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VARIATIONAL AND TOPOLOGICAL METHODS FOR OPERATOR EQUATIONS INVOLVING DUALITY MAPPINGS ON ORLICZ-SOBOLEV SPACES

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Abstract. Let $a : \mathbb{R} \to \mathbb{R}$ be a strictly increasing odd continuous function with $\lim_{t\to+\infty} a(t) = +\infty$ and $A(t) = \int_0^t a(s)\,ds$, $t \in \mathbb{R}$, the N-function generated by a. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $T[u,u]$ a nonnegative quadratic form involving the only generalized derivatives of order m of the function $u \in W_0^m E_A(\Omega)$ and $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, be Carathéodory functions.

We study the problem

$$J_a u = \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha g_\alpha(x, D^\alpha u) \quad \text{in } \Omega,$$
$$D^\alpha u = 0 \text{ on } \partial\Omega, |\alpha| \leq m-1,$$

where J_a is the duality mapping on $\left(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\right)$, subordinated to the gauge function a (given by (1.5)) and

$$\|u\|_{m,A} = \|\sqrt{T[u,u]}\|_{(A)},$$

$\|\cdot\|_{(A)}$ being the Luxemburg norm on $E_A(\Omega)$.

By using the Leray-Schauder topological degree and the mountain pass theorem of Ambrosetti and Rabinowitz, the existence of nontrivial solutions is established. The results of this paper generalize the existence results for Dirichlet problems with p-Laplacian given in [12] and [13].

1. Introduction

Throughout this paper Ω denotes a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. Let $a : \mathbb{R} \to \mathbb{R}$ be a strictly increasing odd continuous function with $\lim_{t\to+\infty} a(t) = +\infty$. For $m \in \mathbb{N}^*$, let us denote by $W_0^m E_A(\Omega)$ the Orlicz-Sobolev space generated by the $N-$ function A, given by

$$A(t) = \int_0^t a(s)\,ds. \tag{1.1}$$

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In this paper we study the existence of solutions of the boundary-value problem

$$J_a u = \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha g_\alpha(x, D^\alpha u) \quad \text{in } \Omega, \tag{1.2}$$

$$D^\alpha u = 0 \quad \text{on } \partial\Omega, |\alpha| \le m-1, \tag{1.3}$$

in the following functional framework:

• $T[u,v]$ is a nonnegative symmetric bilinear form on the Orlicz-Sobolev space $W_0^m E_A(\Omega)$, involving the only generalized derivatives of order m of the functions $u, v \in W_0^m E_A(\Omega)$, satisfying

$$c_1 \sum_{|\alpha|=m} (D^\alpha u)^2 \le T[u,u] \le c_2 \sum_{|\alpha|=m} (D^\alpha u)^2 \quad \forall u \in W_0^m L_A(\Omega), \tag{1.4}$$

with c_1, c_2 being positive constants;

• $\|u\|_{m,A} = \|\sqrt{T[u,u]}\|_{(A)}$ is a norm on $W_0^m E_A(\Omega)$, $\|\cdot\|_{(A)}$ designating the Luxemburg norm on the Orlicz space $L_A(\Omega)$;

• $J_a : \big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big) \to \big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big)^*$ is the duality mapping on $\big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big)$ subordinated to the gauge function a:

$$\langle J_a u, h\rangle = \frac{a\left(\|u\|_{m,A}\right) \cdot \int_\Omega a\Big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\Big) \frac{T[u,h]}{\sqrt{T[u,u]}}\, dx}{\int_\Omega a\Big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\Big) \frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\, dx}, \quad u, h \in W_0^m E_A(\Omega); \tag{1.5}$$

• $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, are Carathéodory functions satisfying some appropriate growth conditions.

The main existence results are contained in Theorems 6.4 and 7.4 and the techniques used are essentially based on Leray-Schauder topological degree and on the mountain pass theorem due to Ambrosetti and Rabinowitz, respectively.

Let us remark that for the particular choice of $a(t) = |t|^{p-2} \cdot t$, $1 < p < \infty$, $m = 1$ and $T[u,v] = \nabla u \cdot \nabla v$, the existence results given by Theorems 6.4 and 7.4 reduce to the well known existence results of the weak solution in $W_0^{1,p}(\Omega)$ for the Dirichlet problem

$$-\Delta_p u = g_0(x,u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

The plan of the paper is as follows: In section 2, some fundamental results concerning the Orlicz-Sobolev spaces are given; these results are taken from Adams [1], Gossez [19], Krasnosel'skij and Rutitskij [22], Tienari [30].

The main results of section 3 concern the smoothness and the uniform convexity of the space $\big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big)$. Note that, in order to prove the uniform convexity of the space $\big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big)$, an inequality given by Proposition 3.9 and playing a similar role to that of Clarkson's inequalities is used. This inequality is a corollary of a result due to Gröger [20] (see, also Langenbach [23]).

The content of section 4 is as follows: the smoothness and the uniform convexity of the space $\big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big)$ allow us to show that the duality mapping on $\big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big)$ corresponding to the gauge function a is given by

$$J_a(0) = 0,$$
$$J_a u = a(\|\cdot\|_{m,A}) \|\cdot\|'_{m,A}(u), \quad u \ne 0.$$

Moreover, J_a is bijective with a continuous inverse, J_a^{-1}.

Section 5 deals with the properties of the so called Nemytskij operator on Orlicz spaces. These properties will be used later coupled with compact imbeddings of Orlicz-Sobolev spaces in some Orlicz spaces (a prototype of such a theorem is Theorem 2.12, due to Donaldson and Trudinger [15] (see, also Adams [1]).

In section 6, the existence of a solution for problem (1.2), (1.3), reduces to a fixed point existence theorem. Since for any $u \in W_0^m E_A(\Omega)$ one has $D^\alpha u\big|_{\partial\Omega} = 0$, $|\alpha| \le m-1$, the approach is realized in $W_0^m E_A(\Omega)$-space. It is shown that if a point $u \in W_0^m E_A(\Omega)$ satisfies

$$J_a u = (i^* \circ N \circ i)u,$$

or, equivalently,

$$u = (J_a^{-1} \circ i^* \circ N \circ i)u,$$

then u satisfies (1.2) (in the sense of $(W_0^m E_A(\Omega))^*$), that is u is a weak solution for (1.2), (1.3). In writing of compact operator $P = J_a^{-1} \circ i^* \circ N \circ i$, i^* is the adjoint of i and the meaning of i and N are given by Propositions 6.2 and 6.3 respectively. In order to prove that P possesses a fixed point in $W_0^m E_A(\Omega)$, an a priori estimate method is used.

In section 7, the existence of a solution for problem (1.2), (1.3), reduces to proving the existence of a critical point for the functional $F : W_0^m E_A(\Omega) \to \mathbb{R}$, given by (7.13). In order to prove that F possesses a critical point in $W_0^m E_A(\Omega)$, we show that F has a mountain-pass geometry. Consequently, the mountain pass theorem of Ambrosetti and Rabinowitz applies.

In section 8, some examples of functions a for which existence results for the problem (1.2), (1.3) may be obtained are given. It would be notice that the same function a appears in examples 8.3 and 8.4; however, the corresponding hypotheses being different, the existence results are obtained by using distinct techniques: the mountain-pass theorem for example 8.3 and a priori estimate method for example 8.4. The same is true for examples 8.6 and 8.7. The only a Leray-Schauder technique can be applied for example 8.8. A slight modification of function a, appearing in example 8.8, enables the use of the mountain-pass theorem, as example 8.10 shows.

2. Orlicz and Orlicz-Sobolev spaces

Definition 2.1. A function $A : \mathbb{R} \to \mathbb{R}_+$ is called an *N-function* if it admits the representation

$$A(t) = \int_0^{|t|} a(s)\, ds,$$

where the function $a : \mathbb{R}_+ \to \mathbb{R}_+$ is right-continuous for $t \ge 0$, positive for $t > 0$ and non-decreasing which satisfies the conditions $a(0) = 0$, $\lim_{t\to\infty} a(t) = \infty$.

It is assumed everywhere below that the function a is continuous.

Remark 2.2. In many applications, it will be convenient to extend the function a for negative values of the argument. Thus, let $\widetilde{a} : \mathbb{R} \to \mathbb{R}_+$ be the function given by

$$\widetilde{a}(s) = \begin{cases} a(t), & \text{if } t \ge 0 \\ -a(-t), & \text{if } t < 0. \end{cases}$$

Then, the function $A : \mathbb{R} \to \mathbb{R}_+$,

$$A(t) = \int_0^t \widetilde{a}(s)\, ds,$$

is an N-function. Obviously, the function $\widetilde{a}$ is continuous and odd.

Throughout this paper, we suppose that $a : \mathbb{R} \to \mathbb{R}$ is a strictly increasing odd continuous function with $\lim_{t\to+\infty} a(t) = +\infty$ and A is the $N-$function given by (1.1).

Let us consider the *Orlicz class*

$$K_A(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable; } \int_\Omega A(u(x))dx < \infty\}.$$

The *Orlicz space* $L_A(\Omega)$ is defined as the linear hull of $K_A(\Omega)$ and it is a Banach space with respect to the *Luxemburg norm*

$$\|u\|_{(A)} = \inf\{k > 0;\ \int_\Omega A\big(\frac{u(x)}{k}\big)\, dx \le 1\}.$$

Remark 2.3. If $a(t) = |t|^{p-2} \cdot t$, $1 < p < \infty$, then $A(t) = \frac{|t|^p}{p}$, $K_A(\Omega) = L_A(\Omega) = L^p(\Omega)$ and $\|u\|_{(A)} = p^{-\frac{1}{p}} \|u\|_{L^p(\Omega)}$.

Generally $K_A(\Omega) \subset L_A(\Omega)$. Moreover, $K_A(\Omega) = L_A(\Omega)$ if and only if A satisfies the Δ_2-*condition*: there exist $k > 0$ and $t_0 > 0$ such that

$$A(2t) \le kA(t), \quad \text{for all } t \ge t_0. \tag{2.1}$$

Theorem 2.4 ([22, p. 24])**.** *A necessary and sufficient condition for the N- function A to satisfy the Δ_2-condition is that there exists a constant α such that, for $u > 0$,*

$$\frac{ua(u)}{A(u)} < \alpha. \tag{2.2}$$

The N-function given by

$$\overline{A}(u) = \int_0^{|u|} a^{-1}(s)\, ds,$$

is called the *complementary N-function* to A.

Remark 2.5. Let p, q be such that $p > 1$ and $p^{-1} + q^{-1} = 1$. If $A(t) = \frac{|t|^p}{p}$, then $\overline{A}(t) = \frac{|t|^q}{q}$. Consequently $K_{\overline{A}}(\Omega) = L_{\overline{A}}(\Omega) = L^q(\Omega)$.

We recall *Young's inequality*

$$uv \le A(u) + \overline{A}(v), \quad \forall u, v \in \mathbb{R}$$

with equality if and only if $u = a^{-1}(|v|) \cdot \operatorname{sign} v$ or $v = a(|u|) \cdot \operatorname{sign} u$.

The space $L_A(\Omega)$ is also a Banach space with respect to the *Orlicz norm*

$$\|u\|_A = \sup\big\{\big|\int_\Omega u(x)v(x)\, dx\big|; v \in K_{\overline{A}}(\Omega), \int_\Omega \overline{A}(v(x))\, dx \le 1\big\}.$$

Moreover [22, p. 80],

$$\|u\|_{(A)} \le \|u\|_A \le 2\|u\|_{(A)}, \quad \forall u \in L_A(\Omega).$$

One also has a *Hölder's type inequality*: if $u \in L_A(\Omega)$ and $v \in L_{\overline{A}}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\Big|\int_\Omega u(x)v(x)\,dx\Big| \le 2\|u\|_{(A)}\|v\|_{(\overline{A})}. \tag{2.3}$$

We shall denote the closure of $L^\infty(\Omega)$ in $L_A(\Omega)$ by $E_A(\Omega)$. One has $E_A(\Omega) \subset K_A(\Omega)$ and $E_A(\Omega) = K_A(\Omega)$ if and only if A satisfies the Δ_2 -condition. We shall denote by $\prod\big(E_A(\Omega), r\big)$ the set of those u from $L_A(\Omega)$ whose distance (with respect to the Orlicz norm) to $E_A(\Omega)$ is strictly less than r. If the N -function A does not satisfy the Δ_2-condition, then

$$\prod(E_A(\Omega), r) \subset K_A(\Omega) \subset \overline{\prod(E_A(\Omega), r)},$$

the inclusions being proper.

Theorem 2.6 ([22, p. 79])**.** *If $u \in L_A(\Omega)$ and $\|u\|_{(A)} \le 1$, then $u \in K_A(\Omega)$ and $\rho(u;A) = \int_\Omega A(u(x))\,dx \le \|u\|_{(A)}$. If $u \in L_A(\Omega)$ and $\|u\|_{(A)} > 1$, then $\rho(u;A) \ge \|u\|_{(A)}$.*

Lemma 2.7 ([18])**.** *If $u \in E_A(\Omega)$, then $a(|u|) \in K_A(\Omega)$.*

The Orlicz-Sobolev space $W^mL_A(\Omega)$ $\big(W^mE_A(\Omega)\big)$ is the space of all $u \in L_A(\Omega)$ whose distributional derivatives $D^\alpha u$ are in $L_A(\Omega)$ $(E_A(\Omega))$ for any α, with $|\alpha| \le m$;

The spaces $W^mL_A(\Omega)$ and $W^mE_A(\Omega)$ are Banach spaces with respect to the norm

$$\|u\|_{W^mL_A(\Omega)} = \Big(\sum_{|\alpha|\le m}\|D^\alpha u\|^2_{(A)}\Big)^{1/2}. \tag{2.4}$$

If Ω has the segment property, then $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $W^mE_A(\Omega)$ [1, Theorem 8.28]. The space $W_0^mE_A(\Omega)$ is defined as the norm-closure of $\mathcal{D}(\Omega)$ in $W^mE_A(\Omega)$.

Now, let us suppose that the boundary $\partial\Omega$ of Ω is $\mathcal{C}^1$. Consider the "restriction to $\partial\Omega$" mapping $\widetilde{\gamma} : \mathcal{C}^\infty(\overline{\Omega}) \to \mathcal{C}(\partial\Omega)$, $\widetilde{\gamma}(u) = u|_{\partial\Omega}$. This mapping is continuous from $\big(\mathcal{C}^\infty(\overline{\Omega}), \|\cdot\|_{W^1L_A(\Omega)}\big)$ to $\big(\mathcal{C}(\partial\Omega), \|\cdot\|_{L_A(\partial\Omega)}\big)$ [19, p. 69]. Consequently, the mapping $\widetilde{\gamma}$ can be extended into a continuous mapping, denoted γ and called the "trace mapping", from $\big(W^1E_A(\Omega), \|\cdot\|_{W^1L_A(\Omega)}\big)$ to $\big(E_A(\partial\Omega), \|\cdot\|_{E_A(\partial\Omega)}\big)$.

Theorem 2.8 ([19, Proposition 2.3])**.** *The kernel of the trace mapping $\gamma : W^1E_A(\Omega) \to E_A(\partial\Omega)$ is $W_0^1E_A(\Omega)$.*

The following results are useful.

Theorem 2.9 ([7])**.** *$W^mL_A(\Omega)$ is reflexive if and only if the N-functions A and $\overline{A}$ satisfy the Δ_2-condition.*

Proposition 2.10 ([18])**.** *There exist constants c_m and $c_{m,\Omega}$ such that*

$$\int_\Omega \sum_{|\alpha|<m} A(D^\alpha u)\,dx \le c_m \int_\Omega \sum_{|\alpha|=m} A(c_{m,\Omega}D^\alpha u)\,dx,$$

for all $u \in W_0^mL_A(\Omega)$.

Corollary 2.11 ([18])**.** *The two norms*

$$\Big(\sum_{|\alpha|\le m}\|D^\alpha u\|^2_{(A)}\Big)^{1/2} \quad \text{and} \quad \Big(\sum_{|\alpha|=m}\|D^\alpha u\|^2_{(A)}\Big)^{1/2}$$

are equivalent on $W_0^mL_A(\Omega)$.

We recall that, if A and B are two N-functions, we say that B *dominates* A *near infinity* if there exist positive constants k and t_0 such that

$$A(t) \le B(kt) \tag{2.5}$$

for all $t \ge t_0$. The two N-functions A and B are *equivalent near infinity* if each dominates the other near infinity. If B dominates A near infinity and A and B are not equivalent near infinity, then we say that A *increases essentially more slowly than* B *near infinity* and we denote $A \prec\prec B$. This is the case if and only if for every $k > 0$

$$\lim_{t\to\infty} \frac{A(kt)}{B(t)} = 0. \tag{2.6}$$

If the N-functions A and B are equivalent near infinity, then A and B define the same Orlicz space [1, p. 234].

Let us now introduce the Orlicz-Sobolev conjugate A_* of the N-function A. We shall always suppose that

$$\lim_{t\to 0} \int_t^1 \frac{A^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\, d\tau < \infty, \tag{2.7}$$

replacing, if necessary, A by another N-function equivalent to A near infinity (which determines the same Orlicz space).

Suppose also that

$$\lim_{t\to \infty} \int_1^t \frac{A^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\, d\tau = \infty. \tag{2.8}$$

With (2.8) satisfied, we define the *Sobolev conjugate* A_* of A by setting

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\, d\tau, t \ge 0. \tag{2.9}$$

Theorem 2.12 ([1]). *If the N-function A satisfies* (2.7) *and* (2.8)*, then*

$$W_0^1 L_A(\Omega) \to L_{A_*}(\Omega).$$

Moreover, if Ω_0 is a bounded subdomain of Ω, then the imbeddings

$$W_0^1 L_A(\Omega) \to L_B(\Omega_0)$$

exist and are compact for any N-function B increasing essentially more slowly than A_* near infinity.

Theorem 2.13 ([30, Theorem 2.7]). *The compact imbedding*

$$W_0^1 L_A(\Omega) \to E_A(\Omega)$$

holds.

3. Geometry and smoothness of the space $(W_0^m E_A(\Omega), \|\cdot\|_{m,A})$

Definition 3.1. The space X is said to be smooth, if for each $x \in X$, $x \ne 0_X$, there exists a unique functional $x^* \in X^*$, such that $\|x^*\| = 1$ and $\langle x^*, x\rangle = \|x\|$.

The following results will be useful.

Theorem 3.2 ([10]). *Let $(X, \|\|)$ be a real Banach space. The norm of X is Gâteaux differentiable if and only if X is smooth.*

In order to study the smoothness of the space $W_0^m E_A(\Omega)$, we recall a result concerning the differentiability of the norm on Orlicz spaces.

Theorem 3.3 ([22]). *The Luxemburg norm $\|\cdot\|_{(A)}$ is Gâteaux-differentiable on $E_A(\Omega)$. For $u \neq 0$, we have*

$$\langle \|\cdot\|'_{(A)}(u), h\rangle = \frac{\int_\Omega a\big(\frac{u(x)}{\|u\|_{(A)}}\big)h(x)\,dx}{\int_\Omega a\big(\frac{u(x)}{\|u\|_{(A)}}\big)\frac{u(x)}{\|u\|_{(A)}}\,dx}, \quad \text{for all } h \in E_A(\Omega). \tag{3.1}$$

Moreover, if the N-function $\overline{A}$ satisfies the Δ_2-condition, then the norm $\|\cdot\|_{(A)}$ is Fréchet-differentiable on $E_A(\Omega)$.

The following results will be also useful.

