^*\alpha^*(\sigma^2(T) - \beta^*)}{2(\sigma^2(T))^2} = R_2^* > 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_\alpha^\beta H(s,\tau)g(\tau, [x_1(\tau) - \omega(\tau)]^*)\Delta\tau \\
&= \int_\alpha^\beta H(s,\tau)g(\tau, [x_1(\tau) - \omega(\tau)])\Delta\tau \\
&\ge \frac{1}{\min_{\alpha^* \le s \le \beta^*} \int_{\alpha^*}^{\beta^*} H(s,\tau)\Delta\tau} \int_{\alpha^*}^{\beta^*} H(s,\tau)(x_1(\tau) - \omega(\tau))\Delta\tau \\
&\ge \frac{R^*\alpha^*(\sigma^2(T) - \beta^*)}{2(\sigma^2(T))^2} = R_2^* \\
&> R_1^* > 0,
\end{aligned}$$

Since $f(t, y)$ is nondecreasing on y,

$$\begin{aligned}
R^* &\ge x_1(t) \ge (Fx_1)(t) \\
&= \int_0^{\sigma(T)} G(t,s)\Big[f\Big(s, \int_0^{\sigma(T)} H(s,\tau)g(\tau, [x_1(\tau) - \omega(\tau)]^*)\Delta\tau\Big) + p_+(s)\Big]\Delta s \\
&\ge \int_\alpha^\beta G(t,s)\Big[f\Big(s, \int_0^{\sigma(T)} H(s,\tau)g(\tau, [x_1(\tau) - \omega(\tau)])\Delta\tau\Big) + p_+(s)\Big]\Delta s \\
&\ge \int_\alpha^\beta G(t,s)f\Big(s, \int_0^{\sigma(T)} H(s,\tau)g(\tau, [x_1(\tau) - \omega(\tau)])\Delta\tau\Big)\Delta s \\
&\ge \int_\alpha^\beta G(t,s)L\Big(\int_0^{\sigma(T)} H(s,\tau)g(\tau, [x_1(\tau) - \omega(\tau)])\Delta\tau\Big)\Delta s \\
&\ge \frac{LR^*\alpha^*(\sigma^2(T) - \beta^*)}{2(\sigma^2(T))^2} \int_{\alpha^*}^{\beta^*} G(t,s)\Delta s.
\end{aligned}$$

So,

$$L \le \frac{2\sigma^2(T)^2}{\alpha^*(\sigma^2 - \beta^*)}\Big[\min_{t\in[0,\sigma^2(T)]_{\mathbb{T}}} \int_{\alpha^*}^{\beta^*} G(t,s)\Delta s\Big]^{-1}.$$

This contradicts the choice of the constant L. Thus from Lemma 2.4, $i(F, K \cap \Omega_{R^*}, K) = 0$. $\square$

Now we present the main result of this paper.

Theorem 3.3. *Suppose that* (C1)–(C3) *are satisfied, then the semipositone dynamic system* (1.1) *has at least one positive solution.*

Proof. By Lemmas 3.1, 3.2 and the properties of the fixed point index, we have

$$i(F, K_{R^*} \backslash \overline{K_r}, K) = -1.$$

Thus the operator F has a fixed point u_0 in $K_{R^*} \backslash \overline{K_r}$ with $r < \|u_0\| < R^*$. Since $\|u_0\| > r$, we have

$$\begin{aligned} u_0(t) - \omega(t) &\geq q(t)\|u_0\| - \int_0^{\sigma(T)} G(t,s)p_-(s)\Delta s \\ &\geq q(t)\|u_0\| - \frac{t(\sigma^2(T) - t)}{\sigma^2(T)} \int_0^{\sigma(T)} p_-(s)\Delta s \\ &\geq q(t)(\|u_0\| - r) > 0. \end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{gathered} u(t) = u_0(t) - \omega(t), \\ v(t) = \int_0^{\sigma(T)} H(t,s)g(s, u_0(s))\Delta s. \end{gathered} \tag{3.10}$$

is the positive solution of system (1.1). The proof is complete. $\square$

Example. Let $\mathbb{T} = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$. We consider the dynamic system

$$\begin{gathered} -x^{\Delta\Delta}(t) = \frac{y}{t+1} - \frac{1}{\sqrt{t}}, \quad t \in (0,1)_{\mathbb{T}}, \\ -y^{\Delta\Delta}(t) = \frac{x^2}{(t - \frac{1}{2})^2}, \quad t \in (0,1)_{\mathbb{T}}, \\ x(0) = x(\sigma^2(1)) = 0, \\ y(0) - y(0) = y(\sigma(1)) + y^{\Delta}(\sigma(1)) = 0. \end{gathered} \tag{3.11}$$

In fact, for $f(t,y) = \frac{y}{t+1} - 3$, $p(t) = -\frac{1}{\sqrt{t}}$, $g(t,x) = \frac{x^2}{(t-\frac{1}{2})^2}$, $\lambda_1 = 3$, $\lambda_2 = \frac{3}{2}$, all conditions of Theorem 3.3 are satisfied. Therefore, problem (3.11) has at least one positive solution.

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