Electronic Journal of Differential Equations, Vol. 2007(2007), No. 137, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ON THE SOLVABILITY OF A FOURTH ORDER OPERATOR DIFFERENTIAL EQUATIONS

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ABSTRACT. We establish sufficient conditions for the solvability of a fourth order operator differential equation. These conditions are expressed in terms of operator coefficients.

1. PRELIMINARIES

In this paper we consider the problem

$$P(d/dt)u = \frac{d^4u}{dt^4} + A^4u + \sum_{j=0}^{4} A_j \frac{d^{4-j}u}{dt^{4-j}} = f, \tag{1.1}$$ $\boxed{\texttt{e1}}$

$$u^{(s_i)}(0) = 0, i = 0, 1, \quad s_i \in \{0, 1, 2, 3\}, \tag{1.2}$$ $\boxed{\texttt{e2}}$

where $f \in L_2(\mathbb{R}_+, H)$, and A_0, A_1, A_2, A_3, A_4 and A are unbounded linear operators in a separable Hilbert space H. By $L_2(\mathbb{R}_+, H)$ we denote the space of H-valued vector functions such that $\int_0^{+\infty} \|f(t)\|_H^2 dt < +\infty$, endowed with the inner product

$$\langle f, g\rangle_{L_2(\mathbb{R}_+,H)} = \int_0^{+\infty} \langle f(t), g(t)\rangle_H dt,$$

and $\mathbb{R}_+ = [0, +\infty)$. Let

$$P(\lambda) = \lambda^4 I + A^4 + \sum_{j=0}^{4} \lambda^{4-j} A_j, \tag{1.3}$$ $\boxed{\texttt{e3}}$

be the operator pencil associated with the operator differential equation (1.1).

Our aim is to establish sufficient conditions for the solvability of problem (1.1), (1.2), expressed in terms of the norms of the operator coefficients of the pencil (1.3). For the general theory of operator pencils see [9, 11]. In particular, unbounded operator pencils of second and fourth orders are treated in [12, 13].

Interest for such equations is justified not only by the relation which they have with the practical problems described by ordinary or partial differential equations but also by their important applications in many theoretical problems. Thus, the study of such equations is connected with the multicompleteness of the eigenvectors and the generalized eigenvectors of the corresponding operator pencil. This relation

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2000 *Mathematics Subject Classification.* 47E05, 35K20.
Key words and phrases. Operator Pencil; operator differential equation; normal operator.

Submitted February 24, 2007. Published October 17, 2007.

has been proved in [1] and permits to answer some questions arising while studying non self-adjoint differential operators. The notion of multicompleteness has been introduced by Keldysh in [2, 3] and many interesting results have been obtained during the last decades by Gasymov [1], Yakubov [9, 10, 11, 13], Markus [5], Mirzoev [6, 7] and others.

2. Definitions

def1 **Definition 2.1.** Let A be a normal operator in a Hilbert space H such that $\sigma(A)$ is contained in the sector

$$S_\varepsilon = \{\lambda : |\arg\lambda| < \varepsilon < \frac{\pi}{12}\}, \quad \inf \operatorname{Re}\sigma(A) = \sigma_0 > 0.$$

We assume also that the operators $B_j = A_j A^{-j}$, $j = \overline{0,4}$ are bounded. Under these conditions we will say that the pencil (1.3) belongs to the class Γ.

It is clear that if the operator A satisfies the conditions of the above definition it admits a polar decomposition in the form: $A = UC$, where C is a positively defined self-adjoint operator and U is a unitary operator, therefore C and U commute and we have for any rational k

$$D(C^k) = D(A^k),$$
$$\|A^k x\|_H = \|C^k x\|_H, \quad x \in D(A^k).$$

In $D(C^k)$ we define the inner product

$$\langle x, y\rangle_{H_k} = \langle C^k x, C^k y\rangle_H, \quad x, y \in D(A^k).$$

Then, we obtain a Hilbert space which we denote by H_k. By $D(\mathbb{R}_+, H)$ we denote the pre-Hilbertian space of the infinitely differentiable H_4-valued functions with a compact support in $\mathbb{R}_+$, relative to the inner product

$$\langle u, v\rangle_W = \langle u^{(4)}, v^{(4)}\rangle_{L_2(\mathbb{R}_+,H)} + \langle C^4 u, C^4 v\rangle_{L_2(\mathbb{R}_+,H)}.$$