Lemma 3.4 ([30, Lemma 2.5]). *If $(u_n)_n \subset E_A(\Omega)$ with $u_n \to u$ in $E_A(\Omega)$, then there exists $h \in K_A(\Omega) \subset L_A(\Omega)$ and a subsequence $(u_{n_k})_{n_k}$ such that $|u_{n_k}(x)| \leq h(x)$ a.e. and $u_{n_k}(x) \to u(x)$ a.e.*

Lemma 3.5 ([22, Lemma 18.2]). *Let A and $\overline{A}$ be mutually complementary N-functions the second of which satisfies the Δ_2-condition. Suppose that the derivative a of A is continuous. Then, the operator N_a, defined by means of the equality $N_a u(x) = a(|u(x)|)$, acts from $\prod(E_A(\Omega), 1)$ into $K_{\overline{A}}(\Omega) = L_{\overline{A}}(\Omega) = E_{\overline{A}}(\Omega)$ and is continuous.*

Now, let $T[u,v]$ be a nonnegative symmetric bilinear form involving the only generalized derivatives of order m of the functions $u, v \in W_0^m E_A(\Omega)$, satisfying the inequalities (1.4). From these inequalities and taking into account Corollary 2.11, we obtain that $W_0^m E_A(\Omega)$ may be (equivalent) renormed by using the norm

$$\|u\|_{m,A} = \|\sqrt{T[u,u]}\|_{(A)}. \tag{3.2}$$

Theorem 3.6. *The space $\big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big)$ is smooth. Thus, the norm $\|\cdot\|_{m,A}$ is Gâteaux-differentiable on $W_0^m E_A(\Omega)$. For $u \neq 0_{W_0^m E_A(\Omega)}$, we have*

$$\langle \|\cdot\|'_{m,A}(u), h\rangle = \frac{\int_\Omega a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big)\frac{T[u,h](x)}{\sqrt{T[u,u](x)}}\,dx}{\int_\Omega a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big)\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\,dx}, \quad \text{for all } h \in W_0^m E_A(\Omega). \tag{3.3}$$

Moreover, if the N-function $\overline{A}$ satisfies the Δ_2-condition, then $u \to \|\cdot\|'_{m,A}(u)$ is continuous thus $\|\cdot\|_{m,A}$ is Fréchet-differentiable.

Proof. Let $u \neq 0$ be in $W_0^m E_A(\Omega)$, that is $\sqrt{T[u,u]} \neq 0_{E_A(\Omega)}$. Let us denote $\psi(u) = \|\sqrt{T[u,u]}\|_{(A)}$. It is obvious that ψ can be written as a product $\psi = QP$, where $Q : E_A(\Omega) \to \mathbb{R}$ is given by $Q(v) = \|v\|_{(A)}$ and $P : W_0^m E_A(\Omega) \to E_A(\Omega)$ is given by $P(u) = \sqrt{T[u,u]}$. The functional Q is Gâteaux differentiable (see Theorem 3.3) and

$$\langle Q'(v), h\rangle = \|v\|'_{(A)}(h), \tag{3.4}$$

for all $v, h \in E_A(\Omega)$, $v \neq 0_{E_A(\Omega)}$. Simple computations show that the operator P is Gâteaux differentiable at u and

$$P'(u)(h) = \frac{T[u,h]}{\sqrt{T[u,u]}}, \tag{3.5}$$

for all $u, h \in W_0^m E_A(\Omega)$, $u \neq 0_{W_0^m E_A(\Omega)}$. Combining (3.4) and (3.5), we obtain that ψ is Gâteaux differentiable at u and

$$\begin{aligned}\langle \psi'(u), h\rangle &= \langle Q'(Pu), P'(u)(h)\rangle \\ &= \langle \|\cdot\|'_{(A)}(Pu), \frac{T[u,h]}{\sqrt{T[u,u]}}\rangle \\ &= \frac{\int_\Omega a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big)\frac{T[u,h](x)}{\sqrt{T[u,u](x)}}\,dx}{\int_\Omega a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big)\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\,dx}.\end{aligned}$$

Now, we will show that the mapping $u \mapsto \psi'(u)$ is continuous. In order to do that it is sufficient to show that any sequence $(u_n)_n \subset W_0^m E_A(\Omega)$ converging to $u \in W_0^m E_A(\Omega)$ contains a subsequence $(u_{n_k})_k \subset (u_n)_n$ such that $\psi'(u_{n_k}) \to \psi'(u)$, as $k \to \infty$, in $\big(W_0^m E_A(\Omega)\big)^*$. We set

$$\langle \psi'(u), h\rangle = \frac{\langle \varphi(u), h\rangle}{q(u)}, \quad \forall h \in W_0^m E_A(\Omega),$$

where $\varphi : W_0^m E_A(\Omega) \to W_0^m E_A(\Omega)$ is defined by

$$\langle \varphi(u), h\rangle = \int_\Omega a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big)\frac{T[u,h](x)}{\sqrt{T[u,u](x)}}\,dx$$

and $q : W_0^m E_A(\Omega) \to \mathbb{R}$ is given by

$$q(u) = \int_\Omega a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big)\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\,dx.$$

First, we show that if $u_n \to u$ in $W_0^m E_A(\Omega)$, then the sequence $(u_n)_n$ contains a subsequence $(u_{n_k})_k \subset (u_n)_n$ such that $q(u_{n_k}) \to q(u)$ as $k \to \infty$. Since

$$|\sqrt{T[u_n,u_n]} - \sqrt{T[u,u]}| \le \sqrt{T[u_n-u, u_n-u]}, \tag{3.6}$$

it follows from

$$\|u_n - u\|_{m,A} = \|\sqrt{T[u_n-u,u_n-u]}\|_{(A)} \to 0 \quad \text{as } n \to \infty, \tag{3.7}$$

that

$$\sqrt{T[u_n,u_n]} \to \sqrt{T[u,u]} \quad \text{as } n \to \infty, \text{ in } E_A(\Omega); \tag{3.8}$$

therefore

$$\frac{\sqrt{T[u_n,u_n]}}{\|u_n\|_{m,A}} \to \frac{\sqrt{T[u,u]}}{\|u\|_{m,A}} \quad \text{as } n \to \infty, \text{ in } E_A(\Omega).$$

By applying Lemma 3.5, and obtain

$$a\big(\frac{\sqrt{T[u_n,u_n]}}{\|u_n\|_{m,A}}\big) \to a\big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\big) \quad \text{as } n \to \infty, \text{ in } E_{\overline{A}}(\Omega).$$

Then, from Lemma 3.4, it follows that there exists a subsequence $(u_{n_k})_k \subset (u_n)_n$ and $w \in K_{\overline{A}}(\Omega) = E_{\overline{A}}(\Omega)$, such that

$$a\big(\frac{\sqrt{T[u_{n_k},u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\big) \to a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big) \quad \text{as } k \to \infty, \text{ for a.e. } x \in \Omega \tag{3.9}$$

and

$$a\big(\frac{\sqrt{T[u_{n_k},u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\big) \le w(x), \quad \text{for a.e. } x \in \Omega. \tag{3.10}$$

Taking into account (3.7), written for $(u_{n_k})_k$, and applying again Lemma 3.4, it follows that there exists a subsequence (also denoted $(u_{n_k})_k$), and $w_1 \in K_A(\Omega)$ such that

$$\sqrt{T[u_{n_k}-u, u_{n_k}-u](x)} \to 0 \quad \text{as } k \to \infty, \text{ for a.e. } x \in \Omega. \tag{3.11}$$

and

$$\sqrt{T[u_{n_k}, u_{n_k}](x)} \le w_1(x), \quad \text{for a.e. } x \in \Omega. \tag{3.12}$$

Out of (3.11) and (3.6), we obtain

$$\sqrt{T[u_{n_k}, u_{n_k}](x)} \to \sqrt{T[u,u](x)} \quad \text{as } k \to \infty, \text{ for a.e. } x \in \Omega. \tag{3.13}$$

Consequently

$$\begin{aligned} &a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\Big)\sqrt{T[u_{n_k}, u_{n_k}](x)} \\ &\to a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\sqrt{T[u,u](x)} \quad \text{as } k \to \infty, \text{ for a.e. } x \in \Omega \end{aligned}$$

and

$$a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\Big)\sqrt{T[u_{n_k}, u_{n_k}](x)} \le w(x)\cdot w_1(x), \quad \text{for a.e. } x \in \Omega.$$

Since $w \cdot w_1 \in L^1(\Omega)$, by using (3.8) and Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned} &\int_\Omega a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\Big)\frac{\sqrt{T[u_{n_k}, u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\,dx \\ &\to \int_\Omega a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}dx, \quad \text{as } k \to \infty, \end{aligned}$$

which is $q(u_{n_k}) \to q(u)$ as $k \to \infty$.

For the $(u_{n_k})_k$ obtained above, we shall show that

$$\varphi(u_{n_k}) \to \varphi(u), \quad \text{as } k \to \infty, \text{ in } \big(W_0^m E_A(\Omega)\big)^*.$$

But

$$T[u,v] = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta}(x) D^\alpha u D^\beta v,$$

where $c_{\alpha\beta} \in \mathcal{C}(\overline{\Omega})$, therefore they are bounded.

First let us remark that, for arbitrary h, one has

$$\begin{aligned} &|\left(\varphi(u_{n_k}) - \varphi(u)\right)(h)| \\ &= \Big|\sum_{|\alpha|=|\beta|=m}\int_\Omega c_{\alpha\beta}\Big[a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\Big)\frac{D^\alpha u_{n_k}}{\sqrt{T[u_{n_k}, u_{n_k}](x)}} \\ &\quad - a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u](x)}}\Big]D^\beta h dx\Big| \\ &\le M\sum_{|\alpha|=|\beta|=m}\Big|\int_\Omega\Big[a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\Big)\frac{D^\alpha u_{n_k}}{\sqrt{T[u_{n_k}, u_{n_k}](x)}} \\ &\quad - a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u](x)}}\Big]D^\beta h dx\Big|. \end{aligned} \tag{3.14}$$

We intend to apply Hölder's inequality (2.3) in (3.14). Since $D^\beta h \in E_A(\Omega)$, for all β with $|\beta| = m$, it is sufficient to show that

$$a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}]}}{\|u_{n_k}\|_{m,A}}\Big)\frac{D^\alpha u_{n_k}}{\sqrt{T[u_{n_k}, u_{n_k}]}} - a\Big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u]}} \in L_{\overline{A}}(\Omega).$$

Moreover, we will show that

$$a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}]}}{\|u_{n_k}\|_{m,A}}\Big)\frac{D^\alpha u_{n_k}}{\sqrt{T[u_{n_k}, u_{n_k}]}} - a\Big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u]}} \in E_{\overline{A}}(\Omega) = K_{\overline{A}}(\Omega). \tag{3.15}$$

Indeed, $a\big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\big)\frac{D^\alpha u}{\sqrt{T[u,u]}} \in K_{\overline{A}}(\Omega)$, because $\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}} \in E_A(\Omega)$, by Lemma 2.7, we obtain $a\big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\big) \in K_{\overline{A}}(\Omega)$. On the other hand, since T satisfies inequalities (1.4), we have

$$\frac{D^\alpha u}{\sqrt{T[u,u].}} \le \frac{1}{\sqrt{c_1}};$$

therefore

$$a\Big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u]}} \le \frac{1}{\sqrt{c_1}}a\Big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\Big) \in K_{\overline{A}}(\Omega) = E_{\overline{A}}(\Omega)$$

(the N-function $\overline{A}$ satisfies the Δ_2-condition). Consequently,

$$a\Big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u]}} \in K_{\overline{A}}(\Omega) = E_{\overline{A}}(\Omega). \tag{3.16}$$

Now, using the same technique, we obtain

$$a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}]}}{\|u_{n_k}\|_{m,A}}\Big)\frac{D^\alpha u_{n_k}}{\sqrt{T[u_{n_k}, u_{n_k}]}} \in K_{\overline{A}}(\Omega) = E_{\overline{A}}(\Omega);$$

therefore we have (3.15). Applying Hölder's inequality in (3.14), we obtain

$$\begin{aligned} |\left(\varphi(u_{n_k}) - \varphi(u)\right)(h)| \le M_1 \sum_{|\alpha|=|\beta|=m} \Big\| a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\Big)\frac{D^\alpha u_{n_k}}{\sqrt{T[u_{n_k}, u_{n_k}](x)}} \\ - a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u](x)}}\|_{(\overline{A})}\Big\|h\|_{m,A}. \end{aligned}$$

Consequently,

$$\begin{aligned} \|\varphi(u_{n_k}) - \varphi(u)\| \le M_1 \sum_{|\alpha|=|\beta|=m} \Big\| a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\Big)\frac{D^\alpha u_{n_k}}{\sqrt{T[u_{n_k}, u_{n_k}](x)}} \\ - a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u](x)}}\|_{(\overline{A})}. \end{aligned}$$

Finally, we show that

$$\Big\|a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}](x)}}{\|u_{n_k}\|_{m,A}}\Big)\frac{D^\alpha u_{n_k}}{\sqrt{T[u_{n_k}, u_{n_k}](x)}} - a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u](x)}}\Big\|_{(\overline{A})} \to 0, \tag{3.17}$$

as $k \to \infty$.

We will use the following result [29, Theorem 14, p. 84]. An element $f \in L_A(\Omega)$ has an absolutely continuous norm if and only if for each measurable f_n such that $f_n \to \widetilde{f}$ a.e. and $|f_n| \leq |f|$, a.e., we have $\|f_n - \widetilde{f}\|_{(A)} \to 0$ as $n \to \infty$. The fact that $f \in L_A(\Omega)$ has an absolutely continuous norm means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|f \cdot \chi_E\|_A < \varepsilon$ provided $\operatorname{mes}(E) < \delta$ $(E \subset \Omega)$. Moreover, any function from $E_A(\Omega)$ has an absolutely continuous norm [22, Theorem 10.3].

Then, (3.17) follows from the above result with the following choices:

$$f_k = a\Big(\frac{\sqrt{T[u_{n_k}, u_{n_k}]}}{\|u_{n_k}\|_{m,A}}\Big)\frac{D^\alpha u_{n_k}}{\sqrt{T[u_{n_k}, u_{n_k}]}} \in E_{\overline{A}}(\Omega),$$

$$\widetilde{f} = a\Big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\Big)\frac{D^\alpha u}{\sqrt{T[u,u]}} \in E_{\overline{A}}(\Omega)$$

From (1.4) and (3.11), it follows

$$D^\alpha u_{n_k}(x) \to D^\alpha u(x), \quad \text{for a.e. } x \in \Omega;$$

therefore, taking into account (3.9), (3.13), we obtain

$$f_k(x) \to \widetilde{f}(x), \quad \text{as } k \to \infty, \text{ for a.e. } x \in \Omega.$$

On the other hand, from (1.4) and (3.10), we have

$$|f_k(x)| \leq \frac{w(x)}{\sqrt{c_1}}, \quad \text{for a.e. } x \in \Omega,$$

with $w \in K_{\overline{A}}(\Omega) = E_{\overline{A}}(\Omega)$. Setting

$$f = \frac{w}{\sqrt{c_1}} \in E_{\overline{A}}(\Omega),$$

it follows (3.17). It follows that $\|\varphi(u_{n_k}) - \varphi(u)\| \to 0$ as $k \to \infty$. □

Now, we will study the uniform convexity of the space $\big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big)$. To do it, we still need some prerequisites. We begin with a technical result due to Gröger ([20]) (see, also [23, p. 153]).

Lemma 3.7. *Let $A(u) = \int_0^{|u|} p(t)\,dt$ and $A_1(u) = \int_0^{|u|} p_1(t)\,dt$ be two N-functions, such that the functions p and p_1 should satisfy the conditions*

$$\frac{p(\tau)}{\tau} \geq \frac{p(t)}{t}, \quad \tau \geq t > 0, \tag{3.18}$$

$$p(t+\tau) - p(\tau) \geq p_1(t), \quad \tau \geq t > 0. \tag{3.19}$$

Then

$$\frac{1}{2}A(a) + \frac{1}{2}A(b) - A(c) \geq A_1(c_*), \tag{3.20}$$

where

$$a \geq b \geq 0, \quad \frac{a-b}{2} \leq c \leq \frac{a+b}{2}, \quad c_* = \sqrt{\frac{a^2+b^2}{2} - c^2}. \tag{3.21}$$

The next corollary is a direct consequence of the preceding lemma.

Corollary 3.8. *Let $A(u) = \int_0^{|u|} p(t)\,dt$ be an N-function. Suppose that the function $p(t)/t$ is nondecreasing on $(0,\infty)$. Then*

$$\frac{1}{2}A(a) + \frac{1}{2}A(b) - A(c) \geq A(c_*),$$

where a, b, c and c_* are as in (3.21).

Proposition 3.9. *Let $A(u) = \int_0^{|u|} p(t)dt$ be an N-function. Suppose that the function $\frac{p(t)}{t}$ is nondecreasing on $(0, \infty)$. Then*

$$\frac{1}{2}A\big(\sqrt{T[u,u]}\big) + \frac{1}{2}A\big(\sqrt{T[v,v]}\big) - A\Big(\sqrt{T[\frac{u+v}{2}, \frac{u+v}{2}]}\Big) \geq A\Big(\sqrt{T[\frac{u-v}{2}, \frac{u-v}{2}]}\Big).$$

Proof. We apply Corollary 3.8 with $a = \sqrt{T[u,u]}$, $b = \sqrt{T[v,v]}$,

$$c = \sqrt{T[\frac{u+v}{2}, \frac{u+v}{2}]}, \quad c_* = \sqrt{T[\frac{u-v}{2}, \frac{u-v}{2}]}.$$

□

Proposition 3.10. *Let A be an N-function. If the N-function A satisfies the Δ_2-condition, then*

$$\rho : L_A(\Omega) = E_A(\Omega) = K_A(\Omega) \to \mathbb{R}, \quad \rho(u) = \int_\Omega A(u(x))dx,$$

is continuous.

Proof. Obviously, ρ is convex, therefore it suffices to show that ρ is upper bounded on a neighborhood of 0. But, if $\|u\|_{(A)} < 1$, then $\rho(u) \leq \|u\|_{(A)} < 1$. □

Proposition 3.11. *Let A be an N-function. Then, one has:*

- (i) *If $\rho(u) = \int_\Omega A(u(x))dx = 1$, then $\|u\|_{(A)} = 1$;*
- (ii) *if, in addition, A satisfies a Δ_2-condition, then $\rho(u) = \int_\Omega A(u(x))dx = 1$ if and only if $\|u\|_{(A)} = 1$.*

Proof. (i) Indeed, we have

$$1 = \rho(u) = \int_\Omega A(\frac{u(x)}{1})dx \geq \|u\|_{(A)},$$

the last inequality being justified by the definition of the $\|\cdot\|_{(A)}$-norm. If $\|u\|_{(A)} < 1$, then (see Theorem 2.6), we have

$$\int_\Omega A(u(x))dx \leq \|u\|_{(A)} < 1,$$

which is a contradiction.

(ii) Taking into account the result given by (i), the "only if" implication has to be proved. Now, since $\|u\|_{(A)} = 1$, we can write

$$\rho(u) = \int_\Omega A(\frac{u(x)}{1})dx = \int_\Omega A(\frac{u(x)}{\|u\|_{(A)}})dx \leq 1.$$

The strict inequality cannot hold. Indeed, if for some u with $\|u\|_{(A)} = 1$, we have $\int_\Omega A(u(x))dx < 1$, then there exists $\varepsilon > 0$ such that $\int_\Omega A(u(x))dx + \varepsilon < 1$. From Proposition 3.10, $\lim_{\lambda \to 1_+} \rho(\lambda u) = \rho(u)$, therefore, there exists $\delta > 0$, such that for each λ with $|\lambda - 1| < \delta$, we have

$$\Big|\int_\Omega A(\lambda u(x))dx - \int_\Omega A(u(x))dx\Big| < \varepsilon.$$

It follows that, for $1 < \lambda < 1 + \delta$, $\int_\Omega A(\lambda u(x))dx < \int_\Omega A(u(x))dx + \varepsilon < 1$. Since $\int_\Omega A(\lambda u(x))dx < 1$, we infer that $\|u\|_{(A)} \leq \frac{1}{\lambda} < 1$, which is a contradiction. □

Proposition 3.12. *Let A be an N-function which satisfies the Δ_2-condition. If $\|u\|_{(A)} > \varepsilon$, then there exists $\eta > 0$ such that $\int_\Omega A(u(x))dx > \eta$.*

Proof. Let u be such that $\|u\|_{(A)} > \varepsilon$. Assume that the assertion in the proposition is not true, therefore for each η we have $\int_\Omega A(u(x))dx \leq \eta$. This means that $\rho(u) = \int_\Omega A(u(x))dx = 0$. Then, the Δ_2-condition implies that, $\rho(2^p u) \leq k^p \rho(u) = 0$, therefore $\rho(2^p u) = 0$. Consequently $\|2^p u\|_A \leq \rho(2^p u) + 1 = 1$, therefore $\|u\|_{(A)} \leq \|u\|_A \leq \frac{1}{2^p} < \varepsilon$ for p large enough, which is a contradiction. □

Definition 3.13. The space $(X, \|\cdot\|_X)$ is called uniformly convex if for each $\varepsilon \in (0, 2]$ there exists $\delta(\varepsilon) \in (0, 1]$ such that for $u, v \in X$ with $\|u\|_X = \|v\|_X = 1$ and $\|u - v\|_X \geq \varepsilon$, one has $\|\frac{u+v}{2}\|_X \leq 1 - \delta(\varepsilon)$.

