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AN APPLICATION OF THE LYAPUNOV-SCHMIDT METHOD TO SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. In this paper we consider the existence of nonzero solutions for the undecoupling elliptic system

$$
\begin{aligned}
-\Delta u &= \lambda u + \delta v + f(u,v),\\
-\Delta v &= \theta u + \gamma v + g(u,v),
\end{aligned}
$$

on a bounded domain of $\mathbb{R}^n$, with zero Dirichlet boundary conditions. We use the Lyapunov-Schmidt method and the fixed-point principle.

1. Introduction

In this present paper we consider the Dirichlet problem

$$
\begin{gathered}
-\Delta u = \lambda u + \delta v + f(u,v) \text{ in } \Omega,\\
-\Delta v = \theta u + \gamma v + g(u,v) \text{ in } \Omega,\\
u = v = 0 \text{ on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain with smooth boundary and subject to Dirichlet boundary conditions; $A = \begin{pmatrix} \lambda & \delta \\ \theta & \gamma \end{pmatrix}$ is a matrix of real entries; $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are globally Lipschitz functions for u, v; i.e.

$$
\begin{aligned}
|f(u,v) - f(\widetilde{u},\widetilde{v})| &\leq k_1(|u-\widetilde{u}| + |v-\widetilde{v}|),\\
|g(u,v) - g(\widetilde{u},\widetilde{v})| &\leq k_2(|u-\widetilde{u}| + |v-\widetilde{v}|),
\end{aligned}
$$

for all $u, \widetilde{u}, v, \widetilde{v} \in \mathbb{R}$.

Our goal is finding non-trivial solutions to the system (1.1) under the above hypothesis, and other suitable conditions on the first two eigenvalues of the Laplacian and on the parameter.

Note that the problem Dirichlet for system (1.1) have been studied by many authors. In [17], under more restrictive conditions, Hoang has considered the system (1.1) in which Ω is an unbounded domain. In [29], the author has considered the case of positivity of solutions in a bounded domain and in [18], the positivity of solutions have been mentioned for an unbounded domain.

Equation (1.1) represents a steady state case of reaction-diffusion systems of interest in biology. Reaction-diffusion systems have been intensively studied during

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recent years, see [35] where many references can be found. There exists a decoupling technique, which consists of reducing the system (1.1) to a single nonlinear equation containing an integral and a differential term. This technique was introduced by Rothe [33], Lazer & McKenna [23] and Brown [4] and has been used thereafter by many authors. For the resonant case many known techniques used to solve the scalar case can be applied to find solutions and positive solutions. See for example Ahmad, Lazer & Paul [1], Ambrosetti & Mancini [2], Anane [3], Bartolo, Benci & Fortunato [6], Berestycki & De Figueiredo [5], Capozzi, Lupo & Solimini [7], Cesari & Kannan [8], Costa & Magalhães [11], De Figueiredo & Gossez [13], Gossez [15], Innacci & Nkashama [19, 20], Landesman & Lazer [22], Lupo & Solimini [24], Omari & Zanolin [31], Rabinowitz [32], Schechter [34], Solimini [36], Vargas & Zuluaga [37, 38], Zuluaga [39, 40] and the references therein.

The decoupling technique has some obvious shortcomings, for example, it is very difficult to apply to systems with three or more equations. Even, in the case of two equations is too restrictive to give conditions to solve the second equation of (1.1) for v in terms of u.

It is known, see [11], using the eigenvalues of the matrix A, we will be able to give a precise description of kernel of operator $-\Delta - A$, and easy to see that this kernel is nonzero if and only if $A - \lambda_j I$ is singular for some eigenvalue λ_j of the operator $-\Delta$.

Zuluaga [41] showed results of existence and nonexistence of solutions for (1.1) under the condition λ_1, the first eigenvalue of $-\Delta$, is also a eigenvalue of matrix A. In this paper, we will extend these results obtained in [41] under the conditions in which λ_1 is not a eigenvalue of A.