By $W(\mathbb{R}_+, H)$ we denote the completion of $D(\mathbb{R}_+, H)$ with respect to the above inner product. We introduce also the following notation

$$W(\mathbb{R}_+, H, s_0, s_1) = \{u \in W(\mathbb{R}_+, H), u^{(s_0)}(0) = u^{(s_1)}(0) = 0\},$$
$$M_{4,j}(s_0, s_1) = \sup_{u\in W(\mathbb{R}_+,H,s_0,s_1)} \frac{\|C^{4-j}u^{(j)}\|_{L_2}}{\|u\|_W}.$$

def2 **Definition 2.2.** We say that the couple (s_0, s_1) belongs to the class S if it takes one of these values: $(0, 1), (0, 2), (1, 3), (2, 3)$.

def3 **Definition 2.3.** We say that the problem (1.1), (1.2) is regular if for any function $f \in L_2(\mathbb{R}_+, H)$ there exists a unique function $u \in W(\mathbb{R}_+, H, s_0, s_1)$ which satisfies the equation (1.1) a.e. and the conditions (1.2). We call such a function u a regular solution of the problem (1.1), (1.2).

3. Main Results

We consider now the pencil operator

$$S_j(\lambda, \gamma, C) = \lambda^8 I + C^8 - \gamma(i\lambda)^{2j} C^{8-2j},$$

defined in H_8 and where γ is a real parameter. We use the notation

$$d_{4,j} = \begin{cases} 1, & j = 0, 4, \\ (\frac{4}{j})^{j/4} (\frac{4}{4-j})^{\frac{4-j}{4}}, & j = 1, 2, 3. \end{cases}$$

thm1 **Theorem 3.1.** *For $\gamma \in (0, d_{4,j})$, $j \in \{0, \dots, 4\}$, the pencil operator $S_j(\lambda, \gamma, C)$ can be represented in the form*

$$S_j(\lambda, \gamma, C) = S_j^-(\lambda, \gamma, C) S_j^+(\lambda, \gamma, C),$$

where

$$S_j^-(\lambda, \gamma, C) = \sum_{i=0}^{4} \alpha_{i,j}(\gamma) \lambda^i C^{4-i},$$
$$S_j^+(\lambda, \gamma, C) = S_j^-(-\lambda, \gamma, C).$$

Moreover, the coefficients $\alpha_{i,j}$ are real and satisfy the following relations

$$\begin{gathered} \alpha_{0,0} = \sqrt{1-\gamma}, \\ \alpha_{4,0} = 1, \\ \alpha_{1,0}^2 - 2\alpha_{0,0}\alpha_{2,0} = 0, \\ \alpha_{2,0}^2 - 2\alpha_{1,0}\alpha_{3,0} + 2\alpha_{0,0}\alpha_{4,0} = 0, \\ \alpha_{3,0}^2 - 2\alpha_{2,0}\alpha_{4,0} = 0. \end{gathered} \tag{3.1}$$ e4

and

$$\begin{gathered} \alpha_{0,4} = 1 \\ \alpha_{4,4} = \sqrt{1-\gamma} \\ \alpha_{1,4}^2 - 2\alpha_{0,4}\alpha_{2,4} = 0 \\ \alpha_{2,4}^2 - 2\alpha_{1,4}\alpha_{3,4} + 2\alpha_{0,4}\alpha_{4,4} = 0 \\ \alpha_{3,4}^2 - 2\alpha_{2,4}\alpha_{4,4} = 0, \end{gathered} \tag{3.2}$$ e5

and for $j = 1, 2, 3$

$$\begin{gathered} \alpha_{0,j} = \alpha_{4,j} = 1, \\ \alpha_{1,j}^2 - 2\alpha_{2,j} = \begin{cases} -\gamma, & j = 1, \\ 0, & j \neq 1. \end{cases} \\ \alpha_{2,j}^2 - 2\alpha_{1,j}\alpha_{3,j} + 2 = \begin{cases} -\gamma, & j = 2, \\ 0, & j \neq 2. \end{cases} \\ \alpha_{3,j}^2 - 2\alpha_{2,j} = \begin{cases} -\gamma, & j = 3, \\ 0, & j \neq 3. \end{cases} \end{gathered} \tag{3.3}$$ e6