Theorem 3.14. *Let $A(u) = \int_0^{|u|} p(t)\,dt$ be an N-function. Suppose that the function $p(t)/t$ is nondecreasing on $(0, \infty)$. If the N-function A satisfies the Δ_2-condition, then $W_0^m E_A(\Omega)$ endowed with the norm*

$$\|u\|_{m,A} = \|\sqrt{T[u,u]}\|_{(A)}$$

is uniformly convex.

Proof. We start with the following technical remark: if the N-function A satisfies a Δ_2-condition and $\int_\Omega A(u(x))dx < 1 - \eta$ for some $0 < \eta < 1$, there is $\delta > 0$ such that $\|u\|_{(A)} < 1 - \delta$. In the contrary case, there is u satisfying $\int_\Omega A(u(x))dx < 1 - \eta$ for which $\|u\|_{(A)} \geq 1 - \delta$ for any $\delta > 0$. In particular inequality $\int_\Omega A(u(x))dx < 1 - \eta$ may be satisfied for some u with $\|u\|_{(A)} > 1/2$. On the other hand, every u satisfying $\int_\Omega A(u(x))dx < 1 - \eta$ has to satisfy $\|u\|_{(A)} < 1$ (see Theorem 2.6 and Proposition 3.11). Put $a = 1/\|u\|_{(A)}$. Clearly $1 < a < 2$, $\|au\|_{(A)} = 1$ and $\int_\Omega A(au(x))dx = 1$ (again by Proposition 3.11).

Now, by the convexity of A we derive that

$$\begin{aligned}
1 &= \int_\Omega A(au(x))\,dx \\
&= \int_\Omega A\left(2(a-1)u(x) + (2-a)u(x)\right)\,dx \\
&\leq (a-1)\int_\Omega A(2u(x))\,dx + (2-a)\int_\Omega A(u(x))\,dx \\
&\leq (a-1)k\int_\Omega A(u(x))\,dx + (2-a)\int_\Omega A(u(x))\,dx;
\end{aligned}$$

therefore

$$1 \leq [(a-1)k + 2 - a]\cdot\int_\Omega A(u(x))\,dx < [(a-1)k + 2 - a]\cdot(1-\eta).$$

On the other hand, from $\frac{1}{2} < a < 1$, $0 < \eta < 1$ and $k > 2$, it follows that $[(a-1)k + 2 - a]\cdot(1-\eta) < 1$, which is a contradiction.

Now, let $\varepsilon > 0$ be and $u, v \in W_0^m E_A(\Omega)$ such that $\|u\|_{m,A} = \|\sqrt{T[u,u]}\|_{(A)} = 1$, $\|v\|_{m,A} = \|\sqrt{T[v,v]}\|_{(A)} = 1$ and $\|u - v\|_{m,A} = \|\sqrt{T[u-v, u-v]}\|_{(A)} > \varepsilon$. Then $\|\frac{u-v}{2}\|_{m,A} = \|\sqrt{T[\frac{u-v}{2}, \frac{u-v}{2}]}\|_{(A)} > \frac{\varepsilon}{2}$. From Proposition 3.12 it follows that there exists $\eta > 0$ such that $\int_\Omega A\big(\sqrt{T[\frac{u-v}{2}, \frac{u-v}{2}]}\big)\,dx > \eta$. On the other hand, from Proposition 3.11, we have $\int_\Omega A\big(\sqrt{T[u,u]}\big)\,dx = \int_\Omega A\big(\sqrt{T[v,v]}\big)\,dx = 1$. Taking into account Proposition 3.9, we obtain that $\int_\Omega A\Big(\sqrt{T[\frac{u+v}{2}, \frac{u+v}{2}]}\Big)\,dx < 1-\eta$. From the above remark, we conclude that there is a $\delta > 0$ depending on ε such that $\|\frac{u+v}{2}\|_{m,A} = \|\sqrt{T[\frac{u+v}{2}, \frac{u+v}{2}]}\|_{(A)} < 1-\delta$. □

4. Duality mapping on $\big(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\big)$

Let X be a real Banach space and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a gauge function, i.e. φ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

By duality mapping corresponding to the gauge function φ we understand the multivalued mapping $J_\varphi : X \to \mathcal{P}(X^*)$, defined as follows:

$$\begin{gathered} J_\varphi 0 = \{0\}, \\ J_\varphi x = \varphi(\|x\|)\{u^* \in X^*; \|u^*\| = 1, \langle u^*, x\rangle = \|x\|\}, \quad \text{if } x \neq 0. \end{gathered} \tag{4.1}$$

According to the Hahn-Banach theorem it is easy to see that the domain of J_φ is the whole space:

$$D(J_\varphi) = \{x \in X; J_\varphi x \neq \emptyset\} = X.$$

Due to Asplund's result [3],

$$J_\varphi = \partial\psi, \psi(x) = \int_0^{\|x\|} \varphi(t)dt, \tag{4.2}$$

for any $x \in X$ and $\partial\psi$ stands for the subdifferential of ψ in the sense of convex analysis.

By the preceding definition, it follows that J_φ is single valued if and only if X is smooth, i.e. for any $x \neq 0$ there is a unique element $u^*(x) \in X^*$ having the metric properties

$$\langle u^*(x), x\rangle = \|x\|, \quad \|u^*(x)\| = 1 \tag{4.3}$$

But it is well known (see, for example, Diestel [10]) that a real Banach space X is smooth if and only if its norm is differentiable in the Gâteaux sense, i.e. at any point $x \in X$, $x \neq 0$ there is a unique element $\|\cdot\|'(x) \in X^*$ such that, for any $h \in X$, the following equality

$$\lim_{t\to 0} \frac{\|x+th\| - \|x\|}{t} = \langle \|\cdot\|'(x), h\rangle$$

holds. Since, at any $x \neq 0$, the gradient of the norm satisfies

$$\|\|\cdot\|'(x)\| = 1, \quad \langle \|\cdot\|'(x), x\rangle = \|x\| \tag{4.4}$$

and it is the unique element in the dual space having these properties, we immediately get that: if X is a smooth real Banach space, then the duality mapping corresponding to a gauge function φ is the single valued mapping $J_\varphi : X \to X^*$, defined as follows:

$$\begin{gathered} J_\varphi 0 = 0, \\ J_\varphi x = \varphi(\|x\|)\|\cdot\|'(x), \quad \text{if } x \neq 0. \end{gathered} \tag{4.5}$$

Remark 4.1. By coupling (4.5) with the Asplund's result quoted above, we get: if X is smooth, then

$$J_\varphi x = \psi'(x) = \begin{cases} 0 & \text{if } x = 0 \\ \varphi(\|x\|)\|\cdot\|'(x) & \text{if } x \neq 0, \end{cases} \tag{4.6}$$

where ψ is given by (4.2).

From (4.4) and (4.5), it follows that

$$\begin{aligned} \|J_\varphi x\| &= \varphi(\|x\|), \\ \langle J_\varphi x, x\rangle &= \varphi(\|x\|)\|x\|, \quad \text{for all } x \in X. \end{aligned} \tag{4.7}$$

The following surjectivity result will play an important role in what follows:

Theorem 4.2. *If X is a real reflexive and smooth Banach space, then any duality mapping $J_\varphi : X \to X^*$ is surjective. Moreover, if X is also strictly convex, then J_φ is a bijection of X onto X^*.*

In proving the surjectivity of J_φ, the main ideas are as follows: (for more details, see Browder [5], Lions [24], Deimling [9])

(i) J_φ is monotone:

$$\langle J_\varphi x - J_\varphi y, x-y\rangle \geq (\varphi(\|x\|) - \varphi(\|y\|))(\|x\| - \|y\|) \geq 0, \forall x,y \in X.$$

The first inequality is a direct consequence of (4.7) while the second one follows from φ being increasing.

(ii) Any duality mapping on a real smooth and reflexive Banach space is demicontinuous:

$$x_n \to x \Rightarrow J_\varphi x_n \rightharpoonup J_\varphi x.$$

Indeed, since $(x_n)_n$ is bounded and $\|J_\varphi x_n\| = \varphi(\|x_n\|)$, it follows that $(J_\varphi x_n)_n$ is bounded in X^*. Since X^* is reflexive, in order to prove $J_\varphi x_n \rightharpoonup J_\varphi x$ it is enough to prove that $J_\varphi x$ is the unique point in the weak closure of $(J_\varphi x_n)_n$.

(iii) J_φ is coercive, in the sense that

$$\frac{\langle J_\varphi x, x\rangle}{\|x\|} = \varphi(\|x\|) \to \infty \text{ as } \|x\| \to \infty.$$

According to a well-known surjectivity result due to Browder (see, for example, Browder [5], Lions [24], Zeidler [31], Deimling [9]), if X is a reflexive real Banach space, then any monotone, demicontinuous and coercive operator $T : X \to X^*$ is surjective.

Consequently, from (i), (ii), (iii) and the Browder's surjectivity result above mentioned it follows that, under the hypotheses of Theorem 4.2, J_φ is surjective.

It can be shown that if X is a strictly convex real Banach space, then any duality mapping $J_\varphi : X \to \mathcal{P}(X^*)$ is strictly monotone, in the following sense: if $x, y \in X$ and $x \neq y$, then, for any $x^* \in J_\varphi x$ and $y^* \in J_\varphi y$ one has $\langle x^* - y^*, x-y\rangle > 0$. Clearly, the strict monotonicity implies the injectivity: if $x, y \in X$ and $x \neq y$ then $J_\varphi x \cap J_\varphi y = \emptyset$. In particular, if the strictly convex real Banach space X is also a smooth one, then any duality mapping $J_\varphi : X \to X^*$ is strictly monotone:

$$\langle J_\varphi x - J_\varphi y, x-y\rangle > 0, \quad \forall x,y \in X, x \neq y,$$

and, consequently, injective.

Corollary 4.3. *If X is a reflexive and smooth real Banach space having the Kadeč-Klee property, then any duality mapping $J_\varphi : X \to X^*$ is bijective and has a continuous inverse. Moreover,*

$$J_\varphi^{-1} = \chi^{-1} J^*_{\varphi^{-1}}, \tag{4.8}$$

where $J^*_{\varphi^{-1}} : X^* \to X^{**}$ is the duality mapping on X^* corresponding to the gauge function φ^{-1} and $\chi : X \to X^{**}$ is the canonical isomorphism defined by $\langle \chi(x), x^*\rangle = \langle x^*, x\rangle$, for all $x \in X$, for all $x^* \in X^*$.

Proof. The existence of J_φ^{-1} follows from Theorem 4.2. As far as formula (4.8) is concerned, first we shall prove that, under the hypotheses of Corollary 4.3, any duality mapping on X^* (in particular, that corresponding to the gauge function φ^{-1}) is single valued. This is equivalent with proving that X^* is smooth.

The smoothness of X^* will be proved by using the (partial) duality between strict convexity and smoothness given by the following theorem due to Klee (see Diestel [10, Chapter 2, §2, Theorem 2]):

$$X^* \text{ smooth (strictly convex) } \Rightarrow X \text{ strictly convex (smooth).}$$

Clearly, if X is reflexive, then

$$X^* \text{ smooth (strictly convex) } \Leftrightarrow X \text{ strictly convex (smooth).}$$

Now, by the hypotheses of Corollary 4.3, X is reflexive and smooth. Also, by the same hypotheses, X possesses the Kadeč-Klee property, that means: X is strictly convex and

$$[x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\|] \Rightarrow x_n \to x. \tag{4.9}$$

Consequently, X being reflexive, smooth and strictly convex so is X^*.

Let us prove that equality (4.8) holds or, equivalently,

$$\chi J_\varphi^{-1} x^* = J^*_{\varphi^{-1}} x^*, \forall x^* \in X^*. \tag{4.10}$$

From the definition of duality mappings, $J^*_{\varphi^{-1}} x^*$ is the unique element in X^{**} having the metric properties

$$\begin{gathered} \langle J^*_{\varphi^{-1}} x^*, x^* \rangle = \varphi^{-1}(\|x^*\|)\|x^*\|, \\ \|J^*_{\varphi^{-1}} x^*\| = \varphi^{-1}(\|x^*\|). \end{gathered} \tag{4.11}$$

We shall show that $\chi J_\varphi^{-1} x^*$ possesses the same metric properties and then the result follows by unicity. Putting $x^* = J_\varphi x$ it follows (by definition of J_φ) that

$$\begin{gathered} x^* = \varphi(\|x\|), \\ \langle x^*, x \rangle = \varphi(\|x\|)\|x\| = \varphi^{-1}(\|x^*\|)\|x^*\| \end{gathered}$$

and, consequently, we deduce that

$$\begin{gathered} \langle \chi J_\varphi^{-1} x^*, x^* \rangle = \langle \chi(x), x^* \rangle = \langle x^*, x \rangle = \varphi^{-1}(\|x^*\|)\|x^*\|, \\ \|\chi J_\varphi^{-1} x^*\| = \|\chi(x)\| = \|x\| = \varphi^{-1}(\|x\|) \end{gathered} \tag{4.12}$$

Equality (4.10) follows by comparing (4.11) and (4.12) and using the uniqueness result evoked above. Formula (4.8) is basic in proving the continuity of J_φ^{-1}. Indeed, let $x_n^* \to x^*$ in X^*. As any duality mapping on a reflexive Banach space, $J^*_{\varphi^{-1}}$ is demicontinuous, $J^*_{\varphi^{-1}} x_n^* \rightharpoonup J^*_{\varphi^{-1}} x^*$. Consequently, we deduce that

$$J_\varphi^{-1} x_n^* = \chi^{-1} J^*_{\varphi^{-1}} x_n^* \rightharpoonup \chi^{-1} J^*_{\varphi^{-1}} x^* = J_\varphi^{-1} x^*. \tag{4.13}$$

On the other hand,

$$\|J_\varphi^{-1} x_n^*\| = \|\chi^{-1} J^*_{\varphi^{-1}} x_n^*\| = \|J^*_{\varphi^{-1}} x_n^*\| = \varphi^{-1}(\|x_n^*\|) \to \varphi^{-1}(\|x\|) = \|J_\varphi^{-1} x^*\|. \tag{4.14}$$

From (4.13), (4.14) and the Kadeč-Klee property of X, we infer that $J_\varphi^{-1} x_n^* \to J_\varphi^{-1} x^*$. □

Corollary 4.4. *If X is a weakly locally uniformly convex, reflexive and smooth real Banach space, then any duality mapping $J_\varphi : X \to X^*$ is bijective and has a continuous inverse given by (4.8).*

Proof. Since any weakly locally uniformly convex Banach space has the Kadeč-Klee property (see Diestel[Chapter 2, §2, Theorems 3 and 4(iii)][10]) the result follows by Corollary 4.3. □

Theorem 4.5. *Let φ be a gauge function. The duality mapping on $(W_0^m E_A(\Omega), \|u\|_{m,A})$ is the single valued operator $J_\varphi : W_0^m E_A(\Omega) \to \big(W_0^m E_A(\Omega)\big)^*$ defined by*

$$J_\varphi u = \psi'(u) = \begin{cases} 0 & \text{if } u = 0 \\ \varphi(\|u\|_{m,A}) \| \cdot \|'_{m,A}(u) & \text{if } u \neq 0, \end{cases}$$

where

$$\psi(u) = \int_0^{\|u\|_{m,A}} \varphi(t)dt = \Phi\left(\|u\|_{m,A}\right), \quad \forall u \in W_0^m E_A(\Omega),$$

where Φ is the N-function generated by φ and, for any $u \neq 0$, $\| \cdot \|'_{m,A}(u)$ being given by (3.3).

This result immediately follows by Theorem 3.6 and Remark 4.1.

5. Nemytskij operator on $L_A(\Omega)$

We recall that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a *Carathéodory function* if it satisfies:

(i) for each $s \in \mathbb{R}$, the function $x \to f(x, s)$ is Lebesgue measurable in Ω;
(ii) for a.e. $x \in \Omega$, the function $s \to f(x, s)$ is continuous in $\mathbb{R}$.

We make convention that in the case of a Carathéodory function, the assertion $x \in \Omega$ to be understood in the sense a.e. $x \in \Omega$.

Proposition 5.1 ([22, Theorem 17.1])**.** *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Then, for each measurable function u, the function $N_f u : \Omega \to R$, given by*

$$(N_f u)(x) = f(x, u(x)), \quad \text{for each } x \in \Omega \tag{5.1}$$

is measurable in Ω.

Definition 5.2. Let $\mathcal{M}$ be the set of all measurable functions $u : \Omega \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. The operator $N_f : \mathcal{M} \to \mathcal{M}$ given by (5.1) is called *Nemytskij operator* defined by Carathéodory function f.

Theorem here below states sufficient conditions when Nemytskij operator maps a Orlicz class $K_A(\Omega)$ into another Orlicz class $K_B(\Omega)$, being at the same time continuous and bounded. The following result is useful.

Theorem 5.3. *Let A and B be two N-functions and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which satisfies the growth condition*

$$|f(x, u)| \leq c(x) + bB^{-1}\left(A(u)\right), x \in \Omega, u \in \mathbb{R}, \tag{5.2}$$

where $c \in K_B(\Omega)$ and $b \geq 0$ is a constant. Then the following statements are true:

(i) *If B satisfies the Δ_2-condition, then N_f is well-defined and mean bounded from $K_A(\Omega)$ into $K_B(\Omega) = E_B(\Omega)$. Moreover, $N_f : \big(E_A(\Omega), \| \cdot \|_{(A)}\big) \to \big(E_B(\Omega), \| \cdot \|_{(B)}\big)$ is continuous;*
(ii) *If both A and B satisfy the Δ_2-condition, then $N_f : \big(E_A(\Omega), \| \cdot \|_{(A)}\big) \to \big(E_B(\Omega), \| \cdot \|_{(B)}\big)$ is norm bounded.*

Proof. Let us first remark that the well-definedness of N_f as well as the continuity and the boundedness on every ball $B(0,r) \subset L_A(\Omega)$, with $r < 1$, may be obtained as consequences Theorem 17.6 in Krasnosel'skij and Rutickij ([22]). The proof of this theorem is quite complicated; that is why a direct proof of Theorem 5.3, including the supplementary result given by (ii), will be given below.

(i) Let $u, v \in \mathbb{R}$. Since B is convex and satisfies the Δ_2-condition, one has

$$B(u+v) = B\big(2 \cdot \frac{1}{2}(u+v)\big) \le \frac{k}{2}\left(B(u) + B(v)\right). \tag{5.3}$$

Let p be such that $2^p \ge b$. Since B satisfies the Δ_2-condition, one has

$$B(bu) \le B(2^p u) \le k^p B(u). \tag{5.4}$$

Now, let $u \in K_A(\Omega)$. By using (5.2), (5.3), (5.4) and integrating on Ω, we have

$$\begin{aligned} \int_\Omega B\left[N_f(u)(x)\right] dx &= \int_\Omega B\left(|f(x,u(x))|\right) dx \\ &\le \frac{k}{2}\int_\Omega B\left(c(x)\right) dx + \frac{k^{p+1}}{2}\int_\Omega A(u(x))\, dx < \infty, \end{aligned} \tag{5.5}$$

saying then $N_f(K_A(\Omega)) \subset L_B(\Omega) = E_B(\Omega)$.

From (5.5) it follows that, if $u \in K_A(\Omega)$ and $\int_\Omega A(u(x))\, dx \le$const., then

$$\int_\Omega B\left[N_f(u)(x)\right] dx \le \frac{k}{2}\int_\Omega B\left(c(x)\right) dx + \text{const.};$$

therefore N_f transforms mean bounded sets in $K_A(\Omega)$ into mean bounded sets in $E_B(\Omega)$.

