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CONCENTRATION PHENOMENA FOR FOURTH-ORDER ELLIPTIC EQUATIONS WITH CRITICAL EXPONENT

MOKHLESS HAMMAMI

Abstract. We consider the nonlinear equation

$$\Delta^2 u = u^{\frac{n+4}{n-4}} - \varepsilon u$$

with $u > 0$ in Ω and $u = \Delta u = 0$ on $\partial\Omega$. Where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 9$, and ε is a small positive parameter. We study the existence of solutions which concentrate around one or two points of Ω. We show that this problem has no solutions that concentrate around a point of Ω as ε approaches 0. In contrast to this, we construct a domain for which there exists a family of solutions which blow-up and concentrate in two different points of Ω as ε approaches 0.

1. Introduction and statement of results

This paper concerns the concentration phenomena for the following nonlinear equation under Navier boundary conditions:

$$\begin{gathered}\Delta^2 u = u^p - \varepsilon u, \quad u > 0 \quad \text{in } \Omega \\ \Delta u = u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 9$, ε is a small positive parameter and $p + 1 = 2n/(n-4)$ is the critical Sobolev exponent of the embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

In the last decades, there have been many works in the study of concentration phenomena for second order elliptic equations with critical exponent; see for example [1, 3, 6, 9, 10, 11, 12, 17, 18, 20, 21, 22, 23, 24, 25, 26] and the references therein. In sharp contrast to this, very little is known for fourth order elliptic equations.

For $\varepsilon = 0$, the situation is complex, Van Der Vorst showed in [27] that if Ω is starshaped (1.1) has no solution whereas Ebobisse and Ould Ahmedou proved in [13] that (1.1) has a solution provided that some homology group of Ω is nontrivial. This topological condition is sufficient, but not necessary, as examples of contractible domains Ω on which a solution exists show [16]. For $-\lambda_1(\Omega) < \varepsilon < 0$, Van der Vost has shown in [28] that (1.1) has a solution, generalizing to (1.1) the famous Brezis-Nirenberg's result [8] concerning the corresponding second order elliptic equation,

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where $\lambda_1(\Omega)$ denotes the first eigenvalue of Δ^2 under the Navier boundary condition. Recently, also for $\varepsilon < 0$, El Mehdi and Selmi [15] have constructed a solution of (1.1) which concentrates around a critical point of Robin's function,

However, as far as the author know, the case of $\varepsilon > 0$ has not been considered before and this is precisely the first aim of the present paper. More precisely, our goal is to study the existence of solutions of (1.1) which concentrate in one or two points of Ω. The similar problems in the case of Laplacian have been considered by Musso and Pistoia [22]. Compared with the second order case, further technical problems arise which are overcome by careful and delicate expansions of the Euler functional associated to (1.1) and its gradient near a neighborhood of highly concentrated functions. Such expansions, which are of self interest, are highly nontrivial and use the techniques developed by Bahri [2] and Rey [23] in the framework of the *Theory of critical points at infinity.*

To state our results, we need to introduce some notations. We denote by G the Green's function of Δ^2, that is, for all $x \in \Omega$,

$$\begin{gathered}\Delta^2 G(x,.) = c'_n \delta_x \quad \text{in } \Omega \\ \Delta G(x,.) = G(x,.) = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where δ_x denotes the Dirac mass at x and $c'_n = (n-4)(n-2)|S^{n-1}|$. We also denote by H the regular part of G, that is,

$$H(x,y) = |x-y|^{4-n} - G(x,y), \quad \text{for } (x,y) \in \Omega \times \Omega.$$

For $\lambda > 0$ and $x \in \mathbb{R}^n$, let

$$\delta_{x,\lambda}(y) = \frac{c_n \lambda^{\frac{n-4}{2}}}{(1+\lambda^2|y-x|^2)^{\frac{n-4}{2}}}, \quad c_n = [(n-4)(n-2)n(n+2)]^{(n-4)/8}. \tag{1.2}$$

It is well known [19] that $\delta_{x,\lambda}$ are the only solutions of

$$\Delta^2 u = u^{\frac{n+4}{n-4}}, \quad u > 0 \text{ in } \mathbb{R}^n$$

with $u \in L^{p+1}(\mathbb{R}^n)$ and $\Delta u \in L^2(\mathbb{R}^n)$. They are also the only minimizers of the Sobolev inequality on the whole space; that is,

$$S = \inf\{\|\Delta u\|^2_{L^2(\mathbb{R}^n)} \|u\|^{-2}_{L^{\frac{2n}{n-4}}(\mathbb{R}^n)} : \Delta u \in L^2, u \in L^{\frac{2n}{n-4}}, u \neq 0\}. \tag{1.3}$$

We denote by $P\delta_{x,\lambda}$ the projection of the $\delta_{x,\lambda}$'s onto $H^2(\Omega) \cap H^1_0(\Omega)$, defined by

$$\Delta^2 P\delta_{x,\lambda} = \Delta^2 \delta_{x,\lambda} \text{ in } \Omega \quad \text{and} \quad \Delta P\delta_{x,\lambda} = P\delta_{x,\lambda} = 0 \text{ on } \partial\Omega,$$

and we set

$$\varphi_{x,\lambda} = \delta_{x,\lambda} - P\delta_{x,\lambda}.$$

The space $\mathcal{H}(\Omega) := H^2(\Omega) \cap H^1_0(\Omega)$ is equipped with the norm $\|\cdot\|$ and its corresponding inner product $(.,.)$ defined by

$$\|u\| = \Big(\int_\Omega |\Delta u|^2\Big)^{1/2}, \quad u \in \mathcal{H}(\Omega), \tag{1.4}$$

$$(u,v) = \int_\Omega \Delta u \Delta v, \quad u,v \in \mathcal{H}(\Omega). \tag{1.5}$$

For $x \in \Omega$, $\lambda > 0$, let

$$E_{x,\lambda} = \{v \in \mathcal{H}(\Omega) : (v, P\delta_{x,\lambda}) = (v, \frac{\partial P\delta_{x,\lambda}}{\partial \lambda}) = (v, \frac{\partial P\delta_{x,\lambda}}{\partial x_j}) = 0,\, j = 1,\dots,n\},$$

where the x_j is the j-th component of x.

Now we state our first result.

Theorem 1.1. *There does not exist any solution of* (1.1) *of the form*

$$u_\varepsilon = \alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon, \tag{1.6}$$

where

$$v_\varepsilon \in E_{x_\varepsilon,\lambda_\varepsilon}, \quad x_\varepsilon \in \Omega \quad \text{and as } \varepsilon \to 0,\ \alpha_\varepsilon \to 1,\ \|v_\varepsilon\| \to 0,\ \lambda_\varepsilon d(x_\varepsilon, \partial\Omega) \to +\infty. \tag{1.7}$$

On the contrary, if Ω is a domain with small "hole", we prove the existence of a family of solutions which blow-up and concentrate in two points. Namely, we have the following result.

Theorem 1.2. *Let D be a bounded smooth domain in $\mathbb{R}^n$ which contains the origin 0. There exists $r_0 > 0$ such that, if $0 < r < r_0$ is fixed and Ω is the domain given by $D\backslash\omega$ for any smooth domain $\omega \subset B(0, r)$, then there exists $\varepsilon_0 > 0$ such that problem* (1.1) *has a solution u_ε for any $0 < \varepsilon < \varepsilon_0$. Moreover, the family of solutions u_ε blows-up and concentrates at two different points of Ω in the following sense:*

$$u_\varepsilon = \sum_{i=1}^{2} \alpha_i^\varepsilon P\delta_{x_i^\varepsilon,\lambda_i^\varepsilon} + v_\varepsilon,$$

where λ_1^ε, $\lambda_2^\varepsilon > 0$, x_1^ε, $x_2^\varepsilon \in \Omega$ with $\lim_{\varepsilon\to 0} x_i^\varepsilon = x_i \in \Omega$, $x_1 \neq x_2$, λ_1^ε and λ_2^ε are of order $\varepsilon^{-1/(n-8)}$, $v_\varepsilon \in E_{x_1^\varepsilon,\lambda_1^\varepsilon} \cap E_{x_2^\varepsilon,\lambda_2^\varepsilon}$ and as $\varepsilon \to 0$, $\alpha_i^\varepsilon \to 1$, $\|v_\varepsilon\| \to 0$.

Note that the construction of solutions which concentrate around k different points of Ω, with $k \geq 2$ is related to suitable critical points of the function $\Psi_k : \mathbb{R}_+^k \times \Omega^k \to \mathbb{R}$ defined by

$$\Psi_k(\Lambda, x) = \frac{1}{2}(M(x)\Lambda, \Lambda) + \frac{1}{2}\sum_{i=1}^{k} \Lambda_i^{\frac{8}{n-4}},$$

where $\Lambda =^T (\Lambda_1, \dots, \Lambda_k)$ and $M(x) = (m_{ij}(x))_{1\leq i,j\leq k}$ is the matrix defined by

$$m_{ii} = H(x_i, x_i), \quad m_{ij} = -G(x_i, x_j) \quad \text{for } i \neq j. \tag{1.8}$$

Let $\rho(x)$ be the least eigenvalue of $M(x)$ and $e(x)$ the eigenvector corresponding to $\rho(x)$ whose norm is 1 and whose components are all strictly positive (see Appendix A of [3]). Now, we define the following subset of $\mathcal{H}(\Omega)$

$$\begin{aligned}\mathcal{M}_\varepsilon =&\{m = (\alpha, \lambda, x, v) \in \mathbb{R}^k \times (\mathbb{R}_+^*)^k \times \Omega_{d_0}^k \times \mathcal{H}(\Omega) : |\alpha_i - 1| < \nu_0,\\ &\lambda_i > \frac{1}{\nu_0}\ \forall i,\ \frac{\lambda_i}{\lambda_j} < c_0,\ |x_i - x_j| > d_0'\ \forall i \neq j,\ v \in E,\ \|v\| < \nu_0\}.\end{aligned}$$

where ν_0, c_0, d_0, d_0' are some suitable positive constants, $\Omega_{d_0} = \{x \in \Omega : d(x, \partial\Omega) > d_0\}$ and $E = \bigcap_{i=1}^k E_{x_i,\lambda_i}$. Then, we have the following necessary condition.