Our paper is organized as follows. Section 2 provides some preliminaries and notation including the Lyapunov-Schmidt method. In Section 3, we consider the problem (1.1) under some special case where some parameters and both Lipschitz constants are equal, problem (3.1). Our main result for such problem is the Theorem 3.4. By the similar arguments, in Section 4, we state our main result of this paper for the problem (1.1).

2. Preliminaries and Notation

In $E = L^2(\Omega) \times L^2(\Omega)$ we use the norm

$$\|U\|^2_{L^2(\Omega)\times L^2(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)},$$

where $U = (u, v)$. To simplify notation, we use $\|\cdot\|$ to denote the norm in $L^2(\Omega)$ or in $L^2(\Omega) \times L^2(\Omega)$.

Solutions of (1.1)**.** We say that $U \in H^1_0(\Omega) \times H^1_0(\Omega)$ is a solution of (1.1) if

$$U = (-\Delta)^{-1}(AU + G(U)), \tag{2.1}$$

where $G(U) = (f(u, v), g(u, v))$. It is clear that $(-\Delta)^{-1} : E \to H^1_0(\Omega) \times H^1_0(\Omega)$ is a linear, self-adjoint, continuous and bijective operator. Also, the embedding $H^1_0(\Omega) \times H^1_0(\Omega) \hookrightarrow E$ is compact, thus $(-\Delta)^{-1} : E \to E$ is compact, self-adjoint and injective as well. Hence, the operator defined by the right hand side of (2.1) is compact.

Throughout this paper we shall denote by λ_1, λ_2 the first two eigenvalues of $-\Delta$ and φ_1, φ_2 are the eigenfunctions associated with the eigenvalue λ_1 and λ_2, respectively.

The Lyapunov-Schmidt method. We will denote by X the subspace of $H_0^1(\Omega)$ spanned by φ_1, that is to say $X = \{t\varphi_1 : t \in \mathbb{R}\}$. We shall also denote $Y = X^\perp = \langle\varphi_1\rangle^\perp$. So, we have the identity

$$H_0^1(\Omega) = X \oplus Y.$$

Then all $U = (u, v) \in E$ can be written as

$$\begin{gathered} u = u_0 + z, u_0 \in X, z \in Y, \\ v = v_0 + w, v_0 \in X, w \in Y, \end{gathered}$$

where $u, v \in H_0^1(\Omega)$. Let us denote by P and Q the projection on X and Y, respectively. Applying P and Q to both sides of (2.1) we obtain a decomposition of it in two systems as follows

$$\begin{aligned} u_0 &= P(-\Delta)^{-1}[\lambda(u_0 + z) + \delta(v_0 + w) + f(u_0 + z, v_0 + w)], \\ v_0 &= P(-\Delta)^{-1}[\theta(u_0 + z) + \gamma(v_0 + w) + g(u_0 + z, v_0 + w)], \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} z &= Q(-\Delta)^{-1}[\lambda(u_0 + z) + \delta(v_0 + w) + f(u_0 + z, v_0 + w)], \\ w &= Q(-\Delta)^{-1}[\theta(u_0 + z) + \gamma(v_0 + w) + g(u_0 + z, v_0 + w)]. \end{aligned} \tag{2.3}$$

For each $(u_0, v_0) \in X \times X$ fixed, we solve (2.3) and have a solution $(z_0, w_0) \in Y \times Y$ which will be plugged into (2.2) to get the solution (u_0, v_0) of (2.2). Thus, the solutions of (1.1) will be of the form $(u_0 + z_0, v_0 + w_0)$.

3. A special case of Problem (1.1)

Before stating our main result, in this Section, we consider problem (1.1) in which $\gamma = \lambda = \lambda_1$, $k_1 = k_2 = k$ and $\delta = \theta > 0$, i.e. we shall deal with the existence of non-trivial solutions of the problem

$$\begin{gathered} -\Delta u = \lambda_1 u + \delta v + f(u, v) \quad \text{in } \Omega, \\ -\Delta v = \delta u + \lambda_1 v + g(u, v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where λ_1 is the first eigenvalue of $-\Delta$. Applying the Lyapunov-Schmidt method we obtain a decomposition of it in two systems as follows