Now, let us consider $u \in E_A(\Omega)$. For the continuity of N_f, it suffices to show that every sequence $(u_n)_n \subset E_A(\Omega)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_A = 0$$

has a subsequence $(u_{n_k})_k$ such that $N_f(u_{n_k}) \to N_f(u)$ as $k \to \infty$, in $L_B(\Omega) = E_B(\Omega)$.

Indeed, let $(u_n)_n$ be a sequence as above. By using Lemma 3.4, it follows that there exists a subsequence $(u_{n_k})_k \subset (u_n)_n$ and $h \in K_A(\Omega)$ such that

$$\lim_{k\to\infty} u_{n_k}(x) = u(x), \quad \text{a.e. } x \in \Omega \tag{5.6}$$

and

$$|u_{n_k}(x)| \le |h(x)|, \quad \text{a.e. } x \in \Omega, k \in \mathbb{N}. \tag{5.7}$$

The function f being a Carathéodory function, it is clear that

$$\lim_{k\to\infty} N_f(u_{n_k})(x) = N_f(u)(x), \quad \text{a.e. } x \in \Omega,$$

therefore,

$$\lim_{k\to\infty} B\left(N_f(u_{n_k})(x) - N_f(u)(x)\right) = 0, \quad \text{a.e. } x \in \Omega. \tag{5.8}$$

On the other hand, from (5.2) it follows that

$$|N_f(u_{n_k})(x)| = |f(x, u_{n_k}(x))| \le c(x) + bB^{-1}\left(A(h(x))\right), \quad \text{a.e. } x \in \Omega, k \in \mathbb{N}.$$

Consequently, by using a similar argument to that in (5.5) and taking into account (5.7), one obtains

$$B\left(N_f(u_{n_k})(x)\right) \le \frac{k}{2} B\left(c(x)\right) + \frac{k^{p+1}}{2} A\left(h(x)\right),$$

and by using a similar argument to that in (5.5), one has

$$B\left(N_f(u)(x)\right) \leq \frac{k}{2} B\left(c(x)\right) + \frac{k^{p+1}}{2} A\left(h(x)\right);$$

therefore, by (5.3) and the preceding two inequalities, one obtains

$$\begin{aligned} B\left(N_f(u_{n_k})(x) - N_f(u)(x)\right) &\leq \frac{k}{2} B\left(N_f(u_{n_k})(x)\right) + \frac{k}{2} B\left(N_f(u)(x)\right) \\ &\leq \frac{k^2}{2} B(c(x)) + \frac{k^{n+2}}{4} [A\left(h(x)\right) + A(u(x))]. \end{aligned}$$

Since the right term of this inequality is in $L^1(\Omega)$ and (5.8) holds, by applying Lebesgue's dominated convergence theorem, it follows that

$$\lim_{k\to\infty} \int_\Omega B\left(N_f(u_{n_k})(x) - N_f(u)(x)\right)\, dx = 0,$$

that is the subsequence $(N_f(u_{n_k}))_k$ converges in mean to $N_f(u)$. The N-function B satisfying the Δ_2 -condition, it follows that the subsequence $(N_f(u_{n_k}))_k$ converges in norm to $N_f(u)$, therefore the operator N_f is continuous.

(ii) Now, let us suppose that A satisfies the Δ_2-condition. If the set $\mathcal{M} \subset E_A(\Omega)$ is norm bounded, then, from Δ_2-condition, it follows that $\mathcal{M}$ is mean bounded, therefore $N_f(\mathcal{M})$ is also mean bounded. But any mean bounded set is norm bounded too. □

Now, let us consider the functional $\mathcal{G} : E_A(\Omega) \to \mathbb{R}$ given by

$$\mathcal{G}(u) = \int_\Omega G\left(x, u(x)\right)\, dx, \tag{5.9}$$

where

$$G\left(x, s\right) = \int_0^s f(x, \tau)\, d\tau. \tag{5.10}$$

We recall the following result concerning the differentiability of the functional F.

Theorem 5.4 ([22, Theorem 18.1])**.** *Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Assume that there exists an N-function M such that*

$$|f(x, u)| \leq c(x) + b\overline{M}^{-1}\left(M(u)\right), x \in \Omega, u \in \mathbb{R}, \tag{5.11}$$

where $\overline{M}$ is the complementary N-function to M, $c \in K_{\overline{M}}(\Omega)$, $b \geq 0$ is a constant and $\overline{M}$ satisfies the Δ_2-condition. Then, the functional $\mathcal{G}$, given by (5.9), is of class $\mathcal{C}^1$ on $E_M(\Omega)$, with Fréchet derivative given by

$$\langle \mathcal{G}'(u), h\rangle = \int_\Omega N_f(u)(x)h(x)\, dx = \int_\Omega f\left(x, u(x)\right)\cdot h(x)\, dx, \quad u, h \in E_M(\Omega). \tag{5.12}$$

6. An existence result for (1.2), (1.3), via a Leray-Schauder technique

Since any $u \in W_0^m E_A(\Omega)$ satisfies the boundary conditions (1.3) (see Theorem 2.8), the idea is to prove the existence of an element $u \in W_0^m E_A(\Omega)$ which satisfies also (1.2) in a sense that will be clarified.

First, we shall prove the following result.

Proposition 6.1. *Let $A(u) = \int_0^{|u|} a(t)\,dt$ be an N-function which satisfies the Δ_2-condition. Suppose that the function $\frac{a(t)}{t}$ is nondecreasing on $(0,\infty)$. Then*

$$J_a : W_0^m E_A(\Omega) \to \big(W_0^m E_A(\Omega)\big)^*$$

is a bijection, with monotone, bounded and continuous inverse.

Proof. According to Theorem 3.14, $W_0^m E_A(\Omega)$ is uniformly convex (in particular, reflexive) and smooth (Theorem 3.6) The result follows by Corollary 4.4. □

In what follows, we give a meaning of the right member in (1.2) as operator acting from $W_0^m E_A(\Omega)$ into $(W_0^m E_A(\Omega))^*$. To do it, let us first remark that if M_α, $|\alpha| < m$, are the N-functions, then, the space

$$X = \bigcap_{|\alpha|<m} W^{m-1} L_{M_\alpha}(\Omega). \tag{6.1}$$

is complete with respect to the norm

$$\|u\|_X = \sum_{|\alpha|<m} \|u\|_{W^{m-1}L_{M_\alpha}(\Omega)}. \tag{6.2}$$

Indeed, if $(u_n)_n$ is a Cauchy sequence in X, then this sequence is Cauchy in $W^{m-1}L_{M_\alpha}(\Omega)$, for each α with $|\alpha| < m$, therefore there exists $v_\alpha \in W^{m-1}L_{M_\alpha}(\Omega)$, $|\alpha| < m$, such that

$$\|u_n - v_\alpha\|_{W^{m-1}L_{M_\alpha}(\Omega)} \to 0 \text{ as } n \to \infty,$$

and α with $|\alpha| < m$. Then

$$\|u_n - v_\alpha\|_{(M_\alpha)} \to 0 \quad \text{as } n \to \infty,$$

and α with $|\alpha| < m$. Consequently, since Ω is bounded, we have the imbeddings

$$L_{M_\alpha}(\Omega) \to L^1(\Omega), \quad \text{for } |\alpha| < m,$$

therefore taking into account the uniqueness of the limit in $L^1(\Omega)$, we can set $u = v_\alpha$, $|\alpha| < m$. Obviously, $u \in X$ and $\|u_n - u\|_X \to 0$ as $n \to \infty$, that is X is complete.

Proposition 6.2. *Let Ω be any domain in $\mathbb{R}^N$. Let $A(u) = \int_0^{|u|} a(t)\,dt$ be an N-function, which satisfies the conditions* (2.7) *and* (2.8). *Let $m \in \mathbb{N}^*$ be given. Suppose that, for each α with $|\alpha| < m$, there exists an N-function M_α which increases essentially more slowly than A_* (the Sobolev conjugate of A) near infinity. Then, the imbedding*

$$W_0^m E_A(\Omega) \overset{i}{\hookrightarrow} X = \bigcap_{|\alpha|<m} W^{m-1} L_{M_\alpha}(\Omega)$$

exists and is compact.

Proof. If $u \in W_0^m E_A(\Omega)$, then $u \in E_A(\Omega)$, $D^\beta u \in E_A(\Omega)$, $|\beta| \le m$ and $D^\beta u = 0$ on $\partial\Omega$, $|\beta| \le m-1$. Therefore, for each β with $|\beta| \le m-1$, $D^\beta u \in W_0^1 E_A(\Omega)$, since, if $|\beta| \le m-1$, $D^\beta u \in E_A(\Omega)$, the first order derivatives of the function $D^\beta u$ are of the form $D^\alpha u$, with $|\alpha| = |\beta| + 1 \le m$. Or $D^\alpha u \in E_A(\Omega)$, $|\alpha| \le m$ and $D^\beta u = 0$ on $\partial\Omega$ for $|\beta| \le m-1$. Therefore, if $|\beta| \le m-1$, $D^\beta u \in W_0^1 E_A(\Omega)$. Under the hypotheses of proposition 6.2, by applying Theorem 2.12, it follows that the imbeddings $W_0^1 E_A(\Omega) \to L_{M_\alpha}(\Omega)$, $|\alpha| < m$, exist and are compact. Consequently, for a fixed α, $|\alpha| < m$, $D^\beta u \in L_{M_\alpha}(\Omega)$, if $|\beta| < m$, that is $u \in W^{m-1}L_{M_\alpha}(\Omega)$, therefore $u \in \bigcap_{|\alpha|<m} W^{m-1}L_{M_\alpha}(\Omega)$. Moreover, from the continuity of the imbeddings $W_0^1 E_A(\Omega) \to L_{M_\alpha}(\Omega)$, $|\alpha| < m$, it follows that there exists a positive constant C, such that

$$\|u\|_X \le C\|u\|_{m,A}.$$

On the other hand, for $|\beta| \le m-1$, we have

$$\|D^\beta u\|_{W_0^1 E_A(\Omega)} \le \|u\|_{W_0^m E_A(\Omega)}.$$

Now, let $(u_n)_n$ be a bounded sequence in $W_0^m E_A(\Omega)$. Then $(D^\beta u_n)_n$ is bounded in $W_0^1 E_A(\Omega)$, for any β, $|\beta| \le m-1$. Since the imbeddings $W_0^1 E_A(\Omega) \to L_{M_\alpha}(\Omega)$, $|\alpha| < m$, are compact, for each $|\beta| < m$, $(D^\beta u_n)_n$ is precompact in $L_{M_\alpha}(\Omega)$, for any α. Consequently, the sequence $(D^\beta u_n)_n$ contains a subsequence $(D^\beta u_{n_k})_k$ which converges in $L_{M_\alpha}(\Omega)$, for any α, $|\alpha| < m$. By finite induction, one can select a subsequence, also denoted $(u_{n_k})_k$ of $(u_n)_n$, such that for $|\beta| < m$, $(D^\beta u_{n_k})_k$ converges in $L_{M_\alpha}(\Omega)$, for all α, $|\alpha| < m$. Therefore

$$D^\beta u_{n_k} \to u_{\beta\ ,_\alpha} \in L_{M_\alpha}(\Omega) \text{ as } k \to \infty.$$

Since Ω is bounded, the imbedding $L_{M_\alpha}(\Omega) \to L^1(\Omega)$ is continuous. It follows that $u_{\beta_\alpha} = u_\beta$, for each α. Therefore,

$$D^\beta u_{n_k} \to u_\beta, \quad k \to \infty,\ \forall \beta,\ |\beta| \le m-1,\ u_\beta \in L_{M_\alpha}(\Omega),\ \forall \alpha, |\alpha| \le m-1.$$

If $u_{n_k} \to u$ as $k \to \infty$ and $D^\beta u_{n_k} \to u_\beta$ as $k \to \infty$, $0 < |\beta| \le m-1$, then, by using the continuity of distributional derivation, it follows that $D^\beta u_{n_k} \to D^\beta u$ as $k \to \infty$, therefore $u_\beta = u$. Thus $(u_{n_k})_k$ converges to $u \in W^{m-1}L_{M_\alpha}(\Omega)$, for every α with $|\alpha| < m$. Therefore, $u \in X$ and $u_{n_k} \to u$ as $k \to \infty$ in X. □

Proposition 6.3. *Let $m \in \mathbb{N}^*$ be given and let $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, be the Carathéodory functions. Suppose that, for each α with $|\alpha| < m$, there exists an N-function M_α, such that M_α and $\overline{M}_\alpha$ satisfy the Δ_2-condition, such that*

$$|g_\alpha(x,s)| \le c_\alpha(x) + d_\alpha \overline{M}_\alpha^{-1}\left(M_\alpha(s)\right), \quad x \in \Omega, s \in \mathbb{R}, |\alpha| < m, \tag{6.3}$$

where $c_\alpha \in K_{\overline{M}_\alpha}(\Omega)$ and d_α is a positive constant. Let us consider the space X given by (6.1), endowed with the norm (6.2). Then, the operator $N : X \to X^*$,

$$(Nu)(h) = \sum_{|\alpha|<m} \int_\Omega g_\alpha(x, D^\alpha u(x)) D^\alpha h(x)dx, \tag{6.4}$$

is well-defined and continuous.

Proof. Indeed, if $u \in \bigcap_{|\alpha|<m} W^{m-1}L_{M_\alpha}(\Omega)$, then, for any α, with $|\alpha| < m$, we have $u \in W^{m-1}L_{M_\alpha}(\Omega)$; therefore, if $|\alpha| < m$, it follows that, for any β, with $|\beta| < m$, $D^\beta u \in L_{M_\alpha}(\Omega) = E_{M_\alpha}(\Omega)$, since M_α satisfy the Δ_2-condition. Consequently

$$\begin{aligned} &\bigcap_{|\alpha|<m} W^{m-1}L_{M_\alpha}(\Omega) \\ &= \big\{u \in \bigcap_{|\alpha|<m} L_{M_\alpha}(\Omega) \mid D^\beta u \in \bigcap_{|\alpha|<m} L_{M_\alpha}(\Omega), \forall |\beta| \le m-1\big\}. \end{aligned}$$

Using Theorem 5.3, it follows that $g_\alpha(x, D^\alpha u(x)) \in K_{\overline{M}_\alpha}(\Omega)$, $|\alpha| < m$. Therefore, N is well-defined.

Now, we will show that the operator N is continuous. Let $u \in X$ and $(u_n)_n \subset X$ be such that $\|u_n - u\|_X \to 0$. Then, for each α with $|\alpha| < m$, we have

$$\|u_n - u\|_{W^{m-1}L_{M_\alpha}(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Consequently, for any β, with $|\beta| \leq |\alpha| < m$, we have

$$\|D^\beta u_n - D^\beta u\|_{(M_\alpha)} \to 0 \text{ as } n \to \infty.$$

In particular

$$\|D^\alpha u_n - D^\alpha u\|_{(M_\alpha)} \to 0 \quad \text{as } n \to \infty,$$

for any α, with $|\alpha| < m$. Using Theorem 5.3, we obtain

$$\|N_{g_\alpha}(D^\alpha u_n) - N_{g_\alpha}(D^\alpha u)\|_{(\overline{M}_\alpha)} \to 0 \text{ as } n \to \infty,$$

for each α, with $|\alpha| < m$. Consequently, by using the generalized Hölder inequality (2.3), we have

$$\begin{aligned}
&|N(u_n)(h) - N(u)(h)| \\
&= \sum_{|\alpha|<m} \int_\Omega \left[g_\alpha(x, D^\alpha u_n(x)) - g_\alpha(x, D^\alpha u(x))\right] D^\alpha h(x) dx \\
&\leq 2 \sum_{|\alpha|<m} \|N_{g_\alpha}(D^\alpha u_n) - N_{g_\alpha}(D^\alpha u)\|_{(\overline{M}_\alpha)} \|D^\alpha h\|_{(M_\alpha)} \\
&\leq 2 \sum_{|\alpha|<m} \|N_{g_\alpha}(D^\alpha u_n) - N_{g_\alpha}(D^\alpha u)\|_{(\overline{M}_\alpha)} \|h\|_{(W^{m-1}L_{M_\alpha})} \\
&\leq 2 \Big(\sum_{|\alpha|<m} \|N_{g_\alpha}(D^\alpha u_n) - N_{g_\alpha}(D^\alpha u)\|_{(\overline{M}_\alpha)} \Big) \|h\|_X;
\end{aligned}$$

therefore

$$\|N(u_n) - N(u)\|_{X^*} \leq 2 \sum_{|\alpha|<m} \|N_{g_\alpha}(D^\alpha u_n) - N_{g_\alpha}(D^\alpha u)\|_{(\overline{M}_\alpha)} \to 0$$

as $n \to \infty$, that is the operator N is continuous. □

Let us suppose that the hypotheses of Propositions 6.2 and 6.3 are satisfied. Then, the diagram

$$W_0^m E_A(\Omega) \overset{i}{\hookrightarrow} \bigcap_{|\alpha|<m} W^{m-1}L_{M_\alpha}(\Omega) \overset{N}{\to} \Big(\bigcap_{|\alpha|<m} W^{m-1}L_{M_\alpha}(\Omega) \Big)^* \overset{i^*}{\hookrightarrow} \Big(W_0^m E_A(\Omega) \Big)^* \tag{6.5}$$

shows that $i^* \circ N \circ i$ is a compact operator from $W_0^m E_A(\Omega)$ to $(W_0^m E_A(\Omega))^*$.

An element $u \in W_0^m E_A(\Omega)$ is said to be *solution* of problem (1.2), (1.3) if

$$J_a u = (i^* \circ N \circ i)(u) \tag{6.6}$$

in the sense of $\big(W_0^m E_A(\Omega)\big)^*$ i.e.

$$\langle J_a u, h \rangle = \langle (i^* \circ N \circ i)(u), h \rangle,$$

for all $h \in W_0^m E_A(\Omega)$ or

$$\frac{a\big(\|u\|_{m,A}\big) \cdot \int_\Omega a\big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\big) \frac{T[u,h]}{\sqrt{T[u,u]}}\, dx}{\int_\Omega a\big(\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\big) \frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}\, dx} = \sum_{|\alpha|<m} \int_\Omega g_\alpha(x, D^\alpha u(x)) D^\alpha h(x) dx \tag{6.7}$$

for all $h \in W_0^m E_A(\Omega)$.

Problem (1.2), (1.3) reduces to a fixed point problem with compact operator. Indeed, by Proposition 6.1, the operator $J_a^{-1} : \big(W_0^m E_A(\Omega)\big)^* \to W_0^m E_A(\Omega)$ is bounded and continuous. Consequently, (6.6) can be equivalently written

$$u = \big(J_a^{-1} \circ i^* \circ N \circ i\big)(u)$$

with $J_a^{-1} \circ i^* \circ N \circ i : W_0^m E_A(\Omega) \to W_0^m E_A(\Omega)$ being a compact operator. We will use the "a priory estimate method" in order to establish the existence of a fixed point for the compact operator $P = J_a^{-1} \circ i^* \circ N \circ i : W_0^m E_A(\Omega) \to W_0^m E_A(\Omega)$.

Theorem 6.4. *Let $A(u) = \int_0^{|u|} a(t)\,dt$ be an N-function, which satisfies the Δ_2-condition and (2.7), (2.8). Suppose that $\frac{a(t)}{t}$ is nondecreasing on $(0,\infty)$. Let $m \in \mathbb{N}^*$ be given and let $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, be Carathéodory functions. Assume that, for each α with $|\alpha| < m$, there exists an N-function M_α which increases essentially more slowly than A_* near infinity and satisfies the Δ_2-condition, such that the growth conditions (6.3) hold. If*

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)}, \quad |\alpha| < m$$

and $\gamma = \max_{|\alpha|<m} \gamma_\alpha$ satisfies

$$\gamma < p_0 = \inf_{t>0} \frac{ta(t)}{A(t)},$$

then the operator $P = J_a^{-1} \circ i^* \circ N \circ i$ has a fixed point in $W_0^m E_A(\Omega)$ or equivalently, problem (1.2), (1.3) has a solution. Moreover, the solution set of problem (1.2), (1.3) is compact in $W_0^m E_A(\Omega)$.