Theorem 1.3. *Assume that u_ε is a solution of* (1.1) *of the form*

$$u_\varepsilon = \sum_{i=1}^{k} \alpha_i^\varepsilon P\delta_{x_i^\varepsilon,\lambda_i^\varepsilon} + v^\varepsilon, \tag{1.9}$$

where $(\alpha^\varepsilon, \lambda^\varepsilon, x^\varepsilon, v^\varepsilon) \in \mathcal{M}_\varepsilon$, then, when $\varepsilon \to 0$, $\alpha_i^\varepsilon \to 1$, $x_i^\varepsilon \to x_i$ for $i = 1, \dots, k$ and we have either $\rho(x) = 0$ and $\rho'(x) = 0$ or $\rho(x) < 0$ and (Λ, x) is a critical point *of* Ψ_k*, where* $\Lambda_i = c\mu_i$*, with* $\mu_i = \lim_{\varepsilon\to 0} \varepsilon^{\frac{-1}{n-8}} \lambda_i^\varepsilon > 0$ *for* $i = 1,\dots,k$ *and* c *is a positive constant.*

The proof of our results is inspired by the methods of [2, 3, 5, 14, 22]. In Section 2, we develop the technical framework needed in the proofs of ours results. Section 3 is devoted to the proof of Theorems 1.1 and 1.3, while Theorem 1.2 is proved in Section 4.

2. The Technical Framework

First of all, let us introduce the general setting. For $\varepsilon > 0$, we define on $\mathcal{H}(\Omega)$ the functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} + \frac{\varepsilon}{2}\int_\Omega u^2. \tag{2.1}$$

If u is a positive critical point of J_ε, u satisfies on Ω the equation (1.1). Conversely, we see that any solution of (1.1) is a critical point of J_ε.

Let us define the functional

$$K_\varepsilon : \mathcal{M}_\varepsilon \to \mathbb{R},\ K_\varepsilon(\alpha,\lambda,x,v) = J_\varepsilon\big(\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i} + v\big). \tag{2.2}$$

Note that (α,λ,x,v) is a critical point of K_ε if and only if $u = \sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i} + v$ is a critical point of J_ε, i.e. if and only if there exist A_i, B_i, $C_{ij} \in \mathbb{R}$, $1 \le i \le k$ and $1 \le j \le n$, such that

$$\frac{\partial K_\varepsilon}{\partial \alpha_i} = 0 \quad \forall i, \tag{2.3}$$

$$\frac{\partial K_\varepsilon}{\partial \lambda_i} = B_i\big(\frac{\partial^2 P\delta_{x_i,\lambda_i}}{\partial \lambda_i^2}, v\big) + \sum_{j=1}^n C_{ij}\big(\frac{\partial^2 P\delta_{x_i,\lambda_i}}{\partial (x_i)_j \partial\lambda_i}, v\big) \quad \forall i, \tag{2.4}$$

$$\frac{\partial K_\varepsilon}{\partial (x_i)_r} = B_i\big(\frac{\partial^2 P\delta_{x_i,\lambda_i}}{\partial \lambda_i \partial (x_i)_r}, v\big) + \sum_{j=1}^n C_{ij}\big(\frac{\partial^2 P\delta_{x_i,\lambda_i}}{\partial (x_i)_j \partial (x_i)_r}, v\big) \quad \forall r,\ \forall i, \tag{2.5}$$

$$\frac{\partial K_\varepsilon}{\partial v} = \sum_{i=1}^k \Big(A_i P\delta_{x_i,\lambda_i} + B_i \frac{\partial P\delta_{x_i,\lambda_i}}{\partial\lambda_i} + \sum_{j=1}^n C_{ij}\frac{\partial P\delta_{x_i,\lambda_i}}{\partial (x_i)_j}\Big), \tag{2.6}$$

where the $(x_i)_r$ is the r-th component of x_i. As usual in this type of problems, we first deal with the v-part of u, in order to show that it is negligible with respect to the concentration phenomenon. Namely, we prove the following.

Proposition 2.1. *There exists* $\varepsilon_1 > 0$ *such that, for* $0 < \varepsilon < \varepsilon_1$*, there exists a* C^1*-map which to any* (α,λ,x) *with* $(\alpha,\lambda,x,0) \in \mathcal{M}_\varepsilon$*, associates* $v_\varepsilon = v_{(\varepsilon,\alpha,\lambda,x)} \in E$*,* $\|v_\varepsilon\| < \nu_0$*, such that (2.6) is satisfied for some* $(A,B,C) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^n)^k$*. Such a* v_ε *is unique, minimizes* $K_\varepsilon(\alpha,\lambda,x,v)$ *with respect to* v *in* E *and we have the estimate*

$$\|v_\varepsilon\| = O\Big(\sum_{i=1}^k \Big(\frac{(\log\lambda_i)^{\frac{n+4}{2n}}}{\lambda_i^{\frac{n+4}{2}}} + (\text{if } n \ge 12)\frac{\varepsilon(\log\lambda_i)^{\frac{n+4}{2n}}}{\lambda_i^4} + (\text{if } n < 12)\big(\frac{\varepsilon}{\lambda_i^{\frac{n-4}{2}}} + \frac{1}{\lambda_i^{n-4}}\big)\Big)\Big)$$

Proof. As in [2] (see also [23]) we write

$$
\begin{aligned}
&K_\varepsilon(\alpha,\lambda,x,v)\\
&= J_\varepsilon\big(\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i}+v\big)\\
&= \frac{1}{2}\|\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i}+v\|^2-\frac{1}{p+1}\int_\Omega |\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i}+v|^{p+1}\\
&\quad+\frac{\varepsilon}{2}\int_\Omega\Big(\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i}+v\Big)^2\\
&= K_\varepsilon(\alpha,\lambda,x,0)-(f_\varepsilon,v)+\frac{1}{2}Q_\varepsilon(v,v)+O\big(\|v\|^{\min(3,p+1)}+\varepsilon\|v\|^2\big),
\end{aligned}
\tag{2.7}
$$

where

$$
(f_\varepsilon,v)=\int_\Omega |\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i}|^p v-\varepsilon\int_\Omega\Big(\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i}\Big)v,
$$

$$
Q_\varepsilon(v,v)=\|v\|^2-p\int_\Omega\big(\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i}\big)^{p-1}v^2=\|v\|^2-p\sum_{i=1}^k\int_\Omega \delta_{x_i,\lambda_i}^{p-1}v^2+o(\|v\|^2).
$$

According to [4], there exists a positive constant c such that

$$
\|v\|^2-p\sum_{i=1}^k\int_\Omega \delta_{x_i,\lambda_i}^{p-1}v^2\ge c\|v\|^2,\quad \forall v\in E. \tag{2.8}
$$

Now, we will estimate (f_ε,v). Using the fact that $(P\delta_{x_i,\lambda_i},v)=0$, we obtain

$$
\begin{aligned}
&\int_\Omega |\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i}|^p v\\
&= O\Big(\sum_i\int_{B_i\cup B_i^c}\delta_{x_i,\lambda_i}^{p-1}\varphi_{x_i,\lambda_i}|v|+\sum_{j\neq i}\int_{B_i\cup B_i^c}\chi_{P\delta_j\le P\delta_i}P\delta_{x_i,\lambda_i}^{p-1}P\delta_{x_j,\lambda_j}|v|\Big)\\
&= O\Big(\sum_{i,j}\Big(\frac{1}{\lambda_j^{(n-4)/2}}\int_{B_i}\delta_{x_i,\lambda_i}^{p-1}|v|+\int_{B_i^c}\delta_{x_i,\lambda_i}^{p}|v|\Big)\Big),
\end{aligned}
\tag{2.9}
$$

where $B_i=\{y:|y-x_i|<d_0/2\}$. Then using the Holder's inequality we have

$$
\int_\Omega |\sum_{i=1}^k \alpha_i P\delta_{x_i,\lambda_i}|^p v=O\Big(\|v\|\sum_{i=1}^k\Big(\frac{(\log\lambda_i)^{(n+4)/2n}}{\lambda_i^{(n+4)/2}}+(\text{if } n<12)\frac{1}{\lambda_i^{n-4}}\Big)\Big). \tag{2.10}
$$

For the second integral, using the Holder's inequality we have

$$
\begin{aligned}
\int_\Omega P\delta_{x_i,\lambda_i}v&=O\Big(\|v\|\Big(\int_\Omega\delta_{x_i,\lambda_i}^{2n/(n+4)}\Big)^{(n+4)/2n}\Big)\\
&=O\Big((\text{if } n\ge 12)\frac{\|v\|(\log\lambda_i)^{(n+4)/2n}}{\lambda_i^4}+(\text{if } n<12)\frac{\|v\|}{\lambda_i^{(n-4)/2}}\Big).
\end{aligned}
\tag{2.11}
$$

It follows from (2.10) and (2.11) that

$$
\begin{aligned}
(f_\varepsilon, v) = O\Big[\|v\| \sum_{i=1}^{k} \Big(& \frac{(\log \lambda_i)^{\frac{n+4}{2n}}}{\lambda_i^{\frac{n+4}{2}}} + (\text{if } n \geq 12) \frac{\varepsilon (\log \lambda_i)^{\frac{n+4}{2n}}}{\lambda_i^4} \\
& + (\text{if } n < 12) \big(\frac{\varepsilon}{\lambda_i^{\frac{n-4}{2}}} + \frac{1}{\lambda_i^{n-4}} \big) \Big) \Big].
\end{aligned} \tag{2.12}
$$

Using (2.8) and the implicit function theorem, we derive the existence of C^1-map which to (α, λ, x) associates $v_\varepsilon \in E$, such that v_ε minimizes $K_\varepsilon(\alpha, \lambda, x, v)$ with respect to $v \in E$ and

$$\|v_\varepsilon\| = O(\|f_\varepsilon\|).$$

Thus the estimate of Proposition 2.1 follows from (2.12). $\square$

Next, we prove a useful expansion of the derivative of the function K_ε associated to (1.1), with respect to $\alpha_i,\ \lambda_i,\ x_i$. For sake of simplicity, we will write δ_i instead of δ_{x_i, λ_i}.