$$\begin{aligned} u_0 &= P(-\Delta)^{-1}[\lambda_1(u_0 + z) + \delta(v_0 + w) + f(u_0 + z, v_0 + w)], \\ v_0 &= P(-\Delta)^{-1}[\delta(u_0 + z) + \lambda_1(v_0 + w) + g(u_0 + z, v_0 + w)], \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} z &= Q(-\Delta)^{-1}[\lambda_1(u_0 + z) + \delta(v_0 + w) + f(u_0 + z, v_0 + w)], \\ w &= Q(-\Delta)^{-1}[\delta(u_0 + z) + \lambda_1(v_0 + w) + g(u_0 + z, v_0 + w)]. \end{aligned} \tag{3.3}$$

Fixing $(u_0, v_0) \in X \times X$, we shall consider (3.3). Letting

$$F_Q(z, w) = (F_Q^{(1)}(z, w), F_Q^{(2)}(z, w)),$$

where

$$\begin{aligned} F_Q^{(1)}(z, w) &:= Q(-\Delta)^{-1}[\lambda_1(u_0 + z) + \delta(v_0 + w) + f(u, v)], \\ F_Q^{(2)}(z, w) &:= Q(-\Delta)^{-1}[\delta(u_0 + z) + \lambda_1(v_0 + w) + g(u, v)]. \end{aligned}$$

Lemma 3.1. *If*

$$(\lambda_1 + k)^2 + (\delta + k)^2 < \frac{\lambda_2^2}{2} \tag{3.4}$$

then F_Q is a contraction in $Y \times Y$.

Proof. Let $z, \widetilde{z}, w, \widetilde{w} \in Y$, by the definition of $F_Q^{(1)}(z, w)$, we find

$$\begin{aligned} F_Q^{(1)}(z,w) - F_Q^{(1)}(\widetilde{z},\widetilde{w}) =& Q(-\Delta)^{-1}\big(\lambda_1(z-\widetilde{z}) + \delta(w-\widetilde{w}) \\ &+ f(u_0+z, v_0+w) - f(u_0+\widetilde{z}, v_0+\widetilde{w})\big). \end{aligned}$$

Therefore, from the characterization of λ_1, we get

$$\begin{aligned} \|F_Q^{(1)}(z,w) - F_Q^{(1)}(\widetilde{z},\widetilde{w})\| \le& \frac{1}{\lambda_2}\big(\lambda_1\|z-\widetilde{z}\| + \delta\|w-\widetilde{w}\| \\ &+ \|f(u_0+z, v_0+w) - f(u_0+\widetilde{z}, v_0+\widetilde{w})\|\big). \end{aligned}$$

By using the Minkowski's inequality and our Lipschitzian assumptions, we obtain

$$\|f(u_0+z, v_0+w) - f(u_0+\widetilde{z}, v_0+\widetilde{w})\| \le k(\|z-\widetilde{z}\| + \|w-\widetilde{w}\|).$$

So

$$\|F_Q^{(1)}(z,w) - F_Q^{(1)}(\widetilde{z},\widetilde{w})\| \le \frac{1}{\lambda_2}((\lambda_1+k)\|z-\widetilde{z}\| + (\delta+k)\|w-\widetilde{w}\|).$$

Therefore,

$$\|F_Q^{(1)}(z,w) - F_Q^{(1)}(\widetilde{z},\widetilde{w})\|^2 \le \frac{2}{\lambda_2^2}((\lambda_1+k)^2\|z-\widetilde{z}\|^2 + (\delta+k)^2\|w-\widetilde{w}\|^2).$$

Similarly, we claim that

$$\|F_Q^{(2)}(z,w) - F_Q^{(2)}(\widetilde{z},\widetilde{w})\|^2 \le \frac{2}{\lambda_2^2}\Big((\delta+k)^2\|z-\widetilde{z}\|^2 + (\lambda_1+k)^2\|w-\widetilde{w}\|^2\Big).$$