Proof. In order to prove that the compact operator P has a fixed point, it suffices to prove that the set

$$\mathcal{S} = \{u \in W_0^m E_A(\Omega) \mid \exists t \in [0,1] \text{ such that } u = tPu\}$$

is bounded in $W_0^m E_A(\Omega)$. To do this, a technical lemma is needed. □

Lemma 6.5. *Let $A(u) = \int_0^{|u|} a(t)\,dt$ be an N-function.*

(a) *If $p_0 = \inf_{t>0} \frac{ta(t)}{A(t)}$, then for any $t > 1$, one has $A(t) \ge A(1)t^{p_0}$;*

(b) *If A satisfies the Δ_2-condition and*

$$p^* = \sup_{t>0} \frac{ta(t)}{A(t)},$$

then $\infty > p^* > 1$ and for any $u \in L_A(\Omega)$ with $\|u\|_{(A)} > 1$, one has

$$\int_\Omega A(u(x))dx \le \|u\|_{(A)}^{p^*}. \tag{6.8}$$

Proof. (a) First, we remark that, from Young's equality, we have

$$\frac{ta(t)}{A(t)} > 1, \quad \textit{for any } t > 0, \tag{6.9}$$

therefore $p_0 \ge 1$. Integrating the inequality

$$\frac{a(\tau)}{A(\tau)} \ge \frac{p_0}{\tau}, \quad \tau > 0.$$

over the interval $[1, t]$, we obtain

$$A(t) \geq A(1)t^{p_0}, \quad \text{for } t > 1. \tag{6.10}$$

(b) According to (6.9), $p^* > 1$. Since A satisfies the Δ_2-condition, $kA(t) \geq A(2t) > ta(t)$; therefore

$$\frac{ta(t)}{A(t)} < k, \quad \text{with } k > 2.$$

Thus, p^* is finite and

$$\frac{a(\tau)}{A(\tau)} \leq \frac{p^*}{\tau}, \quad \tau > 0.$$

Now, let u be such that $\|u\|_{(A)} > 1$. Integrating over the interval $[\frac{|u(x)|}{\|u\|_{(A)}}, |u(x)|]$, we obtain

$$A(u(x)) \leq A(\frac{|u(x)|}{\|u\|_{(A)}})\|u\|_{(A)}^{p^*}. \tag{6.11}$$

Integrating over Ω and taking into account that

$$\int_\Omega A(\frac{|u(x)|}{\|u\|_{(A)}})dx = 1,$$

it follows (6.7).

Now, let $u \in \mathcal{S}$, $u = tJ_a^{-1}(i^*Ni)u$, $t \in (0, 1]$. Then $J_a(\frac{u}{t}) = (i^*Ni)u$, therefore (see (6.7)), we have

$$\langle J_a(\frac{u}{t}), \frac{u}{t}\rangle = \frac{1}{t}\langle (i^*Ni)u, u\rangle = \frac{1}{t}\sum_{|\alpha|<m}\int_\Omega g_\alpha(x, D^\alpha u(x))D^\alpha u(x)dx. \tag{6.12}$$

On the other hand, we have the following estimate (see (6.3)):

$$\begin{aligned}
&\Big|\sum_{|\alpha|<m}\int_\Omega g_\alpha(x, D^\alpha u(x))D^\alpha u(x)dx\Big| \\
&\leq \sum_{|\alpha|<m}\int_\Omega |g_\alpha(x, D^\alpha u(x))||D^\alpha u(x)|\,dx \\
&\leq \sum_{|\alpha|<m}\int_\Omega \Big[c_\alpha(x) + d_\alpha \overline{M}_\alpha^{-1}(M_\alpha(D^\alpha u(x)))\Big]|D^\alpha u(x)|\,dx \\
&\leq \sum_{|\alpha|<m}\int_\Omega c_\alpha(x)|D^\alpha u(x)|\,dx + \sum_{|\alpha|<m} d_\alpha \int_\Omega \overline{M}_\alpha^{-1}(M_\alpha(D^\alpha u(x)))\,|D^\alpha u(x)|\,dx.
\end{aligned} \tag{6.13}$$

Now, for any N-function M_α one has the inequality

$$M_\alpha^{-1}(t)\overline{M}_\alpha^{-1}(t) \leq 2t, \quad \forall t \geq 0;$$

therefore, setting $t = M_\alpha(s)$, we obtain $s \cdot \overline{M}_\alpha^{-1}(M_\alpha(s)) \leq 2M_\alpha(s)$. Consequently, we have

$$\overline{M}_\alpha^{-1}(M_\alpha(D^\alpha u(x)))\,|D^\alpha u(x)| \leq 2M_\alpha(|D^\alpha u(x)|). \tag{6.14}$$

Then, taking into account (6.12), (6.13) and (6.14), it follows that

$$t \cdot |\langle J_a(\frac{u}{t}), \frac{u}{t}\rangle| \leq \sum_{|\alpha|<m}\int_\Omega c_\alpha(x)|D^\alpha u(x)|\,dx + 2\sum_{|\alpha|<m} d_\alpha \int_\Omega M_\alpha(|D^\alpha u(x)|)dx;$$

therefore, we have

$$\begin{aligned} &\frac{1}{t}\|u\|_{m,A}\cdot a\big(\frac{1}{t}\|u\|_{m,A}\big)=\langle J_a(\frac{u}{t}),\frac{u}{t}\rangle \\ &\le \frac{1}{t}\sum_{|\alpha|<m}\int_\Omega c_\alpha(x)|D^\alpha u(x)|\,dx+\frac{2}{t}\sum_{|\alpha|<m}d_\alpha\int_\Omega M_\alpha(|D^\alpha(u(x))|)dx \end{aligned} \tag{6.15}$$

However, for $|\alpha|<m$, we have

$$\|D^\alpha u\|_{(M_\alpha)}\le \|u\|_{W^{m-1}L_{M_\alpha}(\Omega)}\le \|u\|_X\le c\|u\|_{m,A}, \tag{6.16}$$

the space X being given by (6.1). Therefore, from Holder's inequality, we have

$$\Big|\int_\Omega c_\alpha(x)|D^\alpha u(x)|\,dx\Big|\le 2\|c_\alpha\|_{(\overline{M}_\alpha)}\|D^\alpha u\|_{(M_\alpha)}\le 2c\|c_\alpha\|_{(\overline{M}_\alpha)}\|u\|_{m,A}. \tag{6.17}$$

On the other hand, if $\|D^\alpha u\|_{(M_\alpha)}\le 1$, then (see (6.16))

$$\int_\Omega M_\alpha(D^\alpha u(x))dx\le \|D^\alpha u\|_{(M_\alpha)}\le c\|u\|_{m,A}.$$

If $\|D^\alpha u\|_{(M_\alpha)}>1$, then from (6.8),

$$\int_\Omega M_\alpha(D^\alpha(u(x)))dx\le \|D^\alpha(u)\|^{\gamma_\alpha}_{(M_\alpha)}\le c^\gamma\|u\|^\gamma_{m,A},$$

with $\gamma\ge 1$. Consequently, if $u\in W^m_0E_A(\Omega)$, we have

$$\int_\Omega M_\alpha(D^\alpha u(x))dx\le c^\gamma\|u\|^\gamma_{m,A}+c\|u\|_{m,A},|\alpha|<m. \tag{6.18}$$

Then, from (6.15), (6.17), (6.18), we obtain

$$\frac{1}{t}\|u\|_{m,A}\cdot a\big(\frac{1}{t}\|u\|_{m,A}\big)\le \frac{D}{t}\|u\|^\gamma_{m,A}+\frac{E}{t}\|u\|_{m,A},$$

where $D=2c^\gamma\sum_{|\alpha|<m}d_\alpha$ and $E=2c\sum_{|\alpha|<m}\big(d_\alpha+\|c_\alpha\|_{(\overline{M}_\alpha)}\big)$.

On the other hand, from Young's equality and (6.10), we obtain

$$\|\frac{u}{t}\|_{m,A}\cdot a\big(\|\frac{u}{t}\|_{m,A}\big)\ge A\big(\|\frac{u}{t}\|_{m,A}\big)\ge A(1)\|\frac{u}{t}\|^{p_0}_{m,A}=\frac{A(1)}{t^{p_0}}\|u\|^{p_0}_{m,A};$$

thus

$$\frac{A(1)}{t^{p_0}}\|u\|^{p_0}_{m,A}\le \frac{D}{t}\|u\|^\gamma_{m,A}+\frac{E}{t}\|u\|_{m,A}. \tag{6.19}$$

where $D=2c^\gamma\sum_{|\alpha|<m}d_\alpha$and $E=2c\sum_{|\alpha|<m}\big(d_\alpha+\|c_\alpha\|_{(\overline{M}_\alpha)}\big)$. Consequently

$$A(1)\|u\|^{p_0}_{m,A}\le Dt^{p_0-1}\|u\|^\gamma_{m,A}+Et^{p_0-1}\|u\|_{m,A}\le D\|u\|^\gamma_{m,A}+E\|u\|_{m,A}; \tag{6.20}$$

therefore

$$A(1)\|u\|^{p_0-1}_{m,A}-D\|u\|^{\gamma-1}_{m,A}-E\le 0. \tag{6.21}$$

We remark that, since $\gamma<p_0$, inequality (6.21) implies that there exists a constant C such that $\|u\|_{m,A}\le C$. □

7. An existence result for (1.2), (1.3), via mountain pass theorem

In this section, the existence of weak solution for the problem (1.2), (1.3) will be proved by a variational method. First, we recall a version of the Mountain Pass Theorem ([2]) as given in ([28]).

Theorem 7.1. *Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$ with $I(0) = 0$. Suppose that the following conditions hold:*

(G1) *There exist $\rho > 0$ and $r > 0$ such that $I(u) \geq r$ for $\|u\| = \rho$;*

(G2) *There exists $e \in X$ with $\|e\| > \rho$ such that $I(e) \leq 0$.*

Let

$$\Gamma = \{\gamma \in C([0,1]; X); \gamma(0) = 0, \gamma(1) = e\}$$

and set

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) \quad (c \geq r).$$

Then, there is a sequence $(u_n)_n$ in X such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \text{ as } n \to \infty.$$

Remark 7.2. Let A be the N-function given by (1.1). Suppose that A satisfies the Δ_2-condition. Then, according to Lemma 6.5(b),

$$\infty > p^* = \sup_{u > 0} \frac{ua(u)}{A(u)} > 1.$$

Therefore,

$$\frac{a(u)}{A(u)} \leq \frac{p^*}{u}, u > 0.$$

Integrating over the interval $[t_1, t_2]$, $t_2 > t_1 > 0$, we obtain

$$\frac{A(t_2)}{A(t_1)} \leq \big(\frac{t_2}{t_1}\big)^{p^*}. \tag{7.1}$$

In particular, for $0 < t < 1$, one obtains

$$A(t) \geq A(1) t^{p^*} \tag{7.2}$$

and for $1 < t$ it follows that

$$A(t) \leq A(1) t^{p^*}. \tag{7.3}$$

Lemma 7.3. *Let A be an N-function which satisfies the Δ_2 -condition. Then, there exists a positive constant C, such that*

$$\int_\Omega A\left(D^\alpha u(x)\right) dx \leq C \int_\Omega A\left(\sqrt{T[u,u](x)}\right) dx, \tag{7.4}$$

for all $u \in W_0^m E_A(\Omega)$ and any α with $|\alpha| < m$.

Proof. Indeed, from Proposition 2.10 and the left hand side of inequality (1.4), we obtain

$$\int_\Omega A(D^\alpha)\, dx \leq s c_m \int_\Omega A\big(\frac{c_{m,\Omega}}{\sqrt{c_1}} \sqrt{T[u,u](x)}\big)\, dx, \tag{7.5}$$

where s is the number of the multi-index α with $|\alpha| = m$. Let r be such that $2^r \geq \frac{c_{m,\Omega}}{\sqrt{c_1}}$. Since A satisfies the Δ_2-condition, from (7.5) it follows that

$$\int_\Omega A(D^\alpha u)\, dx \leq s c_m k^r \int_\Omega A\left(\sqrt{T[u,u](x)}\right) dx$$

and lemma is proved. ☐

Lemma 7.3 allows us to define

$$\lambda_\alpha = \inf_{u\in W_0^m E_A(\Omega),\, u\neq 0} \frac{\int_\Omega A\left(\sqrt{T[u,u](x)}\right)\,\mathrm{d}x}{\int_\Omega A\left(D^\alpha u(x)\right)\,dx}, \quad |\alpha|<m, \tag{7.6}$$

for any α with $|\alpha|<m$.

It is easy to see that $(\min_{|\alpha|<m}\lambda_\alpha)^{-1}$ is the best constant C in writing inequality (7.4).

Our goal is to prove the following result.

Theorem 7.4. *Let $A(u)=\int_0^u a(t)\,dt$ be an N-function, which satisfies the conditions* (2.7) *and* (2.8)*. Assume that A and $\overline{A}$, the complementary N-function to A, satisfy the Δ_2-condition. Also, let $g_\alpha:\Omega\times\mathbb{R}\to\mathbb{R}$, $|\alpha|<m$, be Carathéodory functions with primitives*

$$G_\alpha(x,s)=\int_0^s g_\alpha(x,\tau)\,d\tau. \tag{7.7}$$

Let us consider the numerical characteristics

$$p_0=\inf_{t>0}\frac{ta(t)}{A(t)},\quad p^*=\sup_{t>0}\frac{ta(t)}{A(t)},\quad p_*=\liminf_{t\to\infty}\frac{tA_*'(t)}{A_*(t)}, \tag{7.8}$$

A_* being the Sobolev conjugate of A.

Suppose that the following conditions hold:

(H1) *there exists a positive constant $C>0$ such that*

$$A(t)\ge C\cdot t^{p_0}, \forall t\in(0,1); \tag{7.9}$$

(H2) *there exist the N-functions M_α, $|\alpha|<m$, which increase essentially more slowly than A_* near infinity and satisfy the Δ_2-condition, such that*

$$|g_\alpha(x,s)|\le c_\alpha + d_\alpha \overline{M}_\alpha^{-1}\left(M_\alpha(s)\right), x\in\Omega, s\in\mathbb{R}, |\alpha|<m, \tag{7.10}$$

where $\overline{M}_\alpha$ are the complementary N-functions to M_α and c_α, d_α are positive constants;

(H3)

$$\limsup_{s\to 0}\frac{g_\alpha(x,s)}{a(s)}<\frac{C\lambda_\alpha}{2N_0}, |\alpha|<m, \tag{7.11}$$

uniformly for almost all $x\in\Omega$, where λ_α are given by (7.6) and $N_0=\sum_{|\alpha|<m}1$;

(H4) *there exist $s_\alpha>0$ and $\theta_\alpha>p^*$ such that*

$$0<\theta_\alpha G_\alpha(x,s)\le s g_\alpha(x,s), \quad \text{for a.e. } x\in\Omega \tag{7.12}$$

and all s with $|s|\ge s_\alpha$ and p^* is given by (7.8);

(H5) $p_0<p_*$.

Then, the problem (1.2), (1.3) *has non-trivial weak solutions in $W_0^m E_A(\Omega)$.*

To prove the theorem, the Mountain Pass Theorem will be applied to the functional $F: W_0^m E_A(\Omega)\to\mathbb{R}$,

$$F(u)=A\big(\|u\|_{m,A}\big)-\sum_{|\alpha|<m}\int_\Omega G_\alpha\left(x,D^\alpha u(x)\right)\,dx. \tag{7.13}$$

Proposition 7.5. *Under the hypotheses of Theorem 7.4, the functional F given by (7.13), is well-defined and $\mathcal{C}^1$ on $W_0^m E_A(\Omega)$.*

Proof. First we shall prove that the functional F is well-defined. This reduces to proving that for any α with $|\alpha| < m$ and any $u \in W_0^m E_A(\Omega)$, $\int_\Omega G_\alpha(x, D^\alpha u(x))\, dx$ makes sense. Indeed, by using (H2) it follows that for any α with $|\alpha| < m$ one has

$$\begin{aligned} |G_\alpha(x,s)| &\le \Big| \int_0^s \Big(c_\alpha + d_\alpha \overline{M}_\alpha^{-1}(M_\alpha(\tau))\Big)\, d\tau \Big| \\ &\le c_\alpha |s| + d_\alpha |s| \overline{M}_\alpha^{-1}(M_\alpha(|s|)), \end{aligned} \tag{7.14}$$

since $\overline{M}_\alpha^{-1}$ and M_α are strictly increasing.

On the other hand, any N-function M_α satisfies

$$M_\alpha^{-1}(t)\overline{M}_\alpha^{-1}(t) \le 2t, \forall t \ge 0$$

and then, from (7.14), one obtains

$$|G_\alpha(x,s)| \le c_\alpha |s| + 2d_\alpha M_\alpha(|s|). \tag{7.15}$$

Thus

$$\int_\Omega G_\alpha(x, D^\alpha u(x))\, dx \le c_\alpha \int_\Omega |D^\alpha u(x)|\, dx + 2d_\alpha \int_\Omega M_\alpha(|D^\alpha u(x)|)\, dx.$$

Since, for $u \in W_0^m E_A(\Omega)$, $D^\alpha u \in E_A(\Omega) \hookrightarrow L^1(\Omega)$, it follows that $\int_\Omega |D^\alpha u(x)|\, dx$ makes sense. Hypothesis (H2) allows us to apply Theorem 2.12. Consequently, $W_0^1 E_A(\Omega) \hookrightarrow L_{M_\alpha}(\Omega) = K_{M_\alpha}(\Omega)$. Since $u \in W_0^m E_A(\Omega)$ and $|\alpha| < m$, we infer that $u \in W_0^1 E_A(\Omega)$, therefore $D^\alpha u \in K_{M_\alpha}(\Omega)$. Consequently, $\int_\Omega M_\alpha(|D^\alpha u(x)|)\, dx$ makes sense.

Now, we shall show that $F \in \mathcal{C}^1$ over $W_0^m E_A(\Omega)$. To do this, we write F as

$$F = \Phi - \Psi,$$

where

$$\Phi(u) = A(\|u\|_{m,A}) \tag{7.16}$$

and

$$\Psi(u) = \sum_{|\alpha|<m} \int_\Omega G_\alpha(x, D^\alpha u(x))\, dx, \tag{7.17}$$

and show that both Φ and Ψ are $\mathcal{C}^1$. As far as Φ is concerned, it follows from Theorem 3.6 that Φ is continuously Fréchet differentiable at any $u \neq 0$ and

$$\langle \Phi'(u), h\rangle = a(\|u\|_{m,A}) \cdot \frac{\int_\Omega a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big) \frac{T[u,h](x)}{\sqrt{T[u,u](x)}}\, dx}{\int_\Omega a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big) \frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\, dx}. \tag{7.18}$$

If $u = 0$, then a direct calculus shows that Φ is Gâteaux differentiable at zero and

$$\langle \Phi'(0), h\rangle = \lim_{t\to 0} \frac{A(|t|\|h\|_{m,A})}{t} = \lim_{t\to 0} a(|t|\|h\|_{m,A}) \cdot sgnt \cdot \|h\|_{m,A} = 0.$$

Moreover, $u \to \Phi'(u)$ is continuous at zero. We start by showing that for any $u \neq 0$,

$$\int_\Omega a\left(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\right) \frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\, dx \ge 1. \tag{7.19}$$

Indeed, from Young's equality and Proposition 3.11, we obtain

$$\begin{aligned}
&\int_\Omega a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\,dx\\
&= \int_\Omega A\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\,dx + \int_\Omega \overline{A}\Big(a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\Big)\,dx\\
&= 1 + \int_\Omega \overline{A}\Big(a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\Big)\,dx \geq 1.
\end{aligned}$$

On the other hand, by using Schwarz's inequality for nonnegative bilinear symmetric forms and Hölder's inequality (2.3), it follows that

$$\begin{aligned}
&|\int_\Omega a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\frac{T[u,h](x)}{\sqrt{T[u,u](x)}}\,dx|\\
&\leq |\int_\Omega a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\sqrt{T[h,h](x)}\,dx| \\
&\leq 2\|h\|_{m,A}\|a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\|_{(\overline{A})}.
\end{aligned} \tag{7.20}$$

From (7.19) and (7.20) we infer that

$$|\langle\Phi'(u),h\rangle| \leq 2a(\|u\|_{m,A})\|h\|_{m,A}\big\|a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big)\big\|_{(\overline{A})}, \tag{7.21}$$

for all $u\neq 0$, and all $h\in W_0^mE_A(\Omega)$. Now, we shall show that, for any $u\in W_0^mE_A(\Omega)\backslash\{0\}$,

$$\big\|a\big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big)\big\|_{(\overline{A})} \leq k+1, \tag{7.22}$$

the constant k occurring in expressing the Δ_2-property of A (see (2.1)). Indeed, for any real t, one has

$$\overline{A}(a(t)) \leq A(2t) \leq kA(t).$$

Consequently, for any $v\in E_A(\Omega)$,

$$\int_\Omega \overline{A}\left(a(v(x))\right)\,dx \leq k\int_\Omega A\left(v(x)\right)\,dx.$$

In particular, for $v=\frac{\sqrt{T[u,u]}}{\|u\|_{m,A}}$ with $u\in W_0^mE_A(\Omega)\backslash\{0\}$, one obtains

$$\int_\Omega \overline{A}\Big(a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\Big)\,dx \leq k\int_\Omega A\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\,dx = k.$$

Consequently,

$$\begin{aligned}
\|a\Big(\frac{\sqrt{T[u_n,u_n](x)}}{\|u_n\|_{m,A}}\Big)\|_{(\overline{A})} &\leq \|a\Big(\frac{\sqrt{T[u_n,u_n](x)}}{\|u_n\|_{m,A}}\Big)\|_{\overline{A}}\\
&\leq \int_\Omega \overline{A}\Big(a\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big)\Big)\,dx + 1 \leq k+1.
\end{aligned}$$

From (7.21) and (7.22) it follows that

$$|\langle\Phi'(u),h\rangle| \leq 2(k+1)a(\|u\|_{m,A})\|h\|_{m,A};$$

thus

$$\|\Phi'(u)\| \le 2(k+1)a(\|u\|_{m,A}) \to 0 \quad \text{as } \|u\|_{m,A} \to 0.$$

To conclude that F is $\mathcal{C}^1$, the $\mathcal{C}^1$-property of Ψ has to be proved. This is a direct consequence of Theorem 5.4, which also gives us the expression of $\Psi'(u)$:

$$\langle \Psi'(u), h\rangle = \sum_{|\alpha|<m} \int_\Omega g_\alpha\left(x, D^\alpha u(x)\right) D^\alpha h(x)\, dx. \tag{7.23}$$

□

In the next lemma, we shall verify the mountain pass theorem conditions.