Lemma 2.2. *Assume that $(\alpha, \lambda, x, v) \in \mathcal{M}_\varepsilon$ and let $v := v_\varepsilon$ be the function obtained in Proposition 2.1. Then the following expansions hold*
(1)

$$\frac{\partial K_\varepsilon}{\partial \alpha_i}(\alpha, \lambda, x, v) = S_n \alpha_i \Big(1 - \alpha_i^{8/(n-4)}\Big) + O\big(\frac{\varepsilon}{\lambda_i^4} + \frac{1}{\lambda_i^{n-4}}\big),$$

(2)

$$
\begin{aligned}
& \lambda_i \frac{\partial K_\varepsilon}{\partial \lambda_i}(\alpha, \lambda, x, v) \\
& = -2\alpha_i c_4 \frac{\varepsilon}{\lambda_i^4} + \alpha_i (1 - 2\alpha_i^{8/(n-4)}) \frac{c_2 (n-4) H(x_i, x_i)}{2\lambda_i^{n-4}} \\
& \quad - c_2 \sum_{j \neq i} \frac{n-4}{2} \alpha_j \Big(1 - \alpha_j^{8/(n-4)} - \alpha_i^{8/(n-4)}\Big) \frac{G(x_i, x_j)}{(\lambda_i \lambda_j)^{(n-4)/2}} \\
& \quad + O\Big(\frac{\varepsilon}{\lambda_i^{n-4}} + \frac{1}{\lambda_i^{n-2}} + (\textit{if } n \geq 12) \frac{\varepsilon^2 (\log \lambda_i)^{\frac{n+4}{n}}}{\lambda_i^8} + (\textit{if } n < 12) \frac{\varepsilon^2}{\lambda_i^{n-4}}\Big),
\end{aligned}
$$

(3)

$$
\begin{aligned}
& \frac{1}{\lambda_i} \frac{\partial K_\varepsilon}{\partial x_i}(\alpha, \lambda, x, v) \\
& = \alpha_i (2\alpha_i^{\frac{8}{n-4}} - 1) \frac{c_2}{2\lambda_i^{n-3}} \frac{\partial H(x_i, x_i)}{\partial x_i} \\
& \quad + c_2 \sum_{j \neq i} \alpha_j (1 - \alpha_j^{8/(n-4)} - \alpha_i^{8/(n-4)}) \frac{1}{\lambda_i^{(n-2)/2} \lambda_j^{(n-4)/2}} \frac{\partial G(x_i, x_j)}{\partial x_i} \\
& \quad + O\Big(\frac{\varepsilon}{\lambda_i^{n-3}} + \frac{1}{\lambda_i^{n-2}} + (\textit{if } n \geq 12) \frac{\varepsilon^2 (\log \lambda_i)^{\frac{n+4}{n}}}{\lambda_i^8} + (\textit{if } n < 12) \frac{\varepsilon^2}{\lambda_i^{n-4}}\Big),
\end{aligned}
$$

where

$$S_n = \int_{\mathbb{R}^n} \delta_{o,1}^{2n/(n-4)} dy, \quad c_2 = c_n^{2n/(n-4} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{(n+4)/2}} dy, \quad c_4 = \int_{\mathbb{R}^n} \delta_{o,1}^2 dy.$$

Proof. To prove Claim 1, we write

$$
\begin{aligned}
&\frac{\partial K_\varepsilon}{\partial \alpha_i}(\alpha,\lambda,x,v)\\
&=\sum \alpha_j\big(P\delta_j,P\delta_i\big)-\int_\Omega\big(\sum\alpha_jP\delta_j+v\big)^{\frac{n+4}{n-4}}P\delta_i+\varepsilon\int_\Omega\big(\sum\alpha_jP\delta_j+v\big)P\delta_i\\
&=\alpha_i\big(P\delta_i,P\delta_i\big)-\alpha_i^{\frac{n+4}{n-4}}\int_\Omega P\delta_i^{\frac{2n}{n-4}}-\frac{n+4}{n-4}\alpha_i^{\frac{8}{n-4}}\int_\Omega P\delta_i^{\frac{n+4}{n-4}}v\\
&\quad+O\Big(\sum_{j\neq i}\int_\Omega\delta_j^{(n+4)/(n-4)}\delta_i+\sum_{j\neq i}\varepsilon\int_\Omega\delta_i\delta_j+\|v\|^2+\varepsilon\int_\Omega\delta_i^2+\varepsilon\int_\Omega\delta_i|v|\Big).
\end{aligned}
$$

Using [7, Proposition 2.1], we have $\varphi_i=c_n\frac{H(x_i,.)}{\lambda_i^{(n-4)/2}}+O\left(\frac{1}{\lambda_i^{n/2}}\right)$. A computation similar to the one performed in [2] and [23] shows that

$$
\big(P\delta_i,P\delta_i\big)=S_n-c_2\frac{H(x_i,x_i)}{\lambda_i^{n-4}}+O\big(\frac{1}{\lambda_i^{n-2}}\big), \tag{2.13}
$$

$$
\int_\Omega P\delta_i^{\frac{2n}{n-4}}=S_n-\frac{2n}{n-4}c_2\frac{H(x_i,x_i)}{\lambda_i^{n-4}}+O\Big(\frac{1}{\lambda_i^{n-2}}\Big), \tag{2.14}
$$

$$
\big(P\delta_i,P\delta_j\big)=c_2\big(\varepsilon_{ij}-\frac{H(x_i,x_j)}{(\lambda_i\lambda_j)^{(n-4)/2}}\big)+O\Big(\sum_{k=i,j}\frac{1}{\lambda_k^{n-2}}\Big)\quad\text{for } i\neq j, \tag{2.15}
$$

$$
\int_\Omega P\delta_iP\delta_j^{\frac{n+4}{n-4}}=\big(P\delta_i,P\delta_j\big)+O\Big(\sum_{k=i,j}\frac{1}{\lambda_k^{n-2}}\Big)\quad\text{for } i\neq j, \tag{2.16}
$$

where $\varepsilon_{ij}=\big(\lambda_i/\lambda_j+\lambda_j/\lambda_i+\lambda_i\lambda_j|x_i-x_j|^2\big)^{(4-n)/2}$. Using the fact that $n\geq 9$ then

$$
\int_\Omega\delta_i^2=\frac{c_4}{\lambda_i^4}+O\big(\frac{1}{\lambda_i^{n-4}}\big), \tag{2.17}
$$

$$
\int_\Omega\delta_i\delta_j=O\big(\frac{1}{(\lambda_i\lambda_j)^{(n-4)/2}}\big)\quad\text{for } i\neq j. \tag{2.18}
$$

From (2.10), (2.11), (2.13)–(2.18) and Proposition 2.1, Claim 1 follows.

Now, we prove Claim 2. As in Claim 1 we have

$$
\begin{aligned}
&\lambda_i\frac{\partial K_\varepsilon}{\partial \lambda_i}(\alpha,\lambda,x,v)\\
&=\sum\alpha_j\big(P\delta_j,\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\big)-\int_\Omega\big(\sum\alpha_jP\delta_j+v\big)^{\frac{n+4}{n-4}}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\\
&\quad+\varepsilon\int_\Omega\big(\sum\alpha_jP\delta_j+v\big)\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\\
&=\sum\alpha_j\big(P\delta_j,\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\big)-\int_\Omega\big(\sum\alpha_jP\delta_j\big)^{\frac{n+4}{n-4}}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\\
&\quad-\frac{n+4}{n-4}\int_\Omega\big(\sum\alpha_jP\delta_j\big)^{\frac{8}{n-4}}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}v+\varepsilon\int_\Omega\big(\sum\alpha_jP\delta_j+v\big)\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}+O(\|v\|^2).
\end{aligned} \tag{2.19}
$$

Note that

$$\int_\Omega \big(\sum \alpha_j P\delta_j\big)^{\frac{8}{n-4}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} v = \int_\Omega \big(\alpha_i P\delta_i\big)^{\frac{8}{n-4}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} v + O\Big(\sum_{k\neq j} \int_{\delta_k \leq \delta_j} \delta_j^{\frac{8}{n-4}} \delta_k |v|\Big). \tag{2.20}$$