Proof. For $\gamma \in (0, d_{4,j})$, $j \in \{0, \dots 4\}$ the polynomial $Q_j(\lambda, \gamma) = \lambda^8 + 1 - \gamma(i\lambda)^{2j}$ has not purely imaginary roots, its roots are simple and symmetrically situated relatively to the real axis and the origin. So it can be represented in the form

$$Q_j(\lambda, \gamma) = S_j(\lambda, \gamma) S_j(-\lambda, \gamma). \tag{3.4}$$ e7

The roots of $S_j(\lambda, \gamma)$ belong to the left half-plane and we can denote them by $\omega_1(\gamma)$, $\omega_2(\gamma)$, $\overline{\omega_1(\gamma)}$, $\overline{\omega_2(\gamma)}$ where $\overline{\omega_1(\gamma)}$ and $\overline{\omega_2(\gamma)}$ are the conjugate of $\omega_1(\gamma)$ and $\omega_2(\gamma)$

respectively. In this case

$$S_j(\lambda,\gamma)=(\lambda-\omega_1(\gamma))(\lambda-\omega_2(\gamma))(\lambda-\overline{\omega_1(\gamma)})(\lambda-\overline{\omega_2(\gamma)}).$$

This proves that if we write $S_j(\lambda,\gamma)=\sum_{i=0}^{4}\alpha_{i,j}\lambda^i$, then the coefficients $\alpha_{i,j}$ are real.

The relations (3.1), (3.2) and (3.3) can be obtained by using the Viete theorem and identifying the coefficients in the left and right parts of the relation (3.4). Let E_σ be the spectral decomposition of the operator C. We have

$$\begin{aligned}
S_j(\lambda,\gamma,C)&=\lambda^8 I+C^8-\gamma(i\lambda)^{2j}C^{8-2j}\\
&=\int_{\sigma_0}^{\infty}(\lambda^8+\sigma^8-\gamma(i\lambda)^{2j}\sigma^{8-2j})dE_\sigma\\
&=\int_{\sigma_0}^{\infty}\sigma^8(\frac{\lambda^8}{\sigma^8}+1-\gamma\frac{(i\lambda)^{2j}}{\sigma^{2j}})dE_\sigma\\
&=\int_{\sigma_0}^{\infty}\sigma^8 Q_j(\frac{\lambda}{\sigma},\gamma)dE_\sigma\\
&=\int_{\sigma_0}^{\infty}\sigma^8 S_j(\frac{\lambda}{\sigma},\gamma)S_j(-\frac{\lambda}{\sigma},\gamma)dE_\sigma\\
&=\int_{\sigma_0}^{\infty}\sigma^4 S_j(\frac{\lambda}{\sigma},\gamma)dE_\sigma\int_{\sigma_0}^{\infty}\sigma^4 S_j(-\frac{\lambda}{\sigma},\gamma)dE_\sigma\\
&=(\sum_{i=0}^{4}\alpha_{i,j}\lambda^i C^{4-i})(\sum_{i=0}^{4}\alpha_{i,j}(-\lambda)^i C^{4-i}).
\end{aligned}$$

Then taking

$$S_j^-(\lambda,\gamma,C)=\sum_{i=0}^{4}\alpha_{i,j}\lambda^i C^{4-i},$$

we obtain $S_j(\lambda,\gamma,C)=S_j^-(\lambda,\gamma,C)S_j^+(\lambda,\gamma,C)$. □

thm2 **Theorem 3.2.** *Let $\gamma\in(0,d_{4,j})$ and $u\in W(\mathbb{R}_+,H)$. Then for $j\in\{0,\dots 4\}$,*

$$\|S_j^-(d/dt,\gamma,C)u\|_{L_2}^2=\|u\|_W^2-\gamma\|C^{4-j}u^{(j)}\|_{L_2}^2-\langle G_j\widetilde{f},\widetilde{f}\rangle_{H^4},$$