Lemma 7.6. *Under the hypotheses of Theorem 7.4, the functional F given by* (7.13) *has the geometry of the mountain pass theorem.*

Proof. We will prove that the hypothesis (G1) of the mountain pass theorem is fulfilled. For the first term in (7.13), according to (H1), we have

$$A(\|u\|_{m,A}) \ge C\|u\|_{m,A}^{p_0}, \tag{7.24}$$

if $\|u\|_{m,A} < 1$.

In what follows, we shall assume that $\|u\|_{m,A} < 1$. We shall now handle the estimations for the second term in (7.13). From (H3) we deduce that for any α with $|\alpha| < m$ there exist $\mu_\alpha \in \big(0, \frac{C\lambda_\alpha}{2N_0}\big)$ and $s_\alpha > 0$ such that

$$G_\alpha(x,s) < \mu_\alpha A(s), \quad \text{for } x \in \Omega,\ 0 < |s| < s_\alpha. \tag{7.25}$$

Indeed, from (7.11) it follows that for any α with $|\alpha| < m$, we can find $\mu_\alpha \in \big(0, \frac{C\lambda_\alpha}{2N_0}\big)$ and $s_\alpha > 0$ such that

$$\frac{g_\alpha(x,s)}{a(s)} < \mu_\alpha, \quad \text{for } x \in \Omega, 0 < |s| < s_\alpha.$$

The above inequalities imply

$$g_\alpha(x,s) < \mu_\alpha a(s), \quad \text{for } x \in \Omega, s \in (0, s_\alpha) \tag{7.26}$$

and, since a is odd

$$g_\alpha(x,s) > -\mu_\alpha a(|s|), \quad \text{for } x \in \Omega, s \in (-s_\alpha, 0). \tag{7.27}$$

Thus

$$|g_\alpha(x,s)| < \mu_\alpha a(|s|), \quad \text{for } x \in \Omega, |s| < s_\alpha. \tag{7.28}$$

(Clearly, (7.28) imply $g_\alpha(x,0) = 0$, for $x \in \Omega$ and any α with $|\alpha| < m$. Consequently, the problem (1.2), (1.3) admits the trivial solution.)

By integrating in (7.26), from 0 to $s \in (0, s_\alpha)$, we obtain that (7.25) is true for $0 < s < s_\alpha$. For $s \in (-s_\alpha, 0)$, taking into account (7.27) and the oddness of a, we have

$$\begin{aligned} G_\alpha(x,s) &= -\int_s^0 g_\alpha(x,\tau)\,d\tau \\ &< \mu_\alpha \int_s^0 a(|\tau|)\,d\tau \\ &= -\mu_\alpha \int_{-s}^0 a(t)\,dt \\ &= \mu_\alpha \int_0^{|s|} a(t)\,dt = \mu_\alpha A(s), \end{aligned}$$

showing that (7.25) is true for $s \in (-s_\alpha, 0)$ too. Now, let us consider $|s| \in [s_\alpha, +\infty)$. The function $\frac{M_\alpha(s)}{s}$ being increasing, we have

$$|s| \le \frac{s_\alpha}{M_\alpha(s_\alpha)} \cdot M_\alpha(|s|).$$

From (7.15), it follows that

$$|G_\alpha(x,s)| \le C_\alpha M_\alpha(|s|), \quad \text{for } |s| \ge s_\alpha, \tag{7.29}$$

where $C_\alpha = c_\alpha \frac{s_\alpha}{M_\alpha(s_\alpha)} + 2d_\alpha$. Since M_α increase essentially more slowly than A_* near infinity, we have

$$\lim_{s\to\infty} \frac{M_\alpha(s)}{A_*(ks)} = 0, \quad \forall k > 0,$$

in particular

$$\lim_{s\to\infty} \frac{M_\alpha(s)}{A_*(s)} = 0.$$

Consequently, there exist $s'_\alpha > s_\alpha$ such that

$$M_\alpha(s) \le C'_\alpha A_*(s), \quad \forall s \ge s'_\alpha \tag{7.30}$$

The definition of p_* implies that there exists $\mu \in (0, p_* - p_0)$ and $s''_\alpha > s'_\alpha$ such that

$$\frac{A'_*(s)}{A_*(s)} \ge \frac{p_* - \mu}{s}, \quad \text{for } s \ge s''_\alpha. \tag{7.31}$$

Let us denote by $S_\alpha = \max(s'_\alpha, s''_\alpha)$, $k_\alpha = \frac{S_\alpha}{s_\alpha} > 1$, $|\alpha| < m$. Taking into account Theorem 2.12, there exists a positive constant K such that

$$\|D^\alpha u\|_{(A_*)} \le K \|u\|_{m,A}, \tag{7.32}$$

for any α with $|\alpha| < m$. If we suppose that

$$\|u\|_{m,A} < \frac{1}{\big(\max_{|\alpha|<m} k_\alpha\big) K}, \tag{7.33}$$

then, it follows from (7.32) that for any α with $|\alpha| < m$,

$$k_\alpha \|D^\alpha u\|_{(A_*)} < 1. \tag{7.34}$$

In what follows we assume that

$$\|u\|_{m,A} < \min\Big(1, \frac{1}{\big(\max_{|\alpha|<m} k_\alpha\big) K}\Big). \tag{7.35}$$

Consequently, the inequalities (7.34) permit us to define for $|\alpha| < m$, the intervals

$$\big[k_\alpha |D^\alpha u(x)|, \frac{|D^\alpha u(x)|}{\|D^\alpha u\|_{(A_*)}}\big], \quad x \in \Omega. \tag{7.36}$$

Now, if we denote $\Omega_\alpha = \{x \in \Omega : |D^\alpha u(x)| \geq s_\alpha\}$, $|\alpha| < m$, then for any $x \in \Omega_\alpha$, we have $k_\alpha |D^\alpha u(x)| \geq s'_\alpha$. Consequently, from (7.30)

$$M_\alpha\left(|D^\alpha u(x)|\right) \leq M_\alpha\left(k_\alpha |D^\alpha u(x)|\right) \leq C'_\alpha A_*\left(k_\alpha |D^\alpha u(x)|\right), \ \forall x \in \Omega_\alpha. \tag{7.37}$$

At the same time, if $x \in \Omega_\alpha$, then $k_\alpha |D^\alpha u(x)| \geq s''_\alpha$, therefore, integrating (7.31) over the intervals (7.36) , we obtain

$$A_*\left(k_\alpha |D^\alpha u(x)|\right) \leq k_\alpha^{p_* - \mu} \|D^\alpha u\|_{(A_*)}^{p_* - \mu} \cdot A_*\big(\frac{|D^\alpha u(x)|}{\|D^\alpha u\|_{(A_*)}}\big). \tag{7.38}$$

for any $x \in \Omega_\alpha$.

Integrating on Ω_α and taking into account the inequality $\int_\Omega A_*\big(\frac{v(x)}{\|v\|_{(A)}}\big)\,dx \leq 1$, we find that

$$\int_{\Omega_\alpha} A_*\left(k_\alpha |D^\alpha u(x)|\right)\,dx \leq k_\alpha^{p_* - \mu} \|D^\alpha u\|_{(A_*)}^{p_* - \mu}, \tag{7.39}$$

for any α with $|\alpha| < m$. Consequently, for every $u \in W_0^m E_A(\Omega)$ satisfying (7.35) and α with $|\alpha| < m$, by using successively (7.29), (7.37), (7.39), we have

$$\begin{aligned} \int_{\Omega_\alpha} G_\alpha\left(x, D^\alpha u(x)\right)\,dx &\leq C_\alpha C'_\alpha \int_{\Omega_\alpha} A_*\left(k_\alpha |D^\alpha u(x)|\right)\,dx \\ &\leq C_\alpha C'_\alpha k_\alpha^{p_* - \mu} \|D^\alpha u\|_{(A_*)}^{p_* - \mu}. \end{aligned} \tag{7.40}$$

Thus, taking into account (7.32), we obtain

$$\sum_{|\alpha| < m} \int_{\Omega_\alpha} G_\alpha\left(x, D^\alpha u(x)\right)\,dx \leq D \cdot \|u\|_{m,A}^{p_* - \mu}, \tag{7.41}$$

where $D = \sum_{|\alpha| < m} D_\alpha$, $D_\alpha = C_\alpha C'_\alpha k_\alpha^{p_* - \mu} K^{p_* - \mu}$, $|\alpha| < m$.

On the other hand, for any α with $|\alpha| < m$, from (7.25) and the definition of λ_α, we deduce

$$\begin{aligned} \int_{\Omega \setminus \Omega_\alpha} G_\alpha\left(x, D^\alpha u(x)\right)\,dx &\leq \mu_\alpha \int_\Omega A\left(D^\alpha u(x)\right)\,dx \\ &\leq \frac{\mu_\alpha}{\lambda_\alpha} \int_\Omega A\left(\sqrt{T[u,u](x)}\right)\,dx \\ &< \frac{C}{2N_0} \int_\Omega A\left(\sqrt{T[u,u](x)}\right)\,dx. \end{aligned} \tag{7.42}$$

From the definition of p_0, we have

$$\frac{a(t)}{A(t)} \geq \frac{p_0}{t}, \quad \forall t > 0. \tag{7.43}$$

Since $\|u\|_{m,A} < 1$, one can consider the interval $\big[\sqrt{T[u,u](x)}, \frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\big]$. By integrating in (7.43) over this interval, we obtain

$$A\left(\sqrt{T[u,u](x)}\right) \leq \|u\|_{m,A}^{p_0} \cdot A\Big(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\Big).$$

By integrating again this inequality on Ω, we find that

$$\int_\Omega A\left(\sqrt{T[u,u](x)}\right)\,dx \le \|u\|_{m,A}^{p_0}. \tag{7.44}$$

Consequently, taking into account (7.42) and (7.44) and summing by α, we have

$$\sum_{|\alpha|<m}\int_{\Omega\setminus\Omega_\alpha} G_\alpha\left(x, D^\alpha u(x)\right)\,dx < \frac{C}{2}\|u\|_{m,A}^{p_0}. \tag{7.45}$$

Then, from (7.24), (7.41), (7.45), we obtain

$$F(u) > C\|u\|_{m,A}^{p_0} - \frac{C}{2}\|u\|_{m,A}^{p_0} - D\cdot\|u\|_{m,A}^{p_*-\mu} = \|u\|_{m,A}^{p_0}\big[\frac{C}{2} - D\|u\|_{m,A}^{p_*-\mu-p_0}\big].$$

So, for

$$\|u\|_{m,A} = \rho \le \min\Big(1, \frac{1}{(\max_{|\alpha|<m} k_\alpha)K}, (\frac{C}{3D})^{\frac{1}{p_*-\mu-p_0}}\Big)$$

it follows that $F(u) > \frac{C}{6}\rho^p > 0$.

Now, we shall verify the hypothesis (G2) of the mountain pass theorem. Let θ_α and s_α be as in (H4). We shall deduce that for any α with $|\alpha|<m$, one has

$$G_\alpha(x,s) \ge \gamma_\alpha(x)\cdot|s|^{\theta_\alpha}, \quad \text{for a.e. } x\in\Omega \text{ and } |s|\ge s_\alpha, \tag{7.46}$$

where the functions γ_α, $|\alpha|<m$, will be specified below.

Indeed, from (7.12) it follows that for any α with $|\alpha|<m$,

$$G_\alpha(x,s) > 0, \quad \text{for a.e. } x\in\Omega \text{ and } |s|\ge s_\alpha. \tag{7.47}$$

Then, for a.e. $x\in\Omega$ and $\tau\ge s_\alpha$, from (7.12), we have

$$\frac{\theta_\alpha}{\tau} \le \frac{g_\alpha(x,\tau)}{G_\alpha(x,\tau)}.$$

Integrating from s_α to $s\ge s_\alpha$, it follows that

$$\frac{s^{\theta_\alpha}}{s_\alpha^{\theta_\alpha}} \le \frac{G_\alpha(x,s)}{G_\alpha(x,s_\alpha)},$$

which implies

$$G_\alpha(x,s) \ge G_\alpha(x,s_\alpha)\cdot\frac{s^{\theta_\alpha}}{s_\alpha^{\theta_\alpha}}, \quad \text{for a.e. } x\in\Omega \text{ and } s\ge s_\alpha, \tag{7.48}$$

for any α with $|\alpha|<m$. On the other hand, for a.e. $x\in\Omega$ and $\tau\le -s_\alpha$, from (7.12) and (7.47), we have

$$\frac{\theta_\alpha}{\tau} \ge \frac{g_\alpha(x,\tau)}{G_\alpha(x,\tau)}.$$

Integrating from $s\le -s_\alpha$ to $-s_\alpha$, it follows that

$$\frac{s_\alpha^{\theta_\alpha}}{|s|^{\theta_\alpha}} \ge \frac{G_\alpha(x,-s_\alpha)}{G_\alpha(x,s)},$$

which implies

$$G_\alpha(x,s) \ge G_\alpha(x,-s_\alpha)\cdot\frac{|s|^{\theta_\alpha}}{s_\alpha^{\theta_\alpha}}, \quad \text{for a.e. } x\in\Omega \text{ and } s\le -s_\alpha, \tag{7.49}$$

for any α with $|\alpha| < m$. Setting

$$\gamma_\alpha(x) = \begin{cases} \frac{G_\alpha(x,s_\alpha)}{s_\alpha^{\theta_\alpha}} & \text{if } s \geq s_\alpha \\ \frac{G_\alpha(x,-s_\alpha)}{s_\alpha^{\theta_\alpha}} & \text{if } s \leq -s_\alpha, \end{cases}$$

from (7.48) and (7.49), we obtain (7.46).

For $\lambda \geq 1$, $|\alpha| < m$ and $u \in W_0^m E_A(\Omega)$, we define

$$\Omega_\lambda^\alpha(u) = \{x \in \Omega \mid \lambda|D^\alpha u(x)| \geq s_\alpha\}.$$

We choose a function $u \in W_0^m E_A(\Omega)$ such that $|D^\alpha u(x)| > 0$, for a.e. $x \in \Omega$, and $\operatorname{vol}\big(\Omega_1^\alpha(u)\big) > 0$, $|\alpha| < m$. It is clear that $\Omega_1^\alpha(u) \subset \Omega_\lambda^\alpha(u)$ and hence $\operatorname{vol}\big(\Omega_1^\alpha(u)\big) \leq \operatorname{vol}\big(\Omega_\lambda^\alpha(u)\big)$, for all $\lambda \geq 1$.

We shall show that $F(\lambda u) \to -\infty$ as $\lambda \to \infty$. For a fixed α with $|\alpha| < m$ and $\lambda \geq 1$, we have

$$\begin{aligned} \int_\Omega G_\alpha\left(x, \lambda D^\alpha u(x)\right) dx &= \int_{\Omega_\lambda^\alpha(u)} G_\alpha\left(x, \lambda D^\alpha u(x)\right) dx \\ &\quad + \int_{\Omega\setminus\Omega_\lambda^\alpha(u)} G_\alpha\left(x, \lambda D^\alpha u(x)\right) dx. \end{aligned}$$

Using (7.46), we obtain

$$\begin{aligned} \int_{\Omega_\lambda^\alpha(u)} G_\alpha\left(x, \lambda D^\alpha u(x)\right) dx &\geq \lambda^{\theta_\alpha} \int_{\Omega_\lambda^\alpha(u)} \gamma_\alpha(x)|D^\alpha u(x)|^{\theta_\alpha}\, dx = \lambda^{\theta_\alpha} K_\alpha(D^\alpha), \\ &\geq \lambda^{\theta_\alpha} \int_{\Omega_1^\alpha(u)} \gamma_\alpha(x)|D^\alpha u(x)|^{\theta_\alpha}\, dx = \lambda^{\theta_\alpha} K_\alpha(D^\alpha), \end{aligned}$$

with

$$K_\alpha(D^\alpha) = \int_{\Omega_1^\alpha(u)} \gamma_\alpha(x)|D^\alpha u(x)|^{\theta_\alpha}\, dx > 0.$$

On the other hand, if $x \in \Omega \setminus \Omega_\lambda^\alpha(u)$, then $\lambda D^\alpha u(x) < s_\alpha$ and by virtue of (7.15), we have

$$\int_{\Omega\setminus\Omega_\lambda^\alpha(u)} \left|G_\alpha\left(x, \lambda D^\alpha u(x)\right)\right| dx \leq \left(c_\alpha s_\alpha + 2d_\alpha M_\alpha(s_\alpha)\right) vol\Omega = K'_\alpha;$$

therefore

$$\int_{\Omega\setminus\Omega_\lambda^\alpha(u)} G_\alpha\left(x, \lambda D^\alpha u(x)\right) dx \geq -K'_\alpha.$$

Consequently,

$$F(\lambda u) \leq A\left(\lambda\|u\|_{m,A}\right) - \sum_{|\alpha|<m} \lambda^{\theta_\alpha} K_\alpha(D^\alpha) + \sum_{|\alpha|<m} K'_\alpha.$$

From (7.3), it follows that for $\|u\|_{m,A} > 1$, we have

$$A\left(\lambda\|u\|_{m,A}\right) \leq A(1)\lambda^{p^*}\|u\|_{m,A}^{p^*} - \sum_{|\alpha|<m} \lambda^{\theta_\alpha} K_\alpha(D^\alpha) + \sum_{|\alpha|<m} K'_\alpha.$$

Since $\theta_\alpha > p^*$ for any α with $|\alpha| < m$, it follows that $F(\lambda u) \to -\infty$ as $\lambda \to \infty$. Consequently, for large λ , say $\lambda \geq \lambda_0$, $F(\lambda u) < 0$. Then, setting $e = \lambda_0 u$, we have $F(\lambda_0 u) < 0$ for some $\lambda_0 > 1$ and the second hypothesis of the mountain pass theorem satisfied. □

Lemma 7.7. *Under the hypotheses of Theorem 7.4, the functional F given by (7.13) has the following property: any sequence $(u_n)_n \subset W_0^m E_A(\Omega)$ for which $(F(u_n))_n$ is bounded and $F'(u_n) \to 0$ as $n \to \infty$, is bounded.*

Proof. Let $(u_n)_n \subset W_0^m E_A(\Omega)$ be a sequence such that $(F(u_n))_n$ is bounded and $F'(u_n) \to 0$ as $n \to \infty$. We shall show that the sequence $(u_n)_n$ is bounded in $W_0^m E_A(\Omega)$. Indeed, let us put

$$\theta = \min_{|\alpha|<m} \theta_\alpha.$$

Since $F = \Phi - \Psi$ with Φ and Ψ given by (7.16) and (7.17) respectively and $F' = \Phi' - \Psi'$ with Φ' and Ψ' given by (7.18) and (7.23) respectively, we have

$$\begin{aligned} &F(u_n) - \frac{1}{\theta} F'(u_n)(u_n) \\ &= A(\|u_n\|_{m,A}) - \frac{1}{\theta}\|u_n\|_{m,A} a(\|u_n\|_{m,A}) \\ &\quad + \sum_{|\alpha|<m} \int_\Omega \big[\frac{1}{\theta} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x) - G_\alpha\left(x, D^\alpha u_n(x)\right)\big]\,dx. \end{aligned} \tag{7.50}$$

Since $(F(u_n))_n$ is bounded, it follows that

$$F(u_n) - \frac{1}{\theta} F'(u_n)(u_n) \le M + \frac{\varepsilon_n}{\theta}\|u_n\|_{m,A}, \tag{7.51}$$

with $\varepsilon_n = \|F'(u_n)\| \to 0$ as $n \to \infty$. Now, we shall give an estimation for

$$\sum_{|\alpha|<m} \int_\Omega \big[\frac{1}{\theta} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x) - G_\alpha\left(x, D^\alpha u_n(x)\right)\big]\,dx,$$

occurring in (7.50).

