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SUFFICIENT CONDITIONS FOR THE OSCILLATION OF SOLUTIONS TO NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

AHMED BERKANE

ABSTRACT. We present sufficient conditions for all solutions to a second-order ordinary differential equations to be oscillatory.

1. Introduction

Kirane and Rogovchenko [4] studied the oscillatory solutions of the equation

$$\left[r(t)\psi(x(t))x'(t)\right]' + p(t)x'(t) + q(t)f(x(t)) = g(t), \quad t \geq t_0, \tag{1.1}$$

where $t_0 \geq 0$, $r(t) \in C^1([t_0,\infty);(0,\infty))$, $p(t) \in C([t_0,\infty);\mathbb{R})$, $q(t) \in C([t_0,\infty);(0,\infty))$, $q(t)$ is not identical zero on $[t_*,\infty)$ for some $t_* \geq t_0$, $f(x), \psi(x) \in C(\mathbb{R},\mathbb{R})$ and $\psi(x) > 0$ for $x \neq 0$. Their results read as follows

Theorem 1.1. *Case $g(t) \equiv 0$: Assume that for some constants K, C, C_1 and for all $x \neq 0$, $f(x)/x \geq K > 0$ and $0 < C \leq \Psi(x) \leq C_1$. Let h, $H \in C(D,R)$, where $D = \{(t,s) : t \geq s \geq t_0\}$, be such that*

- (i) *$H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ in $D_0 = \{(t,s) : t \geq s \geq t_0\}$*
- (ii) *H has a continuous and non-positive partial derivative in D_0 with respect to the second variable, and*
$$-\frac{\partial H}{\partial s} = h(t,s)\sqrt{H(t,s)}$$
for all $(t,s) \in D_0$.

If there exists a function $\rho \in C^1([t_0,\infty);(0,\infty))$ such that

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,s)\Theta(s) - \frac{C_1}{4}\rho(s)r(s)Q^2(t,s)]ds = \infty\,,$$

where

$$\Theta(t) = \rho(t)\Big(Kq(t) - \big(\frac{1}{C} - \frac{1}{C_1}\big)\frac{p^2(t)}{4r(t)}\Big),$$

$$Q(t,s) = h(t,s) + \big[\frac{p(s)}{C_1 r(s)} - \frac{\rho'(s)}{\rho(s)}\big]\sqrt{H(t,s)},$$

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then (1.1) *is oscillatory.*

Theorem 1.2. *Case $g(t) \neq 0$: Let the assumptions of theorem 1 be satisfied and suppose that the function $g(t) \in C([t_0, \infty); \mathbb{R})$ satisfies*

$$\int^{\infty} \rho(s)|g(s)|ds = N < \infty\,.$$

Then any proper solution $x(t)$ of (1.1)*; i.e, a non-constant solution which exists for all $t \geq t_0$ and satisfies $\sup_{t \geq t_0} |x(t)| > 0$, satisfies*

$$\liminf_{t \to \infty} |x(t)| = 0.$$

Note that localization of the zeros is not given in the work by Kirane and Rogovchenko [4]. Here we intend to give conditions that allow us to localize the zeros of solutions to (1.1). Observe that in contrast to [4] where a Ricatti type transform,

$$v(t) = \rho \frac{r(t)\psi(x(t))x'(t)}{x(t)},$$

is used, here we simply use a usual Ricatti transform.

2. Main Results

Differential equation without a forcing term. Consider the second-order differential equation

$$\big[r(t)\psi(x(t))x'(t)\big]' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad t \geq t_0 \tag{2.1}$$

where $t_0 \geq 0$, $r(t) \in C^1([t_0, \infty); (0, \infty))$, $p(t) \in C([t_0, \infty)); \mathbb{R})$, $q(t) \in C([t_0, \infty)); R)$, $p(t)$ and $q(t)$ are not identical zero on $[t_\star, \infty)$ for some $t_\star \geq t_0$, $f(x), \psi(x) \in C(\mathbb{R}, \mathbb{R})$ and $\psi(x) > 0$ for $x \neq 0$.