where

$$\begin{gathered}
\widetilde{f}=\big(C^{7/2}u(0),C^{5/2}u'(0),C^{3/2}u''(0),C^{1/2}u'''(0)\big)\in H\oplus H\oplus H\oplus H=H^4,\\
G_j=\begin{bmatrix}\alpha_1\alpha_0 & \alpha_2\alpha_0 & \alpha_3\alpha_0 & \alpha_4\alpha_0\\ \alpha_2\alpha_0 & \alpha_2\alpha_1-\alpha_3\alpha_0 & \alpha_3\alpha_1-\alpha_4\alpha_0 & \alpha_4\alpha_1\\ \alpha_3\alpha_0 & \alpha_3\alpha_1-\alpha_4\alpha_0 & \alpha_3\alpha_2-\alpha_4\alpha_1 & \alpha_4\alpha_2\\ \alpha_4\alpha_0 & \alpha_4\alpha_1 & \alpha_4\alpha_2 & \alpha_4\alpha_3\end{bmatrix},
\end{gathered}$$

and $\alpha_i\equiv\alpha_{i,j}(\gamma)$.

Proof. By a theorem of the intermediate derivatives [4, pp.19-25], $C^{4-j}u^{(j)}(\cdot) \in L_2(\mathbb{R}_+, H)$ and $u^{(j)}(0) \in D(C^{4-j-\frac{1}{2}})$. We have

$$\begin{aligned}\|S_j^-(d/dt,\gamma,C)u\|^2_{L_2(\mathbb{R}_+,H)} &= \Big\|\sum_{i=0}^{4}\alpha_{i,j}C^{4-i}u^{(i)}\Big\|^2_{L_2(\mathbb{R}_+,H)} \\ &= \sum_{i=0}^{4}\alpha_{i,j}^2\|C^{4-i}u^{(i)}\|^2_{L_2(\mathbb{R}_+,H)} \\ &\quad + 2\sum_{i=1}^{4}\sum_{s=0}^{i-1}\alpha_{i,j}\alpha_{s,j}\operatorname{Re}\langle C^{4-i}u^{(i)}, C^{4-s}u^{(s)}\rangle_{L_2(\mathbb{R}_+,H)}.\end{aligned}$$

Calculating the expressions $\langle C^{4-i}u^{(i)}, C^{4-s}u^{(s)}\rangle_{L_2(\mathbb{R}_+,H)}$ by using successive integrations by parts and taking into account that C is a selfadjoint operator we obtain

$$\begin{aligned}&\|S_j^-(d/dt,\gamma,C)u\|^2_{L_2(\mathbb{R}_+,H)} \\ &= \alpha_{0,j}^2\|C^4u\|^2_{L_2(\mathbb{R}_+,H)} + \alpha_{4,j}^2\|u^{(4)}\|^2_{L_2(\mathbb{R}_+,H)} + (\alpha_{3,j}^2 - 2\alpha_{2,j}\alpha_{4,j})\|Cu^{(3)}\|^2_{L_2(\mathbb{R}_+,H)} \\ &\quad + (\alpha_{2,j}^2 - 2\alpha_{1,j}\alpha_{3,j} + 2\alpha_{0,j}\alpha_{4,j})\|C^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)} \\ &\quad + (\alpha_{1,j}^2 - 2\alpha_{2,j}\alpha_{0,j})\|C^3u^{(1)}\|^2_{L_2(\mathbb{R}_+,H)} - \langle G_j\widetilde{f}, \widetilde{f}\rangle_{H^4}.\end{aligned}$$

Then using the relations (3.1), (3.2) and (3.3) we obtain

$$\|S_j^-(d/dt,\gamma,C)u\|^2_{L_2(\mathbb{R}_+,H)} = \|u\|_W^2 - \gamma\|C^{4-j}u^{(j)}\|^2_{L_2(\mathbb{R}_+,H)} - \langle G_j\widetilde{f}, \widetilde{f}\rangle_{H^4}.$$

$\square$

rmk1 **Remark 3.3.** Theorem 3.2 shows that $M_{4,j}(s_0, s_1)$ can be written with respect to $S_j^-(d/dt,\gamma,C)$ and G_j. So, according to Theorem 3.1 and the definition of G_j, it can be written with respect to γ and $\alpha_{i,j}$. In fact, the pencil $S_j(\lambda,\gamma,C)$ from which we have obtained the pencil $S_j^-(d/dt,\gamma,C)$ has been especially defined to this end.