Using the fact that $v \in E$, we have

$$\int_\Omega P\delta_i^{\frac{8}{n-4}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} v = O\Big(\int_\Omega \frac{\delta_i^{\frac{8}{n-4}}}{\lambda_i^{\frac{n-4}{2}}} |v|\Big) = O\Big(\frac{\|v\|(\log \lambda_i)^{\frac{n+4}{2n}}}{\lambda_i^{\frac{n+4}{2}}} + (\text{if } n < 12) \frac{\|v\|}{\lambda_i^{n-4}}\Big), \tag{2.21}$$

and for $n \geq 8$, we have $\frac{8}{n-4} \leq 2$, then we obtain

$$\begin{aligned} \int_{\delta_k \leq \delta_j} \delta_j^{\frac{8}{n-4}} \delta_k |v| &= O\Big(\int \delta_j^{\frac{8}{n-4}} |v|^2 + \int_{\delta_k \leq \delta_j} \delta_j^{\frac{8}{n-4}} \delta_k^2\Big) \\ &= O\Big(\|v\|^2 + \int_\Omega (\delta_j \delta_k)^{\frac{n}{n-4}}\Big) \\ &= O\Big(\frac{1}{(\lambda_j \lambda_k)^{(n-1)/2}} + \|v\|^2\Big) \end{aligned} \tag{2.22}$$

where we have used $\int_\Omega (\delta_i \delta_j)^{\frac{n}{n-4}} = O\left(\varepsilon_{ij}^{\frac{n}{n-4}} \log \varepsilon_{ij}^{-1}\right)$. Observe that

$$\begin{aligned} \Big(\sum \alpha_j P\delta_j\Big)^{\frac{n+4}{n-4}} &= \sum \big(\alpha_j P\delta_j\big)^{\frac{n+4}{n-4}} + \frac{n+4}{n-4} \sum_{j\neq i} \big(\alpha_i P\delta_i\big)^{\frac{8}{n-4}} \alpha_j P\delta_j \\ &\quad + O\Big(\sum_{j\neq i} P\delta_j^{\frac{8}{n-4}} P\delta_i \chi_{P\delta_i \leq \sum_{j\neq i} P\delta_j} \\ &\quad + \sum_{j\neq i} P\delta_i^{\frac{12-n}{n-4}} P\delta_j^2 \chi_{P\delta_j \leq P\delta_i} + \sum_{k\neq j, k, j\neq i} P\delta_j^{\frac{8}{n-4}} P\delta_k\Big). \end{aligned} \tag{2.23}$$

Using [7, Proposition 2.1], we have $\lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} = -c_n \frac{n-4}{2} \frac{H(x_i,.)}{\lambda_i^{(n-4)/2}} + O\big(\frac{1}{\lambda_i^{n/2}}\big)$. A computation similar to the one performed in [2] and [23] shows that

$$\big(P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\big) = \frac{n-4}{2} c_2 \frac{H(x_i, x_i)}{\lambda_i^{n-4}} + O\big(\frac{1}{\lambda_i^{n-2}}\big), \tag{2.24}$$

$$\int_\Omega P\delta_i^{\frac{n+4}{n-4}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = (n-4) c_2 \frac{H(x_i, x_i)}{\lambda_i^{n-4}} + O\big(\frac{1}{\lambda_i^{n-2}}\big). \tag{2.25}$$

For $i \neq j$, we have

$$\big(P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\big) = c_2 \big(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{2} \frac{H(x_i, x_j)}{(\lambda_i \lambda_j)^{(n-4)/2}}\big) + O\Big(\sum_{k=i,j} \frac{1}{\lambda_k^{n-2}}\Big), \tag{2.26}$$

$$\int_\Omega P\delta_j^{\frac{n+4}{n-4}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = \big(P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\big) + O\Big(\sum_{k=i,j} \frac{1}{\lambda_k^{n-2}}\Big), \tag{2.27}$$

$$\int_\Omega P\delta_j \lambda_i \frac{\partial (P\delta_i)^{\frac{n+4}{n-4}}}{\partial \lambda_i} = \big(P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\big) + O\Big(\sum_{k=i,j} \frac{1}{\lambda_k^{n-2}}\Big). \tag{2.28}$$

We compute now the other integrals

$$
\begin{aligned}
&\int_\Omega P\delta_i\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\\
&=\int_\Omega(\delta_i+\varphi_i)(\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}+\lambda_i\frac{\partial\varphi_i}{\partial\lambda_i})\\
&=\int_\Omega\delta_i\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}+O\left(\int\varphi_i\delta_i+\int\delta_i\lambda_i|\frac{\partial\varphi_i}{\partial\lambda_i}|\right)\\
&=\frac{1}{2}\lambda_i\frac{\partial}{\partial\lambda_i}\Big(\int_{\mathbb{R}^n}\delta_i^2\Big)+O(\frac{1}{\lambda_i^{n-4}})+O\Big(\Big(\int_{B_i}\delta_i\Big)\Big(\|\varphi_i\|_{L^\infty}+\|\lambda_i\frac{\partial\varphi_i}{\partial\lambda_i}\|_{L^\infty}\Big)\Big)\\
&=\frac{-2c_4}{\lambda_i^4}+O(\frac{1}{\lambda_i^{n-4}}),
\end{aligned}
\tag{2.29}
$$

$$
\int_\Omega P\delta_j\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}=O\Big(\int_\Omega\delta_i\delta_j\Big)=O\Big(\frac{1}{(\lambda_i\lambda_j)^{(n-4)/2}}\Big)\quad\text{for } i\neq j,
\tag{2.30}
$$

and as in (2.11)

$$
\int_\Omega\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}v=O\Big((\text{if } n\geq 12)\frac{\|v\|(\log\lambda_i)^{(n+4)/2n}}{\lambda_i^4}+(\text{if } n<12)\frac{\|v\|}{\lambda_i^{(n-4)/2}}\Big).
\tag{2.31}
$$

Using the fact that $|x_i-x_j|>d_0'$ then

$$
\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}=-\frac{n-4}{2}\frac{1}{(\lambda_i\lambda_j|x_i-x_j|^2)^{(n-4)/2}}+O\Big(\sum_{k=i,j}\frac{1}{\lambda_k^{n-2}}\Big).
\tag{2.32}
$$

The Claim 2 follows from Proposition 2.1 and (2.19)–(2.32).

Regarding Claim 3, its proof is similar to Claim 2, so we will omit it. □

Lemma 2.3. *Assume that $(\alpha,\lambda,x,v)\in\mathcal{M}_\varepsilon$ and let $v:=v_\varepsilon$ be the function obtained in Proposition 2.1. Then the following expansion holds*

$$
\begin{aligned}
&K_\varepsilon(\alpha,\lambda,x,v)\\
&=\frac{S_n}{2}\Big(\sum_{i=1}^k\alpha_i^2-\frac{n-4}{n}\sum_{i=1}^k\alpha_i^{p+1}\Big)+\frac{c_2}{2}\sum_{i=1}^k\alpha_i^2\left(2\alpha_i^{p-1}-1\right)\frac{H(x_i,x_i)}{\lambda_i^{n-4}}\\
&\quad+\frac{c_2}{2}\sum_{j\neq i}\alpha_i\alpha_j\left(1-2\alpha_i^{p-1}\right)\frac{G(x_i,x_i)}{(\lambda_j\lambda_i)^{(n-4)/2}}+\frac{\varepsilon}{2}\sum_{i=1}^k\alpha_i^2\frac{c_4}{\lambda_i^4}\\
&\quad+O\Big(\sum_{i=1}^k\Big(\frac{\varepsilon}{\lambda_i^{n-4}}+\frac{1}{\lambda_i^{n-2}}+(\textit{if } n\geq 12)\frac{\varepsilon^2(\log\lambda_i)^{\frac{n+4}{n}}}{\lambda_i^8}+(\textit{if } n<12)\frac{\varepsilon^2}{\lambda_i^{n-4}}\Big)\Big).
\end{aligned}
$$

Proof. Using (2.7) and Proposition 2.1, this lemma follows from (2.13)–(2.18). □

Let

$$
\mathcal{M}_\varepsilon^1=\{(\lambda,x)\in(\mathbb{R}_+^*)^k\times\Omega_{d_0}^k:\lambda_i>\frac{1}{\nu_0}\ \forall i,\ \frac{\lambda_i}{\lambda_j}<c_0,\ |x_i-x_j|>d_0'\ \forall i\neq j\}.
$$

For $(\lambda,x)\in\mathcal{M}_\varepsilon^1$, our aim is to study the α-part of u. Namely, we prove the following result.