The next theorem follows the ideas in Nasr [6]. Assume that there exists an interval $[a, b]$, where $a, b \geq t_\star$, such that $e(t) \geq 0$.

Theorem 2.1. *Assume that for some constants K, C, C_1 and for all $x \neq 0$,*

$$\frac{f(x)}{x} \geq K \geq 0, \tag{2.2}$$

$$0 < C \leq \psi(x) \leq C_1\,. \tag{2.3}$$

Suppose further there exists a continuous function $u(t)$ such that $u(a) = u(b) = 0$, $u(t)$ is differentiable on the open set (a, b), $a, b \geq t_\star$, and

$$\int_a^b \big[\big(Kq(t) - \frac{p^2(t)}{2Cr(t)}\big)u^2(t) - 2C_1 r(t)(u')^2(t)\big]dt \geq 0\,. \tag{2.4}$$

Then every solution of (2.1) *has a zero in $[a, b]$.*

Proof. Let $x(t)$ be a solution of (2.1) that has zero on $[a, b]$. We may assume that $x(t) > 0$ for all $t \in [a, b]$ since the case when $x(t) < 0$ can be treated analogously. Let

$$v(t) = -\frac{x'(t)}{x(t)}, \quad t \in [a, b]. \tag{2.5}$$

Multiplying this equality by $r(t)\psi(x(t))$ and differentiate the result. Using (2.1) we obtain

$$
\begin{aligned}
(r(t)\psi(x(t))v(t))' &= -\frac{(r(t)\psi(x(t))x'(t))'}{x(t)} + r(t)\psi(x(t))v^2(t) \\
&= -p(t)v(t) + q(t)\frac{f(x(t))}{x(t)} + r(t)\psi(x(t))v^2(t) \\
&= \frac{(r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v^2(t) - 2\frac{p(t)}{r(t)\psi(x(t))v(t)}\big) \\
&\quad + q(t)\frac{f(x(t))}{x(t)} \\
&= \frac{(r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{(r(t)\psi(x(t))}\big)^2 \\
&\quad - \frac{p^2(t)}{2r(t)\psi(x(t))} + q(t)\frac{f(x(t))}{x(t)}\,.
\end{aligned}
$$

Using (2.2)-(2.3) and the fact that

$$
\frac{(r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{r(t)\psi(x(t))}\big)^2 \geq 0,
$$

we have

$$
(r(t)\psi(x(t))v(t))' \geq \frac{(r(t)\psi(x(t))}{2}v^2(t) - \frac{p^2(t)}{2Cr(t)} + Kq(t) \tag{2.6}
$$

Multiplying both sides of this inequality by $u^2(t)$ and integrating on $[a,b]$. Using integration by parts on the left side, the condition $u(a) = u(b) = 0$ and (2.3), we obtain

$$
\begin{aligned}
0 &\geq \int_a^b \frac{r(t)\psi(x(t))}{2}v^2(t)u^2(t)dt + 2\int_a^b r(t)\psi(x(t))v(t)u(t)u'(t)dt \\
&\quad + \int_a^b Kq(t)u^2(t)dt - \int_a^b \frac{p^2(t)}{2Cr(t)}u^2(t)dt \\
&\geq \int_a^b \frac{r(t)\psi(x(t))}{2}(v^2(t)u^2(t) + 4v(t)u(t)u'(t))dt \\
&\quad + \int_a^b Kq(t)u^2(t)dt - \int_a^b \frac{p^2(t)u^2(t)}{2Cr(t)}dt \\
&\geq \int_a^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) + 2u'(t)]^2dt - 2\int_a^b r(t)\psi(x(t))u'^2(t)dt \\
&\quad + \int_a^b Kq(t)u^2(t)dt - \int_a^b \frac{p^2(t)}{2Cr(t)}u^2(t)dt \\
&\geq \int_a^b \big[\big(Kq(t) - \frac{p^2(t)}{2Cr(t)}\big)u^2(t) - 2r(t)\psi(x(t))u'^2(t)\big]dt \\
&\quad + \int_a^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) + 2u'(t)]^2dt\,.
\end{aligned}
$$