Let $G_j(s_0, s_1, \gamma)$ be the matrix obtained from the matrix G_j by deleting the lines $s_0 + 1$ and $s_1 + 1$ and the columns $s_0 + 1$ and $s_1 + 1$.

thm3 **Theorem 3.4** ([8]). *Let $j \in \{0, \dots 4\}$, then*

$$M_{4,j}(s_0,s_1) = \begin{cases} d_{4,j}^{-1/2}, & \text{if } \det G_j(s_0,s_1,\gamma) \neq 0, \text{ for any } \gamma \in (0, d_{4,j}) \\ \mu_{4,j}^{-1/2}(s_0,s_1), & \text{otherwise,}\end{cases}$$

where $\mu_{4,j}(s_0, s_1)$ is the smallest positive root from $(0, d_{4,j})$ of the equation

$$\det G_j(s_0,s_1,\gamma) = 0$$

with respect to γ.

rmk2 **Remark 3.5.** Theorem 3.4 implies that to calculate $M_{4,j}(s_0, s_1)$, one must determine the matrix $G_j(s_0, s_1, \gamma)$ from the matrix G_j obtained in Theorem 3.2 and resolve the equation $\det G_j(s_0, s_1, \gamma) = 0$ with respect to γ by using the relations (3.1), (3.2) and (3.3) from Theorem 3.1.

lemme1 **Lemma 3.6.** *Let the operator A be as in Definition 2.1 and $u \in W(\mathbb{R}_+, H, s_0, s_1)$ with $(s_0, s_1) \in S$, then*

$$\|u\|_W \le (1 - 2\sin 2\varepsilon)^{-1/2}\|P_0u\|_{L_2(\mathbb{R}_+,H)},$$

where

$$P_0u = \frac{d^4u}{dt^4} + A^4u.$$

Proof. If $u \in W(\mathbb{R}_+, H, s_0, s_1)$, we have

$$\|P_0u\|^2_{L_2(\mathbb{R}_+,H)} = \|A^4u\|^2_{L_2(\mathbb{R}_+,H)} + \|u^{(4)}\|^2_{L_2(\mathbb{R}_+,H)} + 2\operatorname{Re}\langle A^4u, u^{(4)}\rangle_{L_2(\mathbb{R}_+,H)}.$$

Using integration by parts, we have

$$\begin{aligned}&\langle A^4u, u^{(4)}\rangle_{L_2(\mathbb{R}_+,H)}\\ &= \langle A^{\frac{7}{2}}u, A^{*\frac{1}{2}}u^{(3)}\rangle\big|_0^\infty - \langle A^{\frac{5}{2}}u', A^{*\frac{3}{2}}u^{(2)}\rangle\big|_0^\infty + \langle A^2u^{(2)}, A^{*2}u^{(2)}\rangle_{L_2(\mathbb{R}_+,H)}.\end{aligned}$$

So, for $(s_0, s_1) \in S$, we obtain

$$\|P_0u\|^2_{L_2(\mathbb{R}_+,H)} = \|u\|^2_W + 2\operatorname{Re}\langle A^2u^{(2)}, A^{*2}u^{(2)}\rangle_{L_2(\mathbb{R}_+,H)}.$$

We have

$$\begin{aligned}&\langle A^2u^{(2)}, A^{*2}u^{(2)}\rangle_{L_2(\mathbb{R}_+,H)}\\ &= \langle A^2u^{(2)}, (A^{*2}A^{-2} - I)A^2u^{(2)}\rangle_{L_2(\mathbb{R}_+,H)} + \|A^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)}\\ &\geq \|A^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)} - \|A^{*2}A^{-2} - I\|\|A^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)}.\end{aligned}$$