Proposition 2.4. *There exists $\varepsilon_1 > 0$ such that, for $0 < \varepsilon < \varepsilon_1$, there exists a C^1-map which to any $(\lambda, x) \in \mathcal{M}^1_\varepsilon$, associates $\alpha := \alpha_{(\varepsilon,\lambda,x)}$, which satisfies* (2.3) *for each i and we have the following estimate*

$$|\alpha_i - 1| = O\Big(\frac{\varepsilon}{\lambda_i^4} + \frac{1}{\lambda_i^{n-4}}\Big).$$

Proof. Let $\beta_i = 1 - \alpha_i$. By Lemma 2.2, we have

$$\frac{8}{n-4}\beta_i S_n + O(\beta_i^2) = O\Big(\frac{\varepsilon}{\lambda_i^4} + \frac{1}{\lambda_i^{n-4}}\Big),$$

then $\beta_i = O\big(\frac{\varepsilon}{\lambda_i^4} + \frac{1}{\lambda_i^{n-4}}\big)$. On the other hand, we have

$$\frac{\partial^2 K_\varepsilon}{\partial\alpha_i\partial\alpha_j}(\alpha, \lambda, x, v) = (1-p)S_n\delta_i^j + o(1),$$

with δ_i^j the Kronecker symbol and $o(1)$ tends to zero when $\varepsilon \to 0$, where we have used (2.13), (2.14), Proposition 2.1, the fact that $\partial v/\partial\alpha_i \in E$ and $\|\partial v/\partial\alpha_i\| = o(1)$. Using the implicit function theorem the proposition follows. $\square$

3. Proof of Theorems 1.1 and 1.3

Proof of Theorem 1.3. Assume that u_ε is a family of solutions of (1.1) which has the form (1.9) where $(\alpha^\varepsilon, \lambda^\varepsilon, x^\varepsilon, v^\varepsilon) \in \mathcal{M}_\varepsilon$. The result of the theorem will be obtained through a careful analysis of (2.3), (2.4), (2.5) and (2.6). From Proposition 2.1, there exists v^ε satisfying (2.6). We estimate now the corresponding numbers A_i, B_i, C_{ij} by taking the scalar product of (2.6) with $P\delta_i$, $\partial P\delta_i/\partial\lambda_i$ and $\partial P\delta_i/\partial(x_i)_r$ for $i = 1, \dots, k$ and $r = 1, \dots, n$. Thus from the right side we get a quasi-diagonal system whose coefficients are given by

$$(P\delta_i, P\delta_j) = S_n\delta_i^j + O(\frac{1}{\lambda_i^{n-4}}), (\frac{\partial P\delta_j}{\partial\lambda_j}, P\delta_i) = O\Big(\sum_{k=i,j}\frac{1}{\lambda_k^{n-3}}\Big),$$

$$(\frac{\partial P\delta_j}{\partial(x_j)_r}, P\delta_i) = O\Big(\sum_{k=i,j}\frac{1}{\lambda_k^{n-4}}\Big), (\frac{\partial P\delta_j}{\partial\lambda_j}, \frac{\partial P\delta_i}{\partial\lambda_i}) = \frac{n+4}{n-4}\frac{C_n}{\lambda_i^2}\delta_i^j + O\Big(\sum_{k=i,j}\frac{1}{\lambda_k^{n-2}}\Big),$$

$$(\frac{\partial P\delta_j}{\partial\lambda_j}, \frac{\partial P\delta_i}{\partial(x_i)_r}) = O\Big(\sum_{k=i,j}\frac{1}{\lambda_k^{n-3}}\Big), (\frac{\partial P\delta_i}{\partial x_i}, \frac{\partial P\delta_j}{\partial x_j}) = O\Big(\sum_{k=i,j}\frac{1}{\lambda_k^{n-5}}\Big) \text{ for } i \neq j,$$

$$(\frac{\partial P\delta_i}{\partial(x_i)_r}, \frac{\partial P\delta_i}{\partial(x_i)_j}) = \frac{n+4}{n-4}C'_n\lambda_i^2\delta_r^j + O(\frac{1}{\lambda_i^{n-5}}),$$

where δ_i^j is the Kronecker symbol, S_n is defined in Lemma 2.2,

$$C_n = \frac{(n-4)^2}{4}\int_{\mathbb{R}^n}\frac{(1-|y|^2)^2}{(1+|y|^2)^{n+2}}dy \quad \text{and} \quad C'_n = \frac{(n-4)^2}{4n}\int_{\mathbb{R}^n}\frac{|y|^2}{(1+|y|^2)^{n+2}}dy\,.$$

The left side is given by

$$(\frac{\partial K_\varepsilon}{\partial v}, P\delta_i) = \frac{\partial K_\varepsilon}{\partial\alpha_i}, \quad (\frac{\partial K_\varepsilon}{\partial v}, \frac{\partial P\delta_i}{\partial\lambda_i}) = \frac{1}{\alpha_i}\frac{\partial K_\varepsilon}{\partial\lambda_i}, \quad (\frac{\partial K_\varepsilon}{\partial v}, \frac{\partial P\delta_i}{\partial(x_i)_r}) = \frac{1}{\alpha_i}\frac{\partial K_\varepsilon}{\partial(x_i)_r}.$$

Let $\beta_i = 1-\alpha_i$. By Lemma 2.2, we have

$$\frac{\partial K_\varepsilon}{\partial \alpha_i}(\alpha, x, \lambda, v^\varepsilon) = O\Big(|\beta_i| + \frac{\varepsilon}{\lambda_i^4} + \frac{1}{\lambda_i^{n-4}}\Big),$$

$$\frac{\partial K_\varepsilon}{\partial \lambda_i}(\alpha, x, \lambda, v^\varepsilon) = O\Big(\frac{\varepsilon}{\lambda_i^5} + \frac{1}{\lambda_i^{n-3}}\Big),$$

$$\begin{aligned}&\frac{\partial K_\varepsilon}{\partial (x_i)_r}(\alpha, x, \lambda, v^\varepsilon)\\ &= O\Big(\frac{1}{\lambda_i^{n-4}} + \frac{\varepsilon}{\lambda_i^{n-4}} + (\text{if } n \geq 12)\frac{\varepsilon^2(\log \lambda_i)^{(n+4)/n}}{\lambda_i^7} + (\text{if } n < 12)\frac{\varepsilon^2}{\lambda_i^{n-5}}\Big).\end{aligned}$$

The solution of the system in A_i, B_i, C_{ij} shows that

$$A_i = O\big(|\beta_i| + \frac{\varepsilon}{\lambda_i^4} + \frac{1}{\lambda_i^{n-4}}\big)$$

$$B_i = O\big(\frac{\varepsilon}{\lambda_i^3} + \frac{1}{\lambda_i^{n-5}}\big)$$

$$C_{ij} = O\big(\frac{1}{\lambda_i^{n-2}} + \frac{\varepsilon}{\lambda_i^{n-2}} + (\text{if } n \geq 12)\frac{\varepsilon^2(\log \lambda_i)^{\frac{n+4}{n}}}{\lambda_i^9} + (\text{if } n < 12)\frac{\varepsilon^2}{\lambda_i^{n-3}}\big).$$

This allows us to evaluate the right hand side in the equations (2.3) and (2.5), namely

$$\begin{aligned}&B_i\big(\frac{\partial^2 P\delta_i}{\partial \lambda_i \partial (x_i)_r}, v^\varepsilon\big) + \sum_{j=1}^n C_{ij}\big(\frac{\partial^2 P\delta_i}{\partial (x_i)_j \partial (x_i)_r}, v^\varepsilon\big)\\ &= O\big(|B_i|\|v^\varepsilon\| + \sum_{j=1}^n \lambda_i^2 |C_{ij}|\|v^\varepsilon\|\big) = O\big(\|v^\varepsilon\|(\frac{\varepsilon}{\lambda_i^3} + \frac{1}{\lambda_i^{n-5}})\big) = O\big(\frac{1}{\lambda_i^{n-3}} + \frac{\varepsilon^2}{\lambda_i^5}\big).\end{aligned} \tag{3.1}$$

In the same manner, we obtain

$$B_i\big(\frac{\partial^2 P\delta_i}{\partial \lambda_i^2}, v^\varepsilon\big) + \sum_{j=1}^n C_{ij}\big(\frac{\partial^2 P\delta_i}{\partial (x_i)_j \partial \lambda_i}, v^\varepsilon\big) = O\big(\frac{1}{\lambda_i^{n-1}} + \frac{\varepsilon^2}{\lambda_i^5}\big). \tag{3.2}$$

From Proposition 2.4 , there exists α^ε satisfying (2.3) for each i, and we have

$$1 - \alpha_i^\varepsilon = \beta_i^\varepsilon = O\Big(\frac{\varepsilon}{(\lambda_i^\varepsilon)^4} + \frac{1}{(\lambda_i^\varepsilon)^{n-4}}\Big). \tag{3.3}$$

Using (3.1), (3.2), (3.3) and Lemma 2.2, we deduce that (2.4) and (2.5) are equivalent to

$$\begin{aligned}&-2c_4\frac{\varepsilon}{(\lambda_i^\varepsilon)^4} - \frac{c_2(n-4)H(x_i^\varepsilon, x_i^\varepsilon)}{2(\lambda_i^\varepsilon)^{n-4}} + c_2\sum_{j\neq i}\frac{(n-4)G(x_i^\varepsilon, x_j^\varepsilon)}{2(\lambda_i^\varepsilon\lambda_j^\varepsilon)^{(n-4)/2}}\\ &= O\Big(\frac{\varepsilon}{(\lambda_i^\varepsilon)^{n-4}} + \frac{1}{(\lambda_i^\varepsilon)^{n-2}} + \frac{\varepsilon^2}{(\lambda_i^\varepsilon)^4}\Big),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&-\frac{c_2(n-4)}{2(\lambda_i^\varepsilon)^{n-4}}\frac{\partial H(x_i^\varepsilon, x_i^\varepsilon)}{\partial x_i} + c_2\sum_{j\neq i}\frac{(n-4)}{2(\lambda_i^\varepsilon\lambda_j^\varepsilon)^{(n-4)/2}}\frac{\partial G(x_i^\varepsilon, x_j^\varepsilon)}{\partial x_i}\\ &= O\Big(\frac{\varepsilon}{(\lambda_i^\varepsilon)^{n-4}} + \frac{1}{(\lambda_i^\varepsilon)^{n-3}} + \frac{\varepsilon^2}{(\lambda_i^\varepsilon)^4}\Big).\end{aligned} \tag{3.5}$$