Now, from (2.3) we have

$$\begin{aligned} 0 \geq & \int_a^b \big[\big(Kq(t) - \frac{p^2(t)}{2Cr(t)}\big)u^2(t) - 2r(t)C_1 u'^2(t)\big]dt \\ & + \int_a^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) + 2u'(t)]^2 dt. \end{aligned}$$

If the first integral on the right-hand side of the inequality is greater than zero, then we have directly a contradiction. If the first integral is zero and the second is also zero then $x(t)$ has the same zeros as $u(t)$ at the points a and b; $(x(t) = ku^2(t))$, which is again a contradiction with our assumption. □

Corollary 2.2. *Assume that there exist a sequence of disjoint intervals* $[a_n, b_n]$*, and a sequence of functions* $u_n(t)$ *defined and continuous an* $[a_n, b_n]$*, differentiable on* (a_n, b_n) *with* $u_n(a_n) = u_n(b_n) = 0$*, and satisfying assumption* (2.4)*. Let the conditions of Theorem 2.1. hold. Then* (2.1) *is oscillatory.*

Differential equation with a forcing term. Consider the differential equation

$$\big[r(t)\psi(x(t))x'(t)\big]' + p(t)x'(t) + q(t)f(x(t)) = g(t), \quad t \geq t_0 \tag{2.7}$$

where $t_0 \geq 0$, $g(t) \in C([t_0,\infty);\mathbb{R})$ $r(t) \in C^1([t_0,\infty);(0,\infty))$, $p(t) \in C([t_0,\infty));R)$, $q(t) \in C([t_0,\infty));R)$, $p(t)$ and $q(t)$ are not identical zero on $[t_\star,\infty[$ for some $t_\star \geq t_0$, $f(x), \psi(x) \in C(\mathbb{R},\mathbb{R})$ and $\psi(x) > 0$ for $x \neq 0$.

Assume that there exists an interval $[a,b]$, where $a, b \geq t_\star$, such that $g(t) \geq 0$ and there exists $c \in (a,b)$ such that $g(t)$ has different signs on $[a,c]$ and $[c,b]$. Without loss of generality, let $g(t) \leq 0$ on $[a,c]$ and $g(t) \geq 0$ on $[c,b]$.

Theorem 2.3. *Let* (2.3) *hold and assume that*

$$\frac{f(x)}{x|x|} \geq K, \tag{2.8}$$

for a positive constant K *and for all* $x \neq 0$*. Furthermore assume that there exists a continuous function* $u(t)$ *such that* $u(a) = u(b) = u(c) = 0$*,* $u(t)$ *differentiable on the open set* $(a,c) \cup (c,b)$*, and satisfies the inequalities*

$$\int_a^c \big[\big(\sqrt{Kq(t)g|(t)|} - \frac{p^2(t)}{2Cr(t)}\big)u^2 - 2C_1 r(t)(u')^2(t)\big]d(t) \geq 0, \tag{2.9}$$

$$\int_c^b \big[\big(\sqrt{Kq(t)g|(t)|} - \frac{p^2(t)}{2Cr(t)}\big)u^2 - 2C_1 r(t)(u')^2(t)\big]d(t) \geq 0\,. \tag{2.10}$$

Then every solution of equation (2.7) *has a zero in* $[a,b]$*.*

Proof. Assume to the contrary that $x(t)$, a solution of (2.7), has no zero in $[a,b]$. Let $x(t) < 0$ for example. Using the same computations as in the first part, we obtain:

$$\begin{aligned}(r(t)\psi(x(t))v(t))' &= -\frac{(r(t)\psi(x(t))x'(t))'}{x(t)} + r(t)\psi(x(t))v^2(t) - \frac{g(t)}{x(t)} \\ &= -p(t)v(t) + q(t)\frac{f(x(t))}{x(t)} + r(t)\psi(x(t))v^2(t) - \frac{g(t)}{x(t)} \\ &= \frac{(r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v^2(t) - 2\frac{p(t)v(t)}{r(t)\psi(x(t))}\big) \\ &\quad + q(t)\frac{f(x(t))}{x(t)} - \frac{g(t)}{x(t)} \\ &= \frac{(r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{r(t)\psi(x(t))v(t)}\big)^2 \\ &\quad - \frac{p^2(t)}{2r(t)\psi(x(t))} + q(t)\frac{f(x(t))}{x(t)} - \frac{g(t)}{x(t)}\end{aligned}$$