Now we obtain

$$\|F_Q(z,w) - F_Q(\widetilde{z},\widetilde{w})\|^2 \le \frac{2}{\lambda_2^2}\Big((\lambda_1+k)^2 + (\delta+k)^2\Big)\Big(\|z-\widetilde{z}\|^2 + \|w-\widetilde{w}\|^2\Big).$$

Hence, the assertion follows. □

By using fixed-point principle, we conclude that (3.3) has a unique solution $(z_0(u_0, v_0), w_0(u_0, v_0))$ for each $(u_0, v_0) \in X \times X$ fixed. The assertion of the above Lemma let us to define

$$\begin{aligned} F : X \times X &\to Y \times Y, \\ (u_0, v_0) &\mapsto F(u_0, v_0) := (z_0, w_0), \end{aligned}$$

be the function such that (z_0, w_0) is the only fixed point of F_Q.

Lemma 3.2. *If*

$$(\lambda_1 + k)^2 + (\delta + k)^2 < \frac{\lambda_2^2}{4} \tag{3.5}$$

then

$$\begin{aligned} &\|F(u_0, v_0) - F(\widetilde{u}_0, \widetilde{v}_0)\|^2 \\ &\le \frac{8k^2}{\lambda_2^2 - 4((\lambda_1+k)^2 + (\delta+k)^2)}(\|u_0 - v_0\|^2 + \|\widetilde{u}_0 - \widetilde{v}_0\|^2). \end{aligned} \tag{3.6}$$

for every (u_0, v_0) and $(\widetilde{u}_0, \widetilde{v}_0)$ in $X \times X$.

Proof. Suppose that $F(u_0, v_0) = (z_0, w_0)$ and $F(\widetilde{u}_0, \widetilde{v}_0) = (\widetilde{z}_0, \widetilde{w}_0)$. By the definition of F, we have

$$\begin{aligned} z_0 &= Q(-\Delta)^{-1}[\lambda_1(u_0 + z_0) + \delta(v_0 + w_0) + f(u_0 + z_0, v_0 + w_0)], \\ w_0 &= Q(-\Delta)^{-1}[\delta(u_0 + z_0) + \lambda_1(v_0 + w_0) + g(u_0 + z_0, v_0 + w_0)], \end{aligned}$$

and

$$\begin{aligned} \widetilde{z}_0 &= Q(-\Delta)^{-1}[\lambda_1(\widetilde{u}_0 + \widetilde{z}_0) + \delta(\widetilde{v}_0 + \widetilde{w}_0) + f(\widetilde{u}_0 + \widetilde{z}_0, \widetilde{v}_0 + \widetilde{w}_0)], \\ \widetilde{w}_0 &= Q(-\Delta)^{-1}[\delta(\widetilde{u}_0 + \widetilde{z}_0) + \lambda_1(\widetilde{v}_0 + \widetilde{w}_0) + g(\widetilde{u}_0 + \widetilde{z}_0, \widetilde{v}_0 + \widetilde{w}_0)]. \end{aligned}$$

Because

$$\begin{aligned} Q(-\Delta)^{-1}(u_0) &= Q(-\Delta)^{-1}(v_0) = 0, \\ Q(-\Delta)^{-1}(\widetilde{u}_0) &= Q(-\Delta)^{-1}(\widetilde{v}_0) = 0, \end{aligned}$$

we have

$$\begin{aligned} \|z_0 - \widetilde{z}_0\| \leq & \frac{1}{\lambda_2}\Big(\lambda_1\|z_0 - \widetilde{z}_0\| + \delta\|w_0 - \widetilde{w}_0\| \\ & + \|f(u_0 + z_0, v_0 + w_0) - f(\widetilde{u}_0 + \widetilde{z}_0, \widetilde{v}_0 + \widetilde{w}_0)\|\Big) \\ \leq & \frac{1}{\lambda_2}\Big((\lambda_1 + k)\|z_0 - \widetilde{z}_0\| + (\delta + k)\|w_0 - \widetilde{w}_0\| \\ & + k(\|u_0 - \widetilde{u}_0\| + \|v_0 - \widetilde{v}_0\|)\Big). \end{aligned}$$