Let us perform the change of variables
$$\lambda_i^\varepsilon = (\Lambda_i^\varepsilon)^{\frac{-2}{n-4}} \varepsilon^{\frac{-1}{n-8}} \big(\frac{c_4}{c_2}\big)^{\frac{-1}{n-8}}. \tag{3.6}$$
Note that
$$\Lambda_i^\varepsilon \varepsilon^{\frac{n-4}{2(n-8)}} \to 0 \quad \text{as } \varepsilon \to 0, \quad \frac{\Lambda_i^\varepsilon}{\Lambda_j^\varepsilon} < c_0, \tag{3.7}$$
and that (3.4), (3.5) read
$$\begin{aligned} &\frac{-4}{n-4} (\Lambda_i^\varepsilon)^{\frac{12-n}{n-4}} - H(x_i^\varepsilon, x_i^\varepsilon)\Lambda_i^\varepsilon + \sum_{j\neq i} G(x_i^\varepsilon, x_j^\varepsilon)\Lambda_j^\varepsilon \\ &= O\left(\varepsilon \Lambda_i^\varepsilon + (\Lambda_i^\varepsilon)^{\frac{4}{n-4}} \varepsilon^{\frac{2}{n-8}} \Lambda_i^\varepsilon + \varepsilon (\Lambda_i^\varepsilon)^{\frac{12-n}{n-4}}\right), \end{aligned} \tag{3.8}$$
$$-\Lambda_i^\varepsilon \frac{\partial H(x_i^\varepsilon, x_i^\varepsilon)}{\partial x_i} + \sum_{j\neq i} \Lambda_j^\varepsilon \frac{\partial G(x_i^\varepsilon, x_j^\varepsilon)}{\partial x_i} = O\left(\varepsilon \Lambda_i^\varepsilon + (\Lambda_i^\varepsilon)^{\frac{2}{n-4}} \varepsilon^{\frac{1}{n-8}} \Lambda_i^\varepsilon + \varepsilon (\Lambda_i^\varepsilon)^{\frac{12-n}{n-4}}\right). \tag{3.9}$$
From (3.8), we deduce that
$$\begin{aligned} &\frac{-4}{n-4}{}^T \left((\Lambda_1^\varepsilon)^{\frac{12-n}{n-4}}, \dots, (\Lambda_k^\varepsilon)^{\frac{12-n}{n-4}}\right) - M(x^\varepsilon)\Lambda^\varepsilon \\ &= O\left(\varepsilon \Lambda_i^\varepsilon + (\Lambda_i^\varepsilon)^{\frac{4}{n-4}} \varepsilon^{\frac{2}{n-8}} \Lambda_i^\varepsilon + \varepsilon (\Lambda_i^\varepsilon)^{\frac{12-n}{n-4}}\right), \end{aligned} \tag{3.10}$$
where $\Lambda^\varepsilon =^T (\Lambda_1^\varepsilon, \dots, \Lambda_k^\varepsilon)$. Taking the scalar product of (3.10) with $e(x^\varepsilon)$, we obtain
$$\begin{aligned} &\frac{-4}{n-4} \sum_{i=1}^k (\Lambda_i^\varepsilon)^{\frac{12-n}{n-4}} e_i(x^\varepsilon) - \rho(x^\varepsilon) e(x^\varepsilon).\Lambda^\varepsilon \\ &= O\left(\varepsilon \Lambda_i^\varepsilon + (\Lambda_i^\varepsilon)^{\frac{4}{n-4}} \varepsilon^{\frac{2}{n-8}} \Lambda_i^\varepsilon + \varepsilon (\Lambda_i^\varepsilon)^{\frac{12-n}{n-4}}\right). \end{aligned} \tag{3.11}$$
We distinguish three cases:

1. $\Lambda_i^\varepsilon \to 0$, as $\varepsilon \to 0$ for all i.
2. $\Lambda_i^\varepsilon \to \Lambda_i \in \mathbb{R}_+^*$, as $\varepsilon \to 0$ for all i.
3. $\Lambda_i^\varepsilon \to +\infty$, as $\varepsilon \to 0$ for all i.

Multiplying (3.8) by $(\Lambda_i^\varepsilon)^{-1}$ and using the fact that $n \geq 9$, we see that case (1) cannot occur. Let us consider the second case. Denoting by $x \in \Omega_{d_0}^k$ the limit of x^ε (up to a subsequence), from (3.8) and (3.11), we obtain
$$\frac{4}{n-4} \Lambda_i^{\frac{12-n}{n-4}} + H(x_i, x_i)\Lambda_i - \sum_{j\neq i} G(x_i, x_j)\Lambda_j = 0,$$
$$\frac{4}{n-4} \sum_{i=1}^k \Lambda_i^{\frac{12-n}{n-4}} e_i(x) + \rho(x) e(x).\Lambda = 0.$$
This means that $\rho(x) < 0$ and (Λ, x) satisfied $\frac{\partial \Psi_k}{\partial \Lambda}(\Lambda, x) = 0$. On the other hand, from (3.5) and (3.6) we deduce that
$$-(\Lambda_i^\varepsilon)^2 \frac{\partial H(x_i^\varepsilon, x_i^\varepsilon)}{\partial x_i} + \sum_{j\neq i} \Lambda_j^\varepsilon \Lambda_i^\varepsilon \frac{\partial G(x_i^\varepsilon, x_j^\varepsilon)}{\partial x_i} = O(\varepsilon^{\frac{1}{n-8}}),$$

then, as $\varepsilon \to 0$, we derive

$$-\Lambda_i^2 \frac{\partial H(x_i, x_i)}{\partial x_i} + \sum_{j \neq i} \Lambda_j \Lambda_i \frac{\partial G(x_i, x_j)}{\partial x_i} = 0.$$

This implies that $\frac{\partial \Psi_k}{\partial x}(\Lambda, x) = 0$ i.e. exactly what we want to prove.

Let us now consider the last case. From (3.11) we have

$$\rho(x^\varepsilon) e(x^\varepsilon).\Lambda^\varepsilon = O\left((\Lambda_i^\varepsilon)^{\frac{12-n}{n-4}} + \varepsilon \Lambda_i^\varepsilon + (\Lambda_i^\varepsilon)^{\frac{4}{n-4}} \varepsilon^{\frac{2}{n-8}} \Lambda_i^\varepsilon + \varepsilon (\Lambda_i^\varepsilon)^{\frac{12-n}{n-4}} \right)$$

and therefore, since $\Lambda_i^\varepsilon / \Lambda_j^\varepsilon \leq c_0$ for each $i \neq j$, we obtain

$$\rho(x^\varepsilon) = O\left((\Lambda_i^\varepsilon)^{\frac{2(8-n)}{n-4}} + \varepsilon + (\Lambda_i^\varepsilon)^{\frac{4}{n-4}} \varepsilon^{\frac{2}{n-8}} \right).$$

Thus, using (3.7) we get $\rho(x) = 0$. It remains to prove that $\rho'(x) = 0$. First, we claim that the vector Λ^ε is close to $e(x^\varepsilon)$. In fact Λ^ε may be written under the form

$$\Lambda^\varepsilon = \xi^\varepsilon e(x^\varepsilon) + e'(x^\varepsilon), \tag{3.12}$$

with $e(x^\varepsilon).e'(x^\varepsilon) = 0$. It is easy to get $\xi^\varepsilon = O(|\Lambda^\varepsilon|)$. Now, using the fact that ${}^T\Lambda^\varepsilon M(x^\varepsilon)\Lambda^\varepsilon = o(|\Lambda^\varepsilon|^2)$ (by (3.8)), $\rho(x^\varepsilon) \to 0$ and the fact that zero is a simple eigenvalue of the matrix $M(x)$ then $e'(x^\varepsilon) = o(\xi^\varepsilon)$ and our claim follows. From (3.9) we obtain

$$\frac{\partial M(x^\varepsilon)}{\partial x_i} \Lambda^\varepsilon = o(|\Lambda^\varepsilon|),$$

using (3.12), we obtain

$$\xi^\varepsilon \frac{\partial M(x^\varepsilon)}{\partial x_i} e(x^\varepsilon) + \frac{\partial M(x^\varepsilon)}{\partial x_i} e'(x^\varepsilon) = o(\xi^\varepsilon). \tag{3.13}$$

The matrix $\frac{\partial M(x^\varepsilon)}{\partial x_i}$ being bounded on the set $\{x \in \Omega_{d_0}^k, |x_i - x_j| > d_0'\}$, we get

$$\frac{\partial M(x^\varepsilon)}{\partial x_i} e'(x^\varepsilon) = O(|e'(x^\varepsilon)|) = o(\xi^\varepsilon).$$

The scalar product of (3.13) with $e(x^\varepsilon)$ gives

$${}^T e(x^\varepsilon) \frac{\partial M(x^\varepsilon)}{\partial x_i} e(x^\varepsilon) = o(1). \tag{3.14}$$

Since $|e(x^\varepsilon)|^2 = 1$ and $e(x^\varepsilon).\frac{\partial e(x^\varepsilon)}{\partial x_i} = 0$, therefore

$${}^T e(x^\varepsilon) \frac{\partial M(x^\varepsilon)}{\partial x_i} e(x^\varepsilon) = \frac{\partial \rho}{\partial x_i}(x^\varepsilon). \tag{3.15}$$

Passing to the limit in (3.14) and (3.15), we obtain

$$\frac{\partial \rho}{\partial x_i}(x) = 0.$$

This concludes the proof of Theorem 1.3. □

Proof of Theorem 1.1. Arguing by contradiction, suppose that (1.1) has a solution of the form (1.6) and satisfying (1.7). Multiplying (1.1) by v_ε and integrating over Ω, we obtain

$$\begin{aligned}\|v_\varepsilon\|^2 &= \int_\Omega |\alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon|^p v_\varepsilon - \varepsilon\int_\Omega (\alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon)\, v_\varepsilon \quad (3.16)\\ &= \alpha_\varepsilon^p \int_\Omega P\delta^p_{x_\varepsilon,\lambda_\varepsilon} v_\varepsilon + p\alpha_\varepsilon^{p-1}\int_\Omega P\delta^{p-1}_{x_\varepsilon,\lambda_\varepsilon} v_\varepsilon^2 + o(\|v_\varepsilon\|^2) + O\Big(\varepsilon\int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}|v_\varepsilon|\Big). \quad (3.17)\end{aligned}$$