For $t \in [c, b]$ we have

$$\begin{aligned}(r(t)\psi(x(t))v(t))' = \frac{r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{r(t)\psi(x(t))}\big)^2 \\ - \frac{p^2(t)}{2r(t)\psi(x(t))} + q(t)\frac{f(x(t))}{x(t)|x(t)|}|x(t)| + \frac{|g(t)|}{|x(t)|}\end{aligned}$$

From (2.8), and using the fact that

$$\frac{r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{r(t)\psi(x(t))}\big)^2 \geq 0$$

we deduce

$$(r(t)\psi(x(t))v(t))' \geq \frac{(r(t)\psi(x(t))}{2}v^2(t) - \frac{p^2(t)}{2r(t)\psi(x(t))} + Kq(t)|x(t)| + \frac{|g(t)|}{|x(t)|}\,. \tag{2.11}$$

Using the Hölder inequality in (2.11) we obtain

$$(r(t)\psi(x(t))v(t))' \geq \frac{(r(t)\psi(x(t))}{2}v^2(t) + \sqrt{Kq(t)|g(t)|} - \frac{p^2(t)}{2r(t)\psi(x(t))}\,. \tag{2.12}$$

Multiplying both sides of this inequality by $u^2(t)$ and integrating on $[c, b]$, we obtain after using integration by parts on the left-hand side and the condition $u(c) = u(b) = 0$,

$$\begin{aligned}0 &\geq \int_c^b \frac{r(t)\psi(x(t))}{2}v^2(t)u^2(t)dt + \int_c^b \sqrt{Kq(t)|g(t)|}u^2(t)dt \\ &\quad - \int_c^b \frac{p^2(t)u^2(t)}{2r(t)\psi(x(t))}dt + 2\int_c^b r(t)\psi(x(t))v(t)u(t)u'(t)dt \\ &\geq \int_c^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) - 2u'(t)]^2dt - 2\int_c^b r(t)\psi(x(t))u'^2(t)dt \\ &\quad + \int_c^b \sqrt{Kq(t)|g(t)|}u^2(t)dt - \int_c^b \frac{p^2(t)u^2(t)}{2r(t)\psi(x(t))}dt.\end{aligned}$$

Assumption (2.3) allows us to write

$$
\begin{aligned}
0 \geq & \int_c^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t)+2u'(t)]^2 dt - 2\int_c^b C_1 r(t)(u')^2(t)dt \\
& + \int_c^b \sqrt{Kq(t)|g(t)|}u^2(t)dt - \int_c^b \frac{p^2(t)u^2(t)}{2Cr(t)}dt \\
\geq & \int_c^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t)+2u'(t)]^2 dt \\
& + \int_c^b \big[\big(\sqrt{Kq(t)g|(t)|} - \frac{p^2(t)}{2Cr(t)}\big)u^2(t) - 2C_1 r(t)(u')^2(t)\big]dt.
\end{aligned}
$$

This leads to a contradiction as in Theorem 2.1; the proof is complete. □

Corollary 2.4. *Assume that there exist a sequence of disjoint intervals* $[a_n, b_n]$ *a sequences of points* $c_n \in (a_n, c_n)$*, and a sequence of functions* $u_n(t)$ *defined and continuous on* $[a_n, b_n]$*, differentiable on* $(a_n, c_n) \cup (c_n, b_n)$*with* $u_n(a_n) = u_n(b_n) = u_n(c_n) = 0$*, and satisfying assumptions* (2.9)-(2.10)*. Let the conditions of Theorem 2.3 hold. Then* (2.7) *is oscillatory.*

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Ahmed Berkane

Faculté des Sciences, Département de Mathématiques, Université Badji Mokhtar, 23000 Annaba, Algeria

E-mail address: aberkane20052005@yahoo.fr