Thus

$$\begin{aligned} \|z_0 - \widetilde{z}_0\|^2 \leq & \frac{4}{\lambda_2^2}\Big((\lambda_1 + k)^2\|z_0 - \widetilde{z}_0\|^2 + (\delta + k)^2\|w_0 - \widetilde{w}_0\|^2 \\ & + k^2\big(\|u_0 - \widetilde{u}_0\|^2 + \|v_0 - \widetilde{v}_0\|^2\big)\Big). \end{aligned}$$

Similarly, we find

$$\begin{aligned} \|w_0 - \widetilde{w}_0\|^2 \leq & \frac{4}{\lambda_2^2}\Big((\delta + k)^2\|z_0 - \widetilde{z}_0\|^2 + (\lambda_1 + k)^2\|w_0 - \widetilde{w}_0\|^2 \\ & + k^2\big(\|u_0 - \widetilde{u}_0\|^2 + \|v_0 - \widetilde{v}_0\|^2\big)\Big). \end{aligned}$$

Hence

$$\begin{aligned} \|z_0 - \widetilde{z}_0\|^2 + \|w_0 - \widetilde{w}_0\|^2 \leq & \frac{4}{\lambda_2^2}\Big(((\delta + k)^2 + (\lambda_1 + k)^2)(\|z_0 - \widetilde{z}_0\|^2 + \|w_0 - \widetilde{w}_0\|^2) \\ & + 2k^2(\|u_0 - \widetilde{u}_0\|^2 + \|v_0 - \widetilde{v}_0\|^2)\Big). \end{aligned}$$

Therefore

$$\begin{aligned} & \|z_0 - \widetilde{z}_0\|^2 + \|w_0 - \widetilde{w}_0\|^2 \\ & \leq \frac{8k^2}{\lambda_2^2 - 4((\lambda_1 + k)^2 + (\delta + k)^2)}(\|u_0 - v_0\|^2 + \|\widetilde{u}_0 - \widetilde{v}_0\|^2). \end{aligned}$$

□

Now, we consider the system (3.2). First, the fact that F is a contraction mapping yields

$$\begin{aligned} u_0 &= P(-\Delta)^{-1}[\lambda_1(u_0+z_0)+\delta(v_0+w_0)+f(u_0+z_0,v_0+w_0)],\\ v_0 &= P(-\Delta)^{-1}[\delta(u_0+z_0)+\lambda_1(v_0+w_0)+g(u_0+z_0,v_0+w_0)]. \end{aligned} \tag{3.7}$$

Because

$$\begin{gathered} P(-\Delta)^{-1}(z_0)=P(-\Delta)^{-1}(w_0)=0,\\ P(-\Delta)^{-1}(\lambda_1 u_0)=u_0, P(-\Delta)^{-1}(\lambda_1 v_0)=v_0, \end{gathered}$$

we deduce that

$$\begin{aligned} 0 &= P(-\Delta)^{-1}[\delta v_0+f(u_0+z_0,v_0+w_0)],\\ 0 &= P(-\Delta)^{-1}[\delta u_0+g(u_0+z_0,v_0+w_0)]. \end{aligned} \tag{3.8}$$

On the other hand, from the definition of subspace X,

$$P(-\Delta)^{-1}(\delta u_0)=\frac{\delta}{\lambda_1}u_0,\quad P(-\Delta)^{-1}(\delta v_0)=\frac{\delta}{\lambda_1}v_0.$$

This yields

$$\begin{aligned} 0 &= \frac{\delta}{\lambda_1}v_0+P(-\Delta)^{-1}[f(u_0+z_0,v_0+w_0)],\\ 0 &= \frac{\delta}{\lambda_1}u_0+P(-\Delta)^{-1}[g(u_0+z_0,v_0+w_0)]. \end{aligned} \tag{3.9}$$

Now, (3.9) is equivalent to

$$\begin{aligned} u_0 &= -\frac{\lambda_1}{\delta}P(-\Delta)^{-1}[g(u_0+z_0,v_0+w_0)],\\ v_0 &= -\frac{\lambda_1}{\delta}P(-\Delta)^{-1}[f(u_0+z_0,v_0+w_0)]. \end{aligned} \tag{3.10}$$