From Proposition 3.4 of [4] and the fact that $\alpha_\varepsilon \to 1$, there exists a positive constant c, such that

$$\|v_\varepsilon\|^2 - p\alpha_\varepsilon^{p-1}\int_\Omega P\delta^{p-1}_{x_\varepsilon,\lambda_\varepsilon} v_\varepsilon^2 = \|v_\varepsilon\|^2 - p\int_\Omega \delta^{p-1}_{x_\varepsilon,\lambda_\varepsilon} v_\varepsilon^2 + o\left(\|v_\varepsilon\|^2\right) \ge c\|v_\varepsilon\|^2. \quad (3.18)$$

On the other hand, using the fact that $v_\varepsilon \in E_{x_\varepsilon,\lambda_\varepsilon}$, we obtain

$$\int_\Omega P\delta^p_{x_\varepsilon,\lambda_\varepsilon} v_\varepsilon = O\Big(\int_{B\cup B^c} |\varphi_{x_\varepsilon,\lambda_\varepsilon}|\delta^{p-1}_{x_\varepsilon,\lambda_\varepsilon}|v_\varepsilon|\Big),$$

where $B = \{y : |y - x_\varepsilon| < d_\varepsilon\}$. Then using the Holder's inequality we need to estimate

$$\int_{B^c}\left(\delta^{p-1}_{x_\varepsilon,\lambda_\varepsilon}\varphi_{x_\varepsilon,\lambda_\varepsilon}\right)^{2n/(n+4)} \le \int_{B^c}\delta^{p+1}_{x_\varepsilon,\lambda_\varepsilon} = O\big(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^n}\big) \quad (3.19)$$

and

$$\begin{aligned}&|\varphi_{x_\varepsilon,\lambda_\varepsilon}|_{L^\infty}\Big(\int_B \delta_{x_\varepsilon,\lambda_\varepsilon}^{\frac{8(p+1)}{(n+4)}}\Big)^{\frac{n+4}{2n}}\\ &= O\Big((\text{if} n\ge 12)\frac{(\log\lambda_\varepsilon d_\varepsilon)^{\frac{n+4}{2n}}}{(\lambda_\varepsilon d_\varepsilon)^{\frac{n+4}{2}}} + (\text{if } n<12)\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}\Big).\end{aligned} \quad (3.20)$$

Combining (2.11), (3.16)–(3.20) we get

$$\|v_\varepsilon\| = O\Big(\frac{(\log\lambda_\varepsilon d_\varepsilon)^{\frac{n+4}{2n}}}{(\lambda_\varepsilon d_\varepsilon)^{\frac{n+4}{2}}} + \frac{\varepsilon(\log\lambda_\varepsilon)^{\frac{n+4}{2n}}}{\lambda_\varepsilon^4} + (\text{if } n<12)\big(\frac{\varepsilon}{\lambda_\varepsilon^{(n-4)/2}} + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}\big)\Big). \quad (3.21)$$

Multiplying (1.1) by $\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}/\partial\lambda_\varepsilon$ and integrating over Ω, we derive that

$$\begin{aligned}&\alpha_\varepsilon\big(P\delta_{x_\varepsilon,\lambda_\varepsilon}, \frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon}\big) - \int_\Omega |\alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon|^p\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon}\\ &+\varepsilon\int_\Omega(\alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon)\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon} = 0,\end{aligned}$$

which implies

$$\begin{aligned}&\alpha_\varepsilon\big(P\delta_{x_\varepsilon,\lambda_\varepsilon}, \frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon}\big) - \alpha_\varepsilon^p\int_\Omega P\delta^p_{x_\varepsilon,\lambda_\varepsilon}\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon} - p\alpha_\varepsilon^{p-1}\int_\Omega P\delta^{p-1}_{x_\varepsilon,\lambda_\varepsilon} v_\varepsilon\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon}\\ &+\varepsilon\alpha_\varepsilon\int_\Omega P\delta_{x_\varepsilon,\lambda_\varepsilon}\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon} + O\Big(\frac{\varepsilon}{\lambda_\varepsilon}\int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}|v_\varepsilon| + \frac{\|v_\varepsilon\|^2}{\lambda_\varepsilon}\Big) = 0.\end{aligned} \quad (3.22)$$

According to [7], we have

$$\Big(P\delta_{x_\varepsilon,\lambda_\varepsilon},\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon}\Big)=\frac{n-4}{2}c_2\frac{H(x_\varepsilon,x_\varepsilon)}{\lambda_\varepsilon^{n-3}}+O\big(\frac{1}{\lambda_\varepsilon(\lambda_\varepsilon d_\varepsilon)^{n-2}}\big), \tag{3.23}$$

$$\int_\Omega P\delta_{x_\varepsilon,\lambda_\varepsilon}^{\frac{n+4}{n-4}}\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon}=(n-4)c_2\frac{H(x_\varepsilon,x_\varepsilon)}{\lambda_\varepsilon^{n-3}}+O\big(\frac{1}{\lambda_\varepsilon(\lambda_\varepsilon d_\varepsilon)^{n-2}}\big), \tag{3.24}$$

and as in (2.29)

$$\int_\Omega P\delta_{x_\varepsilon,\lambda_\varepsilon}\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda_\varepsilon}=-2\frac{c_4}{\lambda_\varepsilon^5}+O\big(\frac{1}{\lambda_\varepsilon^5(\lambda_\varepsilon d_\varepsilon)^{n-8}}\big). \tag{3.25}$$

Taking (2.11), (2.21), (3.21), (3.23)–(3.25) in (3.22) we obtain the following relation

$$-2\varepsilon\frac{c_4}{\lambda_\varepsilon^5}-c_2\frac{n-4}{2}\frac{H(x_\varepsilon,x_\varepsilon)}{\lambda_\varepsilon^{n-3}}+o\big(\frac{\varepsilon}{\lambda_\varepsilon^5}+\frac{1}{\lambda_\varepsilon(\lambda_\varepsilon d_\varepsilon)^{n-4}}\big)=0$$

which is a contradiction. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we construct a domain Ω for which (1.1) has a solution which blows-up and concentrates in two points of Ω. More precisely, we will find a solution u_ε of the form

$$u_\varepsilon=\sum_{i=1}^{2}\alpha^\varepsilon_{i,(\lambda^\varepsilon,x^\varepsilon)}P\delta_{x_i^\varepsilon,\lambda_i^\varepsilon}+v^\varepsilon_{(\alpha^\varepsilon,\lambda^\varepsilon,x^\varepsilon)}, \tag{4.1}$$

where $\alpha^\varepsilon_{(\lambda^\varepsilon,x^\varepsilon)}$, $v^\varepsilon_{(\alpha^\varepsilon,\lambda^\varepsilon,x^\varepsilon)}$ are defined in Propositions 2.1, 2.4, $x_i^\varepsilon\in\Omega_{d_0}$, $|x_1^\varepsilon-x_2^\varepsilon|>d_0'$ and λ_i^ε satisfies $\lambda_i^\varepsilon=(\Lambda_i^\varepsilon)^{\frac{-2}{n-4}}\varepsilon^{\frac{-1}{n-8}}(\frac{c_4}{c_2})^{\frac{-1}{n-8}}$. For the rest of this article, we will consider the set

$$\mathcal{M}_\varepsilon^2=\{(\Lambda,x)\in(\mathbb{R}_+^*)^2\times\Omega_{d_0}^2: c<\Lambda_i<\frac{1}{c}\ \forall i,\ |x_1-x_2|>d_0'\}.$$

Let us define the functional

$$K_\varepsilon^2(\Lambda,x)=J_\varepsilon(u_\varepsilon).$$

Lemma 4.1. *We have the expansion*

$$\begin{aligned}K_\varepsilon^2(\Lambda,x)=\frac{4S_n}{n}&+\varepsilon^{\frac{n-4}{n-8}}c_4^{\frac{n-4}{n-8}}c_2^{\frac{-4}{n-8}}\Big[\frac12H(x_1,x_1)\Lambda_1^2+\frac12H(x_2,x_2)\Lambda_2^2\\&-G(x_1,x_2)\Lambda_1\Lambda_2+\frac12\big(\Lambda_1^{\frac{8}{n-4}}+\Lambda_2^{\frac{8}{n-4}}\big)\Big]+o(\varepsilon^{\frac{n-4}{n-8}}),\end{aligned}$$

in the C^1-norm with respect to $(\Lambda,x)\in\mathcal{M}_\varepsilon^2$, where c_2 and c_4 are defined in Lemma 2.2.

The proof of this lemma follows from Propositions 2.1, 2.4 and Lemmas (2.2), (2.3).

To find a solution of (1.1) with two blow-up points in Ω, it is enough to find "sufficiently stable" critical point of the function Ψ defined by

$$\begin{aligned}\Psi&:=\Psi_2(\Lambda,x)\\&=\frac12\big(H(x_1,x_1)\Lambda_1^2+H(x_2,x_2)\Lambda_2^2-2G(x_1,x_2)\Lambda_1\Lambda_2\big)+\frac12\big(\Lambda_1^{\frac{8}{n-4}}+\Lambda_2^{\frac{8}{n-4}}\big).\end{aligned}$$

Here we follow the ideas of [22], [11]. Let D be a bounded domain in $\mathbb{R}^n$ with smooth boundary which contains the origin 0. The following result holds (see Corollary 2.1 of [11] which is analogue corollary for the Laplacian).