Let n be fixed. For any α with $|\alpha| < m$, define $\Omega_{\alpha,n} = \{x \in \Omega \mid |D^\alpha u_n(x)| > s_\alpha\}$, and $\Omega'_{\alpha,n} = \Omega \backslash \Omega_{\alpha,n}$. Clearly

$$\begin{aligned} &\sum_{|\alpha|<m} \int_\Omega \big[\frac{1}{\theta} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x) - G_\alpha\left(x, D^\alpha u_n(x)\right)\big]\,dx \\ &= \sum_{|\alpha|<m} \int_{\Omega_{\alpha,n}} \big[\frac{1}{\theta} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x) - G_\alpha\left(x, D^\alpha u_n(x)\right)\big]\,dx \\ &\quad + \sum_{|\alpha|<m} \int_{\Omega'_{\alpha,n}} \big[\frac{1}{\theta} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x) - G_\alpha\left(x, D^\alpha u_n(x)\right)\big]\,dx. \end{aligned} \tag{7.52}$$

Taking into account (H4),

$$\int_{\Omega_{\alpha,n}} \big[\frac{1}{\theta} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x) - G_\alpha\left(x, D^\alpha u_n(x)\right)\big]\,dx \ge 0. \tag{7.53}$$

Taking into account (7.15), one has

$$\begin{aligned} \big|\int_{\Omega'_{\alpha,n}} G_\alpha\left(x, D^\alpha u_n(x)\right)\,dx\big| &\le \int_{\Omega'_{\alpha,n}} \left[c_\alpha |D^\alpha u_n(x)| + 2 d_\alpha M_\alpha\left(|D^\alpha u_n(x)|\right)\right]\,dx \\ &\le \left[c_\alpha s_\alpha + 2 d_\alpha M_\alpha\left(s_\alpha\right)\right] vol(\Omega) = K'_\alpha. \end{aligned} \tag{7.54}$$

On the other hand, from (7.10), it follows that

$$\begin{aligned} &\Big|\int_{\Omega'_{\alpha,n}} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x)\,dx\Big| \\ &\le \int_{\Omega'_{\alpha,n}} \Big[c_\alpha |D^\alpha u_n(x)| + d_\alpha \overline{M}_\alpha^{-1} M_\alpha\left(D^\alpha u_n(x)\right)|D^\alpha u_n(x)|\Big]\,dx \\ &\le \Big[c_\alpha s_\alpha + d_\alpha s_\alpha \overline{M}_\alpha^{-1} M_\alpha\left(s_\alpha\right)\Big]\, vol(\Omega) = K_\alpha. \end{aligned} \tag{7.55}$$

Thus

$$\begin{aligned} &\Big|\int_{\Omega'_{\alpha,n}} \big[\frac{1}{\theta} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x) - G_\alpha\left(x, D^\alpha u_n(x)\right)\big]\,dx\Big| \\ &\le \frac{K_\alpha}{\theta} + K'_\alpha = C_\alpha. \end{aligned} \tag{7.56}$$

From (7.53), (7.56) and (7.52), we infer that

$$\sum_{|\alpha|<m} \int_\Omega \big[\frac{1}{\theta} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x) - G_\alpha\left(x, D^\alpha u_n(x)\right)\big]\,dx \ge -\sum_{|\alpha|<m} C_\alpha.$$

Consequently (see (7.50)),

$$F(u_n) - \frac{1}{\theta} F'(u_n)(u_n) \ge A\big(\|u_n\|_{m,A}\big) - -\frac{1}{\theta}\|u_n\|_{m,A}\, a\big(\|u_n\|_{m,A}\big) - \sum_{|\alpha|<m} C_\alpha. \tag{7.57}$$

Comparing (7.57) and (7.51), one obtains

$$A\big(\|u_n\|_{m,A}\big) - \frac{1}{\theta}\|u_n\|_{m,A}\, a\big(\|u_n\|_{m,A}\big) - \sum_{|\alpha|<m} C_\alpha \le M + \frac{\varepsilon_n}{\theta}\|u_n\|_{m,A}. \tag{7.58}$$

By definition of p^*, one has

$$\|u_n\|_{m,A}\, a(\|u_n\|_{m,A}) \le p^* A(\|u_n\|_{m,A}). \tag{7.59}$$

Finally, comparing (7.59) and (7.58), one obtains

$$(1 - \frac{p^*}{\theta}) A(\|u_n\|_{m,A}) \le M_1 + \frac{\varepsilon_n}{\theta}\|u_n\|_{m,A}, \tag{7.60}$$

with $M_1 = M + \sum_{|\alpha|<m} C_\alpha$, for any n.

The last inequalities imply the boundedness of $(u_n)_n$. In the contrary case, passing to a subsequence, we may assume that $\|u_n\|_{m,A} \to \infty$, as $n \to \infty$. Dividing by $\|u_n\|_{m,A}$ in (7.60) and making n to tend to infinity, one obtains a contradiction: $\frac{A(\|u_n\|_{m,A})}{\|u_n\|_{m,A}} \to \infty$ as $n \to \infty$ while $\frac{M_1}{\|u_n\|_{m,A}} + \varepsilon_n \to 0$ as $n \to \infty$. □

Proof of Theorem 7.4. From Lemma 7.6, it follows that the functional F satisfies the hypotheses of the Mountain Pass Theorem. Consequently, there exists a sequence $(u_n)_n$ in $W_0^m E_A(\Omega)$ such that

$$F(u_n) \to c \quad \text{as } n \to \infty, \tag{7.61}$$

$$F'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{7.62}$$

By Lemma 7.7, $(u_n)_n$ is bounded. Consequently, passing to a subsequence, we may assume that $u_n \rightharpoonup u \in W_0^m E_A(\Omega)$ as $n \to \infty$ (where "$\rightharpoonup$" denotes the convergence in the weak topology).

Now, we shall show that

$$\Psi(u_n) \to \Psi(u) \quad \text{as } n \to \infty, \tag{7.63}$$

$$\Psi'(u_n) \to \Psi'(u) \quad \text{as } n \to \infty. \tag{7.64}$$

To do this, put (for any α with $|\alpha| < m$)

$$\Psi_\alpha(v) = \int_\Omega G_\alpha(x, v(x))\, dx, \quad \forall v \in E_A(\Omega). \tag{7.65}$$

According to Theorem 5.4, $\Psi_\alpha \in \mathcal{C}^1$. Since $u_n \rightharpoonup u$ in $W_0^m E_A(\Omega)$ as $n \to \infty$ and the imbedding of $W_0^m E_A(\Omega)$ into $W_0^{m-1} E_A(\Omega)$ is compact (Theorem 2.13), we deduce that

$$D^\alpha u_n \to D^\alpha u \quad \text{as } n \to \infty, \text{ in } E_A(\Omega),$$

for $|\alpha| < m$. Consequently

$$\Psi(u_n) = \sum_{|\alpha|<m} \Psi_\alpha(D^\alpha u_n) \to \sum_{|\alpha|<m} \Psi_\alpha(D^\alpha u) = \Psi(u),$$

as $n \to \infty$. Moreover, for any $h \in W_0^m E_A(\Omega)$ one has (by Theorem 5.4 again)

$$\begin{aligned} |\langle \Psi'(u_n) - \Psi'(u), h\rangle| &\le \sum_{|\alpha|<m} |\langle \Psi'_\alpha(D^\alpha u_n) - \Psi'_\alpha(D^\alpha u), D^\alpha h\rangle| \\ &\le \sum_{|\alpha|<m} \|\Psi'_\alpha(D^\alpha u_n) - \Psi'_\alpha(D^\alpha u)\| \|D^\alpha h\|_{(A)} \\ &\le 2\|h\|_{W_0^m E_A(\Omega)} \sum_{|\alpha|<m} \|\Psi'_\alpha(D^\alpha u_n) - \Psi'_\alpha(D^\alpha u)\|, \end{aligned}$$

which implies

$$\|\Psi'(u_n) - \Psi'(u)\| \le \sum_{|\alpha|<m} \|\Psi'_\alpha(D^\alpha u_n) - \Psi'_\alpha(D^\alpha u)\| \to 0 \quad \text{as } n \to \infty,$$

since $D^\alpha u_n \to D^\alpha u$ as $n \to \infty$, $|\alpha| < m$ and $\Psi_\alpha \in \mathcal{C}^1$.

Since $F'(u_n) = \Phi'(u_n) - \Psi'(u_n) \to 0$ as $n \to \infty$ and $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$, it follows that

$$\Phi'(u_n) \to \Psi'(u) \text{ as } n \to \infty.$$

By convexity,

$$\Phi(u_n) - \Phi(u) \le \Phi'(u_n)\,(u_n - u) = \langle \Phi'(u_n) - \Psi'(u), u_n - u\rangle + \langle \Psi'(u), u_n - u\rangle,$$

which implies

$$\limsup_{n\to\infty} \Phi(u_n) \le \Phi(u).$$

On the other hand, being continuous and convex, Φ is weakly lower semicontinuous; therefore

$$\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n).$$

Thus

$$\Phi(u) = \lim_{n\to\infty} \Phi(u_n) \tag{7.66}$$

and, consequently,

$$\lim_{n\to\infty} F(u_n) = F(u) = c.$$

We will show that $F'(u) = 0$. Again by convexity of Φ, one has

$$\langle \Phi'(u_n) - \Phi'(v), u_n - v\rangle \ge 0, \quad \forall n, \forall v \in W_0^m E_A(\Omega).$$

Setting $n \to \infty$, we obtain

$$\langle \Psi'(u) - \Phi'(v), u - v \rangle \geq 0, \quad \forall v \in W_0^m E_A(\Omega).$$

Setting $v = u - th$, $t > 0$, the above inequality gives

$$\langle \Psi'(u) - \Phi'(u - th), h \rangle \geq 0, \quad \forall h \in W_0^m E_A(\Omega).$$

By letting $t \to 0$, one has

$$\langle \Psi'(u) - \Phi'(u), h \rangle \geq 0, \quad \forall h \in W_0^m E_A(\Omega).$$

Thus $F'(u) = \Phi'(u) - \Psi'(u) = 0$. □

8. Examples

In this section, some examples illustrating the above existence results are given. To do this, some prerequisites are needed.

Lemma 8.1. *Let $A : \mathbb{R} \to \mathbb{R}_+$, $A(t) = \int_0^{|t|} a(s)\,ds$, be an N-function and $\overline{A}$, the complementary N-function of A.*

(i) *Suppose that*

$$p^* = \sup_{t>0} \frac{ta(t)}{A(t)} < \infty \quad \text{and} \quad p_0 = \inf_{t>0} \frac{ta(t)}{A(t)} > 1.$$

Then both A and $\overline{A}$ satisfy the Δ_2-condition.

(ii) *Suppose, in addition, that $p^* < N$ and there are constants $0 < \gamma < N$ and $\delta > 0$ such that*

$$A(t) \geq Ct^\gamma, \quad \forall t \in (0, A^{-1}(\delta)). \tag{8.1}$$

Then A_*, the Sobolev conjugate of A, can be defined.

Proof. (i) Since $p^* < \infty$, A satisfies the Δ_2-condition. (see [22, Theorem 4.1]). Since $p_0 > 1$, $\overline{A}$ satisfies the Δ_2-condition (see [22, Theorem 4.3]).

(ii) It is sufficient to prove that conditions (2.8) and (2.9) are satisfied (see Theorem 2.12). Indeed, it follows from (8.1) that

$$A^{-1}(\tau) \leq c_1 \cdot \tau^{1/\gamma}, \tau \in (0, \delta),$$

with $C_1 = C^{-1/\gamma}$. Consequently,

$$\int_t^\delta \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau \leq c_1 \cdot \frac{N\gamma}{N-\gamma}\Big(\delta^{\frac{N-\gamma}{N\gamma}} - t^{\frac{N-\gamma}{N\gamma}}\Big).$$

Without loss of generality, we may assume that $0 < \delta < 1$ and then

$$\begin{aligned} \lim_{t\to 0} \int_t^1 \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau &= \lim_{t\to 0}\Big(\int_t^\delta \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau + \int_\delta^1 \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau\Big) \\ &\leq c_1 \cdot \frac{N\gamma}{N-\gamma}\delta^{\frac{N-\gamma}{N\gamma}} + \int_\delta^1 \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau < \infty. \end{aligned}$$

Thus (2.8) is satisfied.

To prove that (2.9) is also satisfied, we first remark that, from (7.1), one has $A(t) \leq \frac{t^{p^*}}{(A^{-1}(1))^{p^*}}$, for any $t > A^{-1}(1)$. It follows that $A^{-1}(\tau) \geq c' \cdot \tau^{1/p^*}$, for any $\tau > 1$, with $c' = A^{-1}(1)$. Consequently, for any $t > 1$,

$$\int_1^t \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau \geq c' \cdot \frac{Np^*}{N-p^*}\Big(t^{\frac{N-p^*}{Np^*}} - 1\Big)$$

and, since $p^* < N$,
$$\lim_{t\to\infty}\int_1^t \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau = \infty,$$
thus (2.9) is also satisfied. $\square$

The next lemma summarizes some arguments used in [8, p. 55].

Lemma 8.2. *Let $A:\mathbb{R}\to\mathbb{R}_+$, $A(t)=\int_0^{|t|}a(s)\,ds$, be an N-function, which satisfies the conditions* (2.7) *and (2.8). If $\lim_{t\to\infty}\frac{ta(t)}{A(t)}=l<\infty$, then*
$$\lim_{t\to\infty}\frac{tA_*'(t)}{A_*(t)}=\frac{Nl}{N-l}.$$

Example 8.3. Consider the problem (1.2), (1.3), under the following hypotheses:

(i) the function $a:\mathbb{R}\to\mathbb{R}$ is defined by $a(t)=\sum_{i=1}^n a_i|t|^{p_i-2}t$, where $a_i>0$, $1\le i\le n$, $p_{i+1}>p_i>1$, $1\le i\le n-1$, $p_n<N$;

(ii) The Carathéodory functions $g_\alpha:\Omega\times\mathbb{R}\to\mathbb{R}$, $|\alpha|<m$, satisfy the following conditions:
$$\limsup_{s\to 0}\frac{g_\alpha(x,s)}{a(s)}<\frac{a_1\lambda_\alpha}{2p_1N_0}, \tag{8.2}$$
uniformly for almost all $x\in\Omega$, where λ_α are given by (7.6) and $N_0=\sum_{|\alpha|<m}1$;

(iii) $p_n<N$ and there exist q_α, $1<q_\alpha<\frac{Np_n}{N-p_n}$, $|\alpha|<m$, such that
$$|g_\alpha(x,s)|\le c_\alpha+d_\alpha|s|^{q_\alpha-1},\quad x\in\Omega,\ s\in\mathbb{R}; \tag{8.3}$$

(iv) there exist $s_\alpha>0$ and $\theta_\alpha>p_n$ such that
$$0<\theta_\alpha G_\alpha(x,s)\le sg_\alpha(x,s),\quad \text{for a.e. } x\in\Omega \tag{8.4}$$
and all s with $|s|\ge s_\alpha$.

Under these conditions, the problem (1.2), (1.3) has a nontrivial weak solution.

Proof. Before giving the proof, we underline that the function a given by hypothesis (i) appears in [17], in the following context (see [17, example 3.1]: if a is given by (i) and
$$f(s)=\sum_{j=1}^m\beta_j|s|^{\delta_j-1}s,$$
with $\beta_j>0$, for $j=1,\dots,m$, and $\delta_{j+1}>\delta_j>1$, for $j=1,\dots,m-1$, satisfying $N>p_n$ and
$$p_n-1<\delta_m<\frac{N(p_n-1)+p_n}{N-p_n},$$
then, the problem
$$-\big(r^{N-1}a(u')\big)'=r^{N-1}f(u)\quad\text{in } (0,R)$$
$$u'(0)=0=u(R)$$
has a positive solution and therefore the problem
$$-\operatorname{div}\Big(\frac{a(|Du|)}{|Du|}Du\Big)=f(u)\quad\text{in } \Omega=B_{\mathbb{R}^N}(0,R)$$
$$u=0\quad\text{on } \partial\Omega$$

has a positive radial solution of class $\mathcal{C}^1$.

The idea of the proof is as follows: The preceding assumptions entail that the hypotheses of Theorem 7.4 are fulfilled. To do this, first we compute the numerical characteristics p_0, p^* and p_*, given by (7.8). Let $h : (0,\infty) \to \mathbb{R}$, defined by $h(t) = \frac{ta(t)}{A(t)}$, with

$$A(t) = \sum_{i=1}^{n} \frac{a_i}{p_i} |t|^{p_i}. \tag{8.5}$$

By direct calculus, we obtain $\lim_{t\to 0} h(t) = p_1$ and $p_1 < h(t)$, for all $t > 0$. Thus $p_0 = p_1 > 1$. Analogously, $\lim_{t\to\infty} h(t) = p_n$ and $h(t) < p_n$, for all $t > 0$. Thus $p^* = p_n < N$.

Clearly, from (8.5), it follows that

$$A(t) \geq \frac{a_1}{p_1} t^{p_1}, \forall t > 0. \tag{8.6}$$

Since $p^* = p_n < N$ and (8.6) holds, we deduce (Lemma 8.1, (ii)) that A_* exists. Consequently, according to Lemma 8.2, we can compute p_* and we obtain

$$p_* = \liminf_{t\to\infty} \frac{tA'_*(t)}{A_*(t)} = \frac{Np_n}{N - p_n}.$$

Since $p_0 > 1$ and $p^* < N$, it follows (Lemma 8.1, (i)) that A and $\overline{A}$ satisfy the Δ_2-condition.