Letting

$$F_P(u_0,v_0)=(F_P^{(1)}(u_0,v_0),F_P^{(2)}(u_0,v_0)),$$

where

$$\begin{aligned} F_P^{(1)}(u_0,v_0) &:= -\frac{\lambda_1}{\delta}P(-\Delta)^{-1}[g(u_0+z_0,v_0+w_0)],\\ F_P^{(2)}(u_0,v_0) &:= -\frac{\lambda_1}{\delta}P(-\Delta)^{-1}[f(u_0+z_0,v_0+w_0)]. \end{aligned}$$

Lemma 3.3. *If* $(\lambda_1+k)^2+(\delta+k)^2<\lambda_2^2/4$ *and*

$$\frac{8k^2}{\delta^2}\Big(1+\frac{8k^2}{\lambda_2^2-4((\delta+k)^2+(\lambda_1+k)^2)}\Big)<1 \tag{3.11}$$

then F_P *is a contraction in* $X\times X$.

Proof. Letting $(\widetilde{u}_0,\widetilde{v}_0)$ in $X\times X$. Corresponding to $(\widetilde{u}_0,\widetilde{v}_0)$, from Lemma 3.1, we have $(\widetilde{z}_0,\widetilde{w}_0)$ in $Y\times Y$. From the definition of $F_P^{(1)}(u_0,v_0)$ we find

$$\begin{aligned} &F_P^{(1)}(u_0,v_0)-F_P^{(1)}(\widetilde{u}_0,\widetilde{v}_0)\\ &= -\frac{\lambda_1}{\delta}P(-\Delta)^{-1}[g(u_0+z_0,v_0+w_0)-g(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)]. \end{aligned}$$

Using our Lipschitzian assumptions we obtain

$$
\begin{aligned}
&\|F_P^{(1)}(u_0,v_0)-F_P^{(1)}(\widetilde{u}_0,\widetilde{v}_0)\| \\
&\le \frac{\lambda_1}{\delta}\frac{1}{\lambda_1}\|g(u_0+z_0,v_0+w_0)-g(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)\| \\
&\le \frac{k}{\delta}(\|u_0-\widetilde{u}_0\|+\|v_0-\widetilde{v}_0\|+\|z_0-\widetilde{z}_0\|+\|w_0-\widetilde{w}_0\|).
\end{aligned}
$$

By (3.6), we have

$$
\begin{aligned}
&\|F_P^{(1)}(u_0,v_0)-F_P^{(1)}(\widetilde{u}_0,\widetilde{v}_0)\|^2 \\
&\le \frac{4k^2}{\delta^2}\Big(\|u_0-\widetilde{u}_0\|^2+\|v_0-\widetilde{v}_0\|^2+\|z_0-\widetilde{z}_0\|^2+\|w_0-\widetilde{w}_0\|^2\Big) \\
&\le \frac{4k^2}{\delta^2}\Big(\|u_0-\widetilde{u}_0\|^2+\|v_0-\widetilde{v}_0\|^2\Big) \\
&\quad+\frac{4k^2}{\delta^2}\frac{8k^2}{\lambda_2^2-4((\lambda_1+k)^2+(\delta+k)^2)}(\|u_0-\widetilde{u}_0\|^2+\|v_0-\widetilde{v}_0\|^2) \\
&\le \frac{4k^2}{\delta^2}\Big(1+\frac{8k^2}{\lambda_2^2-4((\lambda_1+k)^2+(\delta+k)^2)}\Big)(\|u_0-\widetilde{u}_0\|^2+\|v_0-\widetilde{v}_0\|^2).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\|F_P^{(2)}(u_0,v_0)-F_P^{(2)}(\widetilde{u}_0,\widetilde{v}_0)\|^2 \\
&\le \frac{4k^2}{\delta^2}\Big(1+\frac{8k^2}{\lambda_2^2-4((\lambda_1+k)^2+(\delta+k)^2)}\Big)(\|u_0-\widetilde{u}_0\|^2+\|v_0-\widetilde{v}_0\|^2).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|F_P(u_0,v_0)-F_P(\widetilde{u}_0,\widetilde{v}_0)\|^2 \\
&\le \frac{8k^2}{\delta^2}\Big(1+\frac{8k^2}{\lambda_2^2-4((\lambda_1+k)^2+(\delta+k)^2)}\Big)(\|u_0-\widetilde{u}_0\|^2+\|v_0-\widetilde{v}_0\|^2).
\end{aligned}
$$

So, the proof is complete. □

The main result in this section is the following theorem, whose proof follows the arguments above.