Now, we have also

$$\begin{aligned}\|A^{*2}A^{-2} - I\| &= \|\int_{\sigma(A)} \big(\frac{\overline{\sigma}^2}{\sigma^2} - 1\big) dE_\sigma\|\\ &= \|\int_{\sigma(A)} \frac{\overline{\sigma}^2 - \sigma^2}{\sigma^2} dE_\sigma\|\\ &= \|\int_{\sigma(A)} \big(\frac{e^{-2i\theta} - e^{2i\theta}}{e^{2i\theta}}\big) dE_\sigma\|\end{aligned}$$

where $\theta = \arg\sigma$. The spectrum $\sigma(A)$ of the operator A is contained in the sector S_ε defined by Definition 2.1 and this means that

$$\big|\frac{e^{-2i\theta} - e^{2i\theta}}{e^{2i\theta}}\big| \leq 2\sin 2\varepsilon,$$

this proves that

$$\langle A^2u^{(2)}, A^{*2}u^{(2)}\rangle_{L_2(\mathbb{R}_+,H)} \geq (1 - 2\sin 2\varepsilon)\|A^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)}.$$

Hence

$$\|P_0u\|^2_{L_2(\mathbb{R}_+,H)} \geq \|u\|^2_W + 2(1 - 2\sin 2\varepsilon)\|A^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)},$$

and so

$$\|P_0u\|^2_{L_2(\mathbb{R}_+,H)} \geq \|u\|^2_W + 2\|C^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)} - 4\sin 2\varepsilon\|C^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)}.$$

Since

$$\begin{aligned}\|C^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)} &= \int_0^\infty \langle C^2u^{(2)}, C^2u^{(2)}\rangle dt\\ &= \langle C^{\frac{5}{2}}u', C^{\frac{3}{2}}u^{(2)}\rangle\big|_0^\infty - \langle C^{\frac{7}{2}}u, C^{\frac{1}{2}}u^{(3)}\rangle\big|_0^\infty + \int_0^\infty \langle C^4u, u^{(4)}\rangle dt\\ &= \int_0^\infty \langle C^4u, u^{(4)}\rangle dt\\ &\le \|C^4u\|_{L_2(\mathbb{R}_+,H)}\|u^{(4)}\|_{L_2(\mathbb{R}_+,H)}\\ &\le \frac{1}{2}\big(\|C^4u\|^2_{L_2(\mathbb{R}_+,H)} + \|u^{(4)}\|^2_{L_2(\mathbb{R}_+,H)}\big)\\ &= \frac{1}{2}\|u\|^2_W\end{aligned}$$

we conclude that

$$\begin{aligned}\|P_0u\|^2_{L_2(\mathbb{R}_+,H)} &\ge (1-2\sin 2\varepsilon)\|u\|^2_W + 2\|C^2u^{(2)}\|^2_{L_2(\mathbb{R}_+,H)}\\ &\ge (1-2\sin 2\varepsilon)\|u\|^2_W,\end{aligned}$$

and finally, we have

$$\|u\|_W \le (1-2\sin 2\varepsilon)^{-\frac{1}{2}}\|P_0u\|_{L_2(\mathbb{R}_+,H)}.$$

□

lemme2 **Lemma 3.7.** *Let the operator A be as in Definition 2.1, then for $(s_0,s_1)\in S$, the operator P_0 is a bijection between $W(\mathbb{R}_+,H,s_0,s_1)$ and $L_2(\mathbb{R}_+,H)$.*

Proof. Consider the problem

$$\begin{gathered}P_0u = g,\\ u^{(s_0)}(0) = u^{(s_1)}(0) = 0,\end{gathered}$$

where $g\in L_2(\mathbb{R}_+,H)$ and $(s_0,s_1)\in S$. The unique solution of this problem is given by (see [1])

$$u(t) = V_1(t)\Psi_0 + U_1(t)\Psi_1 + \int_0^\infty G(t,x)g(x)dx,$$

where $G(t,x)$ is the Green function of the problem

$$\begin{gathered}P_0u = g,\\ u(0) = u^{'}(0) = 0,\end{gathered}$$

and $V_1(t)$ and $U_1(t)$ are analytic semigroups generated by the operators w_1A and w_2A respectively, where $w_1$and w_2 are the roots on the left half plane of the equation $w^4=-1$. The elements $\Psi_0$and $\Psi_1$can be obtained by solving the system

$$\begin{gathered}V_1^{(s_0)}(t)\Psi_0 + U_1^{(s_0)}(t)\Psi_1 + \int_0^\infty G^{(s_0)}(t,x)g(x)dx = 0,\\ V_1^{(s_1)}(t)\Psi_0 + U_1^{(s_1)}(t)\Psi_1 + \int_0^\infty G^{(s_1)}(t,x)g(x)dx = 0.\end{gathered}$$