Now, we are in position to properly show that the above hypotheses entail the fulfillment of those of Theorem 7.4. Indeed, since $p_1 = p_0$, (8.6) says that (H1) in Theorem 7.4 is satisfied. The hypothesis (H2) in Theorem 7.4 is satisfied with $M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}$, $|\alpha| < m$. Clearly, M_α satisfies the Δ_2 -condition. In order to prove that M_α, $|\alpha| < m$, increase essentially more slowly than A_* near infinity, it follows that [1, p. 231]),

$$\lim_{t\to\infty} \frac{A_*^{-1}(t)}{M_\alpha^{-1}(t)} = 0.$$

Indeed, using l'Hôspital rule,

$$\lim_{t\to\infty} \frac{A_*^{-1}(t)}{M_\alpha^{-1}(t)} = \lim_{t\to\infty} \underline{c}_\alpha \frac{A^{-1}(t)}{t^{\frac{1}{q_\alpha}+\frac{1}{N}}} = \lim_{s\to\infty} \underline{c}_\alpha \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} = 0, \underline{c}_\alpha = q_\alpha^{(q_\alpha-1)/q_\alpha}, \tag{8.7}$$

since from (ii), the degree of denominator is $p_n(\frac{1}{q_\alpha} + \frac{1}{N}) > 1$. To conclude that (H2) in Theorem 7.4 is also satisfied, we have to prove that inequalities (7.10) hold. Indeed, since $\overline{M}_\alpha(s) = \frac{|s|^{q'_\alpha}}{q'_\alpha}$, $\frac{1}{q_\alpha} + \frac{1}{q'_\alpha} = 1$ (see Remark 2.5), then

$$|s|^{q_\alpha - 1} = (q_\alpha - 1)^{\frac{1}{q'_\alpha}} \, \overline{M}_\alpha^{-1} \left(M_\alpha(s)\right).$$

Consequently, (8.3) rewrites as

$$|g_\alpha(x,s)| \leq c_\alpha + d_\alpha \left(q_\alpha - 1\right)^{\frac{1}{q'_\alpha}} \overline{M}_\alpha^{-1} \left(M_\alpha(s)\right), x \in \Omega, s \in \mathbb{R}, |\alpha| < m, \tag{8.8}$$

showing that (H2) is satisfied. Hypotheses (H3) and (H4) in Theorem 7.4 are fulfilled by virtue of (8.2) and (8.4) respectively; since $p^* = p_n$, (8.4) implies the fulfillment of (H4); finally, since $p_0 = p_1 < p_n < \frac{Np_n}{N-p_n} = p_*$, (H5) is satisfied too. □

Example 8.4. Consider the problem (1.2), (1.3), under the following hypotheses:

(i) the function $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = \sum_{i=1}^{n} a_i |t|^{p_i - 2} t$, where $a_i > 0$, $1 \le i \le n$, $p_{i+1} > p_i > 1$, $1 \le i \le n-1$, $p_1 \ge 2$, $p_n < N$;

(ii) there exist q_α, $1 < q_\alpha < p_1$, $|\alpha| < m$, such that the growth conditions (8.3) hold.

Under these conditions, the problem (1.2), (1.3) has a solution. Moreover, the solution set of problem (1.2), (1.3) is compact in $W_0^m E_A(\Omega)$.

Proof. The idea of the proof is as follows: The preceding assumptions entail that the hypotheses of Theorem 6.4 are fulfilled.

Indeed, A satisfies the Δ_2-condition inasmuch as $p^* = p_n < N$ (see Lemma 8.1, (i)). Conditions (2.8) and (2.9) are satisfied (the arguments are those used in the case of Example 8.3). Since $p_i > 2$, $2 \le i \le n$, it easily follows that $\frac{a(t)}{t}$ is strictly increasing for $t > 0$. As M_α functions, which increase essentially more slowly than A_* near infinity and satisfy the Δ_2-condition as well as the growth conditions (6.3), we shall take $M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}$, $|\alpha| < m$ (the arguments are those used for Example 8.3). Finally, the last condition in Theorem 6.4 is satisfied since

$$\gamma_\alpha = \sup_{t>0} \frac{t M_\alpha'(t)}{M_\alpha(t)} = q_\alpha, |\alpha| < m$$

and, by hypothesis (ii), $q_\alpha < p_1 = p_0$. $\square$

Remark 8.5. Comparing the existence results provided by Examples 8.3 and 8.4, it can be seen that the results obtained in Example 8.3 are stronger than those obtained in Example 8.4.

Indeed, under the hypotheses from Example 8.3, we obtain (via the mountain pass theorem) the existence of a nontrivial solution for the problem (1.2), (1.3), while Example 8.4 states only the existence of a solution without specifying if it is nontrivial.

Comparing the hypotheses of these two examples it can be seen that:

- the hypotheses about a in Example 8.4 are stronger than those formulated in Example 8.3;
- one part of the hypotheses about the functions g_α (referring to growth conditions (8.3)) are common to both examples;
- in Example 8.3 are formulated other supplementary conditions about the functions g_α (see (8.2) and (8.4)).

These supplementary conditions have as consequence the fact that functional F (defined by (7.13)) has a mountain pass type geometry.

Example 8.6. Consider the problem (1.2), (1.3), under the following hypotheses:

(i) the function $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = |t|^{p-2} t \sqrt{t^2 + 1}$, $1 < p < N - 1$;

(ii) The Carathéodory functions $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, satisfy the following conditions:
$$\limsup_{s \to 0} \frac{g_\alpha(x,s)}{a(s)} < \frac{\lambda_\alpha}{2pN_0}, \tag{8.9}$$
uniformly for almost all $x \in \Omega$, where λ_α are given by (7.6) and $N_0 = \sum_{|\alpha|<m} 1$;

(iii) there exist q_α, $1 < q_\alpha < \frac{Np}{N-p}$, $|\alpha| < m$, such that the growth conditions (8.3) hold;

(iv) there exist $s_\alpha > 0$ and $\theta_\alpha > p + 1$ such that the conditions (8.4) hold.

Under these conditions, the problem (1.2), (1.3) has a nontrivial weak solution.

Proof. The idea of the proof is that used for Example 8.3, namely, we shall show that the preceding assumptions entail the fulfillment of those of Theorem 7.4. To do this, first we compute the numerical characteristics p_0, p^* and p_*, given by (7.8). Let $h : (0,\infty) \to \mathbb{R}$, defined by $h(t) = \frac{ta(t)}{A(t)}$, with

$$A(t) = \frac{t^p}{p}\sqrt{t^2+1} - \frac{1}{p}\int_0^t \frac{\tau^p}{\sqrt{\tau^2+1}}\,d\tau, t > 0. \tag{8.10}$$

First, by direct calculus, we obtain $\lim_{t\to 0} h(t) = p$ and, since

$$h(t) = p + \frac{pI(t)}{t^p\sqrt{t^2+1} - I(t)}, I(t) = \int_0^t \frac{\tau^p}{\sqrt{\tau^2+1}}\,d\tau, \tag{8.11}$$

one has $p < h(t)$, for all $t > 0$. Consequently $p_0 = p > 1$.

Secondly, one has

$$\frac{pI(t)}{t^p\sqrt{t^2+1} - I(t)} < 1, \quad \forall t > 0. \tag{8.12}$$

Indeed, let $f(t) = (p+1)I(t) - t^p\sqrt{t^2+1}$. Since $f(0) = 0$ and $f'(t) < 0$, for all $t > 0$, inequality (8.12) follows. From (8.11) and (8.12), we infer that $h(t) < p+1$, for all $t > 0$ and, since $\lim_{t\to\infty} h(t) = p+1$, we conclude that $p^* = p+1 < N$.

To compute p_*, first we prove the existence of A_*. Indeed, since $a(t) \geq t^{p-1}$, for all $t \geq 0$, we obtain that

$$A(t) \geq \frac{1}{p}t^p, \quad \forall t \geq 0. \tag{8.13}$$

Condition $p+1 < N$ being also satisfied (by hypothesis), the existence of A_* follows by Lemma 8.1, (ii). According to Lemma 8.2,

$$p_* = \liminf_{t\to\infty} \frac{tA_*'(t)}{A_*(t)} = \frac{N(p+)}{N-p-1}.$$

Since $p_0 > 1$ and $p^* < N$, it follows (Lemma 8.1, (i)) that A and $\overline{A}$ satisfy the Δ_2-condition.

Now, we are in position to properly show that the above hypotheses entail the fulfillment of those of Theorem 7.4. Indeed, since $p = p_0$, (8.13) says that (H1) in Theorem 7.4 is satisfied. The hypothesis (H2) in Theorem 7.4 is satisfied with $M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}$, $|\alpha| < m$, which, obviously, satisfy the Δ_2-condition. Since $a(t) \geq t^p$, for all $t \geq 0$, it follows that $A(t) \geq \frac{t^{p+1}}{p+1}$, for all $t \geq 0$; therefore

$$\lim_{s\to\infty} \frac{s}{\big(A(s)\big)^{\frac{1}{q_\alpha}+\frac{1}{N}}} \leq \lim_{s\to\infty} \frac{s}{(p+)^{\frac{1}{q_\alpha}+\frac{1}{N}}s^{\left(\frac{1}{q_\alpha}+\frac{1}{N}\right)(p+1)}} = 0.$$

Consequently, from (8.7), one obtains $\lim_{t\to\infty} \frac{A_*^{-1}(t)}{M_\alpha^{-1}(t)} = 0$, that is, M_α, $|\alpha| < m$, increase essentially more slowly than A_* near infinity.

As in Example 8.3, (8.8) shows that (H2) is satisfied. Hypotheses (H3) and (H4) in Theorem 7.4 are fulfilled by virtue of (8.9), (8.13) and (8.4) respectively; since $p^* = p+1$, (8.4) implies the fulfillment of (H4); finally, since $p_0 = p < p+1 < \frac{N(p+1)}{N-p-1} = p_*$, (H5) is satisfied too. □

Example 8.7. Consider the problem (1.2), (1.3), under the following hypotheses:

(i) the function $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = |t|^{p-2}t\sqrt{t^2+1}$, $2 \le p < N-1$;
(ii) there exist q_α, $1 < q_\alpha < p$, $|\alpha| < m$, such that the growth conditions (8.3) hold.

Under these conditions, problem (1.2), (1.3) has a solution. Moreover, the solution set of (1.2), (1.3) is compact in $W_0^m E_A(\Omega)$.

Proof. The idea of the proof is as follows: the preceding assumptions entail that the hypotheses of Theorem 6.4 are fulfilled.

Indeed, A satisfies the Δ_2-condition inasmuch as $p^* = p+1 < N$ (see Lemma 8.1, (i)). Conditions (2.8) and (2.9) are satisfied (the arguments are those used for Example 8.6). Since $p \ge 2$, it easily follows that $\frac{a(t)}{t}$ is strictly increasing on $(0,\infty)$. As M_α functions, which increase essentially more slowly than A_* near infinity and satisfy the Δ_2-condition as well as the growth conditions (6.3), we shall take $M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}$, $|\alpha| < m$ (the arguments are those used for Example 8.6). Finally, the last condition in Theorem 6.4 is satisfied since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha, \quad |\alpha| < m$$

and, by hypothesis (ii), $q_\alpha < p = p_0$. □

Example 8.8. Consider problem (1.2), (1.3), under the following hypotheses:

(i) the function $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = |t|^{p-2}t\ln(1+|t|)$, $2 \le p > N-1$;
(ii) there exist q_α, $1 < q_\alpha < p$, $|\alpha| < m$, such that the growth conditions (8.3) hold.

Under these conditions, the problem (1.2), (1.3) has a solution. Moreover, the solution set of problem (1.2), (1.3) is compact in $W_0^m E_A(\Omega)$.

Proof. Before giving the proof, we underline that the function a given by hypothesis (i) appears in [8], in the following context (see [8], example 1 in the introduction): if a is given by (i) and

$$1 < p < N-1 \quad \text{and} \quad p < \delta < \frac{N(p-1)+p}{N-p},$$

then, the problem

$$-\operatorname{div}\big(a(|\nabla u|)\frac{\nabla u}{|\nabla u|}\big) = |u|^{\delta-1}u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

has a nontrivial nonnegative weak solution in $W_0^1 E_A(\Omega)$.

The idea of the proof is as follows: the preceding assumptions entail that the hypotheses of Theorem 6.4 are fulfilled. To do it, first we compute the numerical characteristics p_0 and p^*, given by (7.8). Let $h : (0,\infty) \to \mathbb{R}$, defined by $h(t) = \frac{ta(t)}{A(t)}$, with

$$A(t) = \frac{t^p}{p}\ln(1+t) - \frac{1}{p}\int_0^t \frac{\tau^p}{1+\tau}\,d\tau, \quad t > 0. \tag{8.14}$$

By direct calculus, we obtain $\lim_{t\to 0} h(t) = p+1$ and, since

$$h(t) = p + \frac{pI(t)}{t^p\ln(1+t) - I(t)}, I(t) = \int_0^t \frac{\tau^p}{1+\tau}\,d\tau, \tag{8.15}$$

one has $p < h(t)$, for all $t > 0$. Consequently $p_0 = p > 1$. To compute p^*, we shall show that

$$\frac{pI(t)}{t^p \ln(1+t) - I(t)} < 1, \quad \forall t > 0. \tag{8.16}$$

Indeed, let $f(t) = f(t) = (p+)I(t) - t^p \ln(1+t)$. Since $f(0) = 0$ and $f'(t) < 0$, for all $t > 0$, inequality (8.16) follows. From (8.15) and (8.16), we infer that $h(t) < p+1$, for all $t > 0$ and, since $\lim_{t\to 0} h(t) = p+1$, we conclude that $p^* = p+1 < N$. Therefore, A satisfies the Δ_2-condition inasmuch as $p^* = p+1 < N$ (see Lemma 8.1(i)). On the other hand, by a direct calculus, one has $\lim_{t\to 0} \frac{A(t)}{t^{p+1}} = \frac{1}{p+1}$; therefore, for $\varepsilon = \frac{1}{p+1}$ there exists $\underline{\delta} = A^{-1}(\delta)$ such that

$$A(t) > \frac{2}{p+1} t^{p+1}, \quad \forall t \in (0, \underline{\delta} = A^{-1}(\delta)). \tag{8.17}$$

Since $p^* = p+1 < N$ and (8.17) holds, we deduce (Lemma 8.1, (ii)) that conditions (2.8) and (2.9) are satisfied. Since $p \geq 2$, it easily follows that $\frac{a(t)}{t}$ is strictly increasing on $(0, \infty)$. As M_α functions, which increase essentially more slowly than A_* near infinity and satisfy the Δ_2-condition as well as the growth conditions (6.3), we shall take $M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}$, $|\alpha| < m$. Indeed, from Lemma 6.5, a), it follows that $A(t) \geq A(1)t^p$, $t > 1$, therefore

$$\lim_{s\to\infty} \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} \leq \lim_{s\to\infty} \frac{s}{(A(1))^{\frac{1}{q_\alpha}+\frac{1}{N}} s^{\left(\frac{1}{q_\alpha}+\frac{1}{N}\right)p}} = 0.$$

Consequently (see Example 8.6), the arguments continue. Finally, the last condition in Theorem 6.4 is satisfied since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha, |\alpha| < m$$

and, by hypothesis (ii), $q_\alpha < p = p_0$. □

Remark 8.9. We will show that, under the hypotheses adopted for Example 8.8, Theorem 7.4 cannot be applied. Thus, the hypothesis (H1) of Theorem 7.4 is never verified. Indeed, if $f(t) = A(t) - Ct^p$, then, it is easy to see that, for $t \in (0, t_0)$, $t_0 = e^{Cp} - 1$, we have always $f(t) < 0$.

Example 8.10. Consider the problem (1.2), (1.3), under the following hypotheses:

- (i) the function $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = |t|^{p-2} t \ln(1 + \alpha + |t|)$, $1 < p \leq N - 1$, $\alpha > 0$;
- (i) The Carathéodory functions $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, satisfy the following conditions:
$$\limsup_{s\to 0} \frac{g_\alpha(x,s)}{a(s)} < \frac{\ln(1+\alpha)\cdot\lambda_\alpha}{2pN_0}, \tag{8.18}$$
uniformly for almost all $x \in \Omega$, where λ_α are given by (7.6) and $N_0 = \sum_{|\alpha|<m} 1$;
- (iii) there exist q_α, $1 < q_\alpha < \frac{Np}{N-p}$, $|\alpha| < m$, such that the growth conditions (8.3) hold;
- (iv) there exist $s_\alpha > 0$ and $\theta_\alpha \geq p+1$ such that conditions (8.4) hold.

Under these conditions, the problem (1.2), (1.3) has a nontrivial weak solution.

Proof. The idea of the proof is as follows: the preceding assumptions entail that the hypotheses of Theorem 7.4 are fulfilled. To do this, first we compute the numerical characteristics p_0, p^* and p_*, given by (7.8). Let $h:(0,\infty)\to\mathbb{R}$, defined by $h(t)=\frac{ta(t)}{A(t)}$, with

$$A(t)=\frac{t^p}{p}\ln(1+\alpha+t)-\frac{1}{p}\int_0^t\frac{\tau^p}{1+\alpha+\tau}\,d\tau, t>0. \tag{8.19}$$

By direct calculus, we obtain $\lim_{t\to 0}h(t)=p$ and, since

$$h(t)=p+\frac{pI(t)}{t^p\ln(1+\alpha+t)-I(t)}, I(t)=\int_0^t\frac{\tau^p}{1+\alpha+\tau}\,d\tau, \tag{8.20}$$

one has $p<h(t)$, for all $t>0$. Consequently, $p_0=p>1$. To estimate p^*, we shall prove the existence of a constant $0<C_0<1$, such that

$$p^*=\sup_{t>0}h(t)\le p+C_0.$$

Indeed, the system

$$\begin{gathered} t-C(1+\alpha+t)\ln(1+\alpha+t)=0\\ 1-C-C\ln(1+\alpha+t)=0 \end{gathered} \tag{8.21}$$

admits a unique solution (t_0,C_0). Clearly

$$0<C_0=\frac{1}{1+\ln(1+\alpha+t_0)}<1,$$

where $t_0-(1+\alpha)\ln(1+\alpha+t_0)=0$. Moreover $h(t)\le p+C_0$, for all $t>0$. This is true, since

$$\frac{pI(t)}{t^p\ln(1+\alpha+t)-I(t)}\le C_0, \forall t>0. \tag{8.22}$$

Indeed, let $f(t)=(p+C_0)\,I(t)-t^p\ln(1+\alpha+t)$, for all $t\ge 0$. One has $f'(t)=\frac{pt^{p-1}}{1+\alpha+t}g(t)$ with $g(t)=t-C_0(1+\alpha+t)\ln(1+\alpha+t)$, $t\ge 0$. Since (t_0,C_0) is the unique solution of (8.21), it may easily show that $t_0>0$, $g'(t)=0$ and $g(t_0)=0=\max_{t\ge 0}g(t)$. Consequently, $g(t)\le 0$, $t\ge 0$, which implies $f'(t)\le 0$, for all $t\ge 0$. Since $f(0)=0$, it follows that $f(t)\le 0$, for all $t\ge 0$, which is equivalent with (8.22). This calculus explicit the claim concerning inequality (6.19) in [8].

Clearly, since $a(t)\ge\ln(1+\alpha)\cdot t^{p-1}$, for all $t\ge 0$, it follows that

$$A(t)\ge\frac{\ln(1+\alpha)}{p}t^p, \quad \forall t\ge 0. \tag{8.23}$$

Since $p^*<p+1\le N$ and (8.23) holds, we deduce (Lemma 8.1, (ii)) that A_* exists. Consequently, according to Lemma 8.2, we can compute p_* and we obtain

$$p_*=\liminf_{t\to\infty}\frac{tA_*'(t)}{A_*(t)}=\frac{Np}{N-p}.$$

Since $p_0>1$ and $p^*<N$, it follows (Lemma 8.1, (i)) that A and $\overline{A}$ satisfy the Δ_2-condition.

Now, we are in position to properly show that the above hypotheses entail the fulfillment of those of Theorem 7.4. Indeed, since $p=p_0$, (8.23) says that (H1) in Theorem 7.4 is satisfied. The hypothesis (H2) in Theorem 7.4 is satisfied with

$M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}$, $|\alpha| < m$, which, obviously satisfy the Δ_2-condition. From (8.23) it follows that

$$\lim_{s\to\infty} \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} \le \lim_{s\to\infty} \frac{s}{\left(\frac{\ln(1+\alpha}{p}\right)^{\frac{1}{q_\alpha}+\frac{1}{N}} \cdot s^{\left(\frac{1}{q_\alpha}+\frac{1}{N}\right)p}} = 0.$$

Consequently, from (8.7), one obtains $\lim_{t\to\infty} \frac{A_*^{-1}(t)}{M_\alpha^{-1}(t)} = 0$; that is, M_α, $|\alpha| < m$, increase essentially more slowly than A_* near infinity.

As in Example 8.3, (8.8) shows that (H2) is satisfied. Hypotheses (H3) and (H4) in Theorem 7.4 are fulfilled by virtue of (8.18), (8.13) and (8.4) respectively; since $p^* = p + 1$, (8.4) implies the fulfillment of (H4); finally, since $p_0 = p < \frac{Np}{N-p} = p_*$, (H5) is satisfied too. □

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