Theorem 3.4. *If* $(\lambda_1+k)^2+(\delta+k)^2<\lambda_2^2/4$, *and*

$$
\frac{8k^2}{\delta^2}\Big(1+\frac{8k^2}{\lambda_2^2-4((\delta+k)^2+(\lambda_1+k)^2)}\Big)<1,
$$

then (3.1) has a solution. Furthermore, this solution is unique.

4. Main results

In this Section, we establish existence result for the cases in which $A-\lambda_1 I$ is regular. Letting

$$
l:=(|\lambda|+k_1)^2+(|\delta|+k_1)^2+(|\theta|+k_2)^2+(|\gamma|+k_2)^2.
$$

Our main result is as follows.

Theorem 4.1. *Suppose that λ_1 is not a eigenvalue of matrix A, $l < \lambda_2^2/2$, and*
$$\frac{4(k_1^2+k_2^2)\big((\lambda_1-\lambda)^2+(\lambda_1-\gamma)^2+\theta^2+\delta^2\big)}{((\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta)^2}\Big(1+\frac{4(k_1^2+k_2^2)}{\lambda_2^2-2l}\Big)<1.$$
Then, (1.1) *has a unique solution (u,v) in $L^2(\Omega)\times L^2(\Omega)$.*

For the proof of the above theorem we need some lemmas.

Lemma 4.2. *For each $(u_0,v_0)\in X\times X$ fixed, if $l<\lambda_2^2$ then (2.3) has a unique solution $(z_0,w_0)\in Y\times Y$.*

As in Lemma 3.1, it is easy to verify the statement of the above lemma. This result let us to define
$$\begin{gathered} T: X\times X\to Y\times Y,\\ (u_0,v_0)\mapsto T(u_0,v_0):=(z_0,w_0), \end{gathered}$$
where (z_0,w_0) is the unique solution of (2.3).

Lemma 4.3. *If $l<\lambda_2^2/2$ then*
$$\|T(u_0,v_0)-T(\widetilde{u}_0,\widetilde{v}_0)\|^2\le\frac{4(k_1^2+k_2^2)}{\lambda_2^2-2l}\big(\|u_0-v_0\|^2+\|\widetilde{u}_0-\widetilde{v}_0\|^2\big). \tag{4.1}$$

It is easy to check the statement of the above lemma.

Lemma 4.4. *If $l<\lambda_2^2/2$ and*
$$\frac{4(k_1^2+k_2^2)((\lambda_1-\lambda)^2+(\lambda_1-\gamma)^2+\theta^2+\delta^2)}{((\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta)^2}\Big(1+\frac{4(k_1^2+k_2^2)}{\lambda_2^2-2l}\Big)<1 \tag{4.2}$$
then (2.2) has a unique solution in $X\times X$.