□

Theorem 3.8. *Let pencil* (1.3) *belongs to the class* Γ *and the couple* (s_0, s_1) *be in the class* S *and*

$$\sum_{j=0}^{4} \beta_j \|A_{4-j} A^{j-4}\| < 1,$$

where $\beta_j = M_{4,j}(s_0, s_1)(1 - 2\sin 2\varepsilon)^{-1/2}$. *Then, problem* (1.1), (1.2) *admits a unique regular solution.*

Proof. According to the Lemma 3.7 we can look for the solution of (1.1), (1.2) in the form $u = P_0^{-1} v$, where $v \in L_2(\mathbb{R}_+, H)$. In this case, problem (1.1), (1.2) takes the form

$$(I + P_1 P_0^{-1}) v = f, \tag{3.5}$$

e8

where $v \in L_2(\mathbb{R}_+, H)$,

$$\begin{gathered} P_0 = \frac{d^4}{dt^4} + A^4, \\ D(P_0) = W(\mathbb{R}_+, H, s_0, s_1), \\ P_1 = \sum_{j=0}^{4} A_{4-j} \frac{d^j}{dt^j}. \end{gathered}$$

A sufficient condition for the existence of a unique solution of (3.5) is that

$$\|P_1 P_0^{-1}\| < 1.$$

Let $h \in L_2(\mathbb{R}_+, H)$, we have

$$\begin{aligned} \|P_1 P_0^{-1} h\|_{L_2(\mathbb{R}_+, H)} &= \|\sum_{j=0}^{4} A_{4-j} \frac{d^j}{dt^j} (P_0^{-1} h)\|_{L_2(\mathbb{R}_+, H)} \\ &= \|\sum_{j=0}^{4} A_{4-j} A^{j-4} A^{4-j} \frac{d^j}{dt^j} (P_0^{-1} h)\|_{L_2(\mathbb{R}_+, H)} \\ &\le \sum_{j=0}^{4} \|A_{4-j} A^{j-4}\| \|A^{4-j} \frac{d^j}{dt^j} (P_0^{-1} h)\|_{L_2(\mathbb{R}_+, H)} \\ &= \sum_{j=0}^{4} \|A_{4-j} A^{j-4}\| \|C^{4-j} \frac{d^j}{dt^j} (P_0^{-1} h)\|_{L_2(\mathbb{R}_+, H)}. \end{aligned}$$

Since $P_0^{-1} h \in W(\mathbb{R}_+, H, s_0, s_1)$,

$$\|C^{4-j} \frac{d^j}{dt^j} (P_0^{-1} h)\|_{L_2(\mathbb{R}_+, H)} \le M_{4,j}(s_0, s_1) \|P_0^{-1} h\|_W.$$

Moreover, according to Lemma 3.6, we have

$$\|P_0^{-1} h\|_W \le (1 - 2\sin 2\varepsilon)^{-\frac{1}{2}} \|h\|_{L_2(\mathbb{R}_+, H)}.$$

In this case

$$\|P_1 P_0^{-1} h\|_{L_2(\mathbb{R}_+, H)} \le \sum_{j=0}^{4} \|A_{4-j} A^{j-4}\| M_{4,j}(s_0, s_1)(1 - 2\sin 2\varepsilon)^{-\frac{1}{2}} \|h\|_{L_2(\mathbb{R}_+, H)}.$$

Then

$$\|P_1 P_0^{-1}\| \leq \sum_{j=0}^{4} M_{4,j}(s_0, s_1)(1 - 2\sin 2\varepsilon)^{-1/2} \|A_{4-j} A^{j-4}\|.$$

In view of the corresponding condition of the theorem, the above inequality implies

$$\|P_1 P_0^{-1}\| < 1.$$

□

rmk3 **Remark 3.9.** The value of $M_{4,j}(s_0, s_1)$ obtained in Theorem 3.4 gives us the best estimation for the norm of the operator $P_1 P_0^{-1}$ in the sense that the coefficients β_j are optimal in the inequality

$$\sum_{j=0}^{4} \beta_j \|A_{4-j} A^{j-4}\| < 1.$$

Acknowledgements. We would like to thank the anonymous referees for their constructive and helpful comments which improved the quality of this paper.

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