Proof. By Lemma 4.2, we obtain
$$\begin{aligned} u_0&=P(-\Delta)^{-1}[\lambda(u_0+z_0)+\delta(v_0+w_0)+f(u_0+z_0,v_0+w_0)],\\ v_0&=P(-\Delta)^{-1}[\theta(u_0+z_0)+\gamma(v_0+w_0)+g(u_0+z_0,v_0+w_0)]. \end{aligned} \tag{4.3}$$
It follows from the properties of P that
$$\begin{aligned} u_0&=P(-\Delta)^{-1}[\lambda u_0+\delta v_0+f(u_0+z_0,v_0+w_0)],\\ v_0&=P(-\Delta)^{-1}[\theta u_0+\gamma v_0+g(u_0+z_0,v_0+w_0)], \end{aligned} \tag{4.4}$$
which implies
$$\begin{aligned} u_0&=\frac{\lambda}{\lambda_1}u_0+\frac{\delta}{\lambda_1}v_0+P(-\Delta)^{-1}[f(u_0+z_0,v_0+w_0)],\\ v_0&=\frac{\theta}{\lambda_1}u_0+\frac{\gamma}{\lambda_1}v_0+P(-\Delta)^{-1}[g(u_0+z_0,v_0+w_0)]. \end{aligned} \tag{4.5}$$
By solving the system (4.5), we have
$$\begin{aligned} u_0&=\frac{\lambda_1(\lambda_1-\gamma)P(-\Delta)^{-1}[f]+\lambda_1\delta P(-\Delta)^{-1}[g]}{(\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta}=:F_P^{(1)}(u_0,v_0),\\ v_0&=\frac{\lambda_1(\lambda_1-\lambda)P(-\Delta)^{-1}[g]+\lambda_1\theta P(-\Delta)^{-1}[f]}{(\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta}=:F_P^{(2)}(u_0,v_0). \end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
&F_P^{(1)}(u_0,v_0)-F_P^{(1)}(\widetilde{u}_0,\widetilde{v}_0)\\
&=\frac{\lambda_1(\lambda_1-\gamma)}{(\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta}P(-\Delta)^{-1}[f(u_0+z_0,v_0+w_0)-f(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)]\\
&+\frac{\lambda_1\delta}{(\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta}P(-\Delta)^{-1}[g(u_0+z_0,v_0+w_0)-g(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)].
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|F_P^{(1)}(u_0,v_0)-F_P^{(1)}(\widetilde{u}_0,\widetilde{v}_0)\|\\
&\le\frac{|\lambda_1-\gamma|}{|(\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta|}\|f(u_0+z_0,v_0+w_0)-f(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)\|\\
&\quad+\frac{|\delta|}{|(\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta|}\|g(u_0+z_0,v_0+w_0)-g(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)\|.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\|F_P^{(2)}(u_0,v_0)-F_P^{(2)}(\widetilde{u}_0,\widetilde{v}_0)\|\\
&\le\frac{|\lambda_1-\lambda|}{|(\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta|}\|f(u_0+z_0,v_0+w_0)-f(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)\|\\
&\quad+\frac{|\theta|}{|(\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta|}\|g(u_0+z_0,v_0+w_0)-g(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)\|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|F_P(u_0,v_0)-F_P(\widetilde{u}_0,\widetilde{v}_0)\|^2\\
&\le\frac{(\lambda_1-\gamma)^2+(\lambda_1-\lambda)^2+\delta^2+\theta^2}{((\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta)^2}\Big(\|f(u_0+z_0,v_0+w_0)-f(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)\|^2\\
&\quad+\|g(u_0+z_0,v_0+w_0)-g(\widetilde{u}_0+\widetilde{z}_0,\widetilde{v}_0+\widetilde{w}_0)\|^2\Big),
\end{aligned}
$$

where

$$
F_P(u_0,v_0)=(F_P^{(1)}(u_0,v_0),F_P^{(2)}(u_0,v_0)).
$$

Using our Lipschitzian assumptions,

$$
\begin{aligned}
&\|F_P(u_0,v_0)-F_P(\widetilde{u}_0,\widetilde{v}_0)\|^2\\
&\le 4(k_1^2+k_2^2)\frac{(\lambda_1-\gamma)^2+(\lambda_1-\lambda)^2+\delta^2+\theta^2}{((\lambda_1-\lambda)(\lambda_1-\gamma)-\theta\delta)^2}\Big(1+\frac{4(k_1^2+k_2^2)}{\lambda_2^2-2l}\Big)\\
&\quad\times(\|u_0-\widetilde{u}_0\|^2+\|v_0-\widetilde{v}_0\|^2)
\end{aligned}
$$

which completes the proof. □

The proof of Theorem 4.1 is similar to the proof of Theorem 3.4; therefore, we omit it.

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