Corollary 4.2. *For any sufficiently small $\sigma > 0$ there exists $r_0 > 0$ such that if $0 < r < r_0$ is fixed and Ω is a domain given by $D\backslash\omega$ for any smooth domain $\omega \subset B(0,r)$, then*
$$\rho(x) < 0 \quad \forall x \in \mathcal{S}^2,$$
where the manifold $\mathcal{S}$ is defined by $\mathcal{S} = \{x_1 \in \Omega : |x_1| = \sigma\}$.

Here $\rho(x)$ denotes the least eigenvalue of the matrix $M(x)$ defined in (1.8) ($\rho(x) = -\infty$ if $x_1 = x_2$). Let $e(x)$ be the eigenvector corresponding to $\rho(x)$ whose norm is 1 and whose all components are strictly positive.

In the following we will construct a critical point of the "min-max" type of the function Ψ. Let us introduce for $\delta > 0$ and $\rho > 0$ the following manifold
$$W_\rho^\delta = \{x \in \Omega_\delta^2 : \rho(x) < -\rho\}.$$
Let $\rho_0 = -\max_{x\in\mathcal{S}^2}\rho(x)$ and $\delta_0 = \mathrm{dist}(\mathcal{S}, \partial\Omega)$. It holds for any $0 < \rho < \rho_0$ and $0 < \delta < \delta_0$ that $\mathcal{S}^2 \subset W_\rho^\delta$. Since $\frac{8}{n-4} < 2$, there exist $R_0 > 0$ such that
$$b = \max_{x\in\mathcal{S}^2, 0\le R\le R_0} \Psi(Re(x), x) > 0 \quad \text{and} \quad \max_{x\in\mathcal{S}^2, R=0,R_0} \Psi(Re(x), x) = 0. \tag{4.2}$$
Next we let Γ be the class of continuous function $\gamma : [0, R_0]\times\mathcal{S}^2\times[0,1] \to \mathbb{R}_+^2 \times W_\rho^\delta$, such that

(1) $\gamma(0, x, t) = (0, x)$ and $\gamma(R_0, x, t) = (R_0 e(x), x)$ for all $x \in \mathcal{S}^2, t \in [0,1]$.
(2) $\gamma(R, x, 0) = (Re(x), x)$ for all $(R, x) \in [0, R_0] \times \mathcal{S}^2$.

For every $(R, x, t) \in [0, R_0]\times\mathcal{S}^2\times[0,1]$ we denote $\gamma(R, x, t) = (\tilde\Lambda(R, x, t), \tilde x(R, x, t))$ and, for $\tau > 0$, we define the set
$$\mathcal{I}_\tau = \{(R, x) \in [0, R_0] \times \mathcal{S}^2 : \tilde\Lambda_1(R, x, 1)\tilde\Lambda_2(R, x, 1) = \tau\}.$$
In the following we prove that Ψ has a critical level between a and b where b is defined in (4.2) and a will be defined in Corollary 4.4. The first step in this direction is the following topological result which is similar to [11, Lemma 7.1].

Lemma 4.3. *For every open neighborhood $\mathcal{U}$ of $\mathcal{I}_\tau$ in $\mathbb{R}_+^2 \times \mathcal{S}^2$, the projection $g : \mathcal{U} \to \mathcal{S}^2$ induces a monomorphism in cohomology, that is $g^* : H^*(\mathcal{S}^2) \to H^*(\mathcal{U})$ is a monomorphism.*

Corollary 4.4. *For $\tau > 0$ small, there exist $a = a(\tau) > 0$, such that*
$$\sup_{x\in\mathcal{S}^2, 0\le R\le R_0} \Psi(\gamma(R, x, 1)) \ge a \text{ for all } \gamma \in \Gamma.$$

Proof. Since Ω is smooth, there is $c_0 > 0$ such that if $x_1, x_2 \in \Omega_\delta$ and $|x_1 - x_2| < c_0$ then the line segment $[x_1, x_2] \subset \Omega$. Then we let $K > 0$ so that $G(x_1, x_2) \ge K$ implies $|x_1 - x_2| < c_0$. Assume, by contradiction, for each $a > 0$, there exists $\gamma \in \Gamma$ such that
$$\Psi(\gamma(R, x, 1)) < a \quad \text{for all } (R, x) \in [0, R_0] \times \mathcal{S}^2.$$
This implies that, for a small neighborhood $\mathcal{U}$ of $\mathcal{I}_\tau$ in $[0, R_0] \times \mathcal{S}^2$, we have
$$-G(\tilde x(R, x, 1))\tau + \tau^{4/(n-4)} \le a,$$
and therefore
$$G(\tilde x(R, x, 1)) \ge \frac{1}{2}\tau^{\frac{8-n}{n-4}} \ge K \tag{4.3}$$
if we choose $2a < \tau^{4/(n-4)}$ and τ small. Let $D_0 = \mathbb{R}_+^2 \times \Omega \times \Omega$ and $\gamma_1 = \gamma(., 1)$. Consider the inclusion $i_2 : \gamma_1(\mathcal{U}) \to D_0$ and the maps $p : \gamma_1(\mathcal{U}) \to \mathbb{R}_+^2 \times \Omega$ and

$f:\mathbb{R}^2_+\times\Omega\to D_0$ defined as $p(\Lambda,x_1,x_2)=(\Lambda,x_1)$ and $f(\Lambda,x_1)=(\Lambda,x_1,x_1)$. From (4.3) we find that the function $h:\gamma_1(\mathcal{U})\times[0,1]\to D_0$ defined as $h(\Lambda,x_1,x_2,t)=(\Lambda,x_1,x_2+t(x_1-x_2))$ is a homotopy between i_2 and fop. We consider the commutative diagram

$$\begin{array}{ccc} H^*([0,R_0]\times\mathcal{S}^2) & \xleftarrow{\gamma_1^*} & H^*(D_0) \\ \downarrow i_1^* & & \downarrow i_2^* \\ H^*(\mathcal{U}) & \xleftarrow{\gamma_2^*} & H^*(\gamma_1(\mathcal{U})) \end{array}$$

where i_1 is the inclusion map and $\gamma_2=\gamma_1/\mathcal{U}$. Let $u\in H^{n-1}(\mathcal{S})$ and $v\in H^{n-1}(\Omega)$ nontrivial elements such that $i(v)=u$. If $\hat v\times\hat v\in H^{2(n-1)}(D_0)$ is the corresponding element, then by homotopy axiom and Lemma 4.3 we have $i_1^*o\gamma_1^*(\hat v\times\hat v)\neq 0$. On the other hand we see that $f^*(\hat v\times\hat v)=\hat v\smile\hat v\in H^{2(n-1)}(\mathbb{R}^2_+\times\Omega)$ is zero, because $H^{2(n-1)}(\Omega)=0$. Then we have $\gamma_2^*oi_2^*(\hat v\times\hat v)=0$, providing a contradiction. $\square$

Let $T_\delta=\{x\in\mathcal{S}^2:|x_1-x_2|\le\delta\}$. We can choose δ small such that

$$\Psi(Re(x),x)<\frac{a}{2}\quad\text{for each } x\in T_\delta\quad\text{and } 0\le R\le R_0. \tag{4.4}$$

Let us introduce the manifold

$$V_\delta=\{x\in\Omega^2_\delta:|x_1-x_2|>\delta\}.$$

To prove that the function Ψ constrained to $\mathbb{R}^2_+\times(W^\delta_\rho\cap V_\delta)$ has a critical level between a and b we need to care about the fact that the domain $\mathbb{R}^2_+\times(W^\delta_\rho\cap V_\delta)$ is not necessarily closed for the gradient flow of Ψ. The following lemma, is the first step in this direction.

Lemma 4.5. *There exists $\delta_0'>0$ such that for any $\delta\in(0,\delta_0')$ and for any $(\Lambda,x)\in\mathbb{R}^2_+\times(W^\delta_\rho\cap V_\delta)$ with $\Psi(\Lambda,x)\in[a,b]$, $\nabla_\Lambda\Psi(\Lambda,x)=0$ and $x=(x_1,x_2)\in\partial V_\delta$, then there exists a vector T tangent to $\mathbb{R}^2_+\times\partial V_\delta$ at the point (Λ,x) such that $\nabla\Psi(\Lambda,x).T\neq 0$.*

Proof. The proof will be given in two steps.
Step1. We argue by contradiction. Let $(\Lambda_\delta,x_\delta)\in\mathbb{R}^2_+\times\Omega^2$ be such that $\Psi(\Lambda_\delta,x_\delta)\in[a,b]$, $\nabla_\Lambda\Psi(\Lambda_\delta,x_\delta)=0$, $\rho(x_\delta)<-\rho$, $\operatorname{dist}(x_{1_\delta},\partial\Omega)=\delta$, $\operatorname{dist}(x_{2_\delta},\partial\Omega)\ge\delta$, $|x_{1_\delta}-x_{2_\delta}|\ge\delta$ and for any vector T tangent to $\mathbb{R}^2_+\times\partial V_\delta$ at the point $(\Lambda_\delta,x_\delta)$ it holds

$$\nabla\Psi(\Lambda_\delta,x_\delta).T=0.$$

Set $\tilde\Omega_\delta=\frac{\Omega-\tilde x_{1_\delta}}{\delta}$, $y=\frac{x-\tilde x_{1_\delta}}{\delta}$ and $\mu_\delta=\delta^{\frac{-(n-4)^2}{2(n-8)}}\Lambda_\delta$, where $\tilde x_{1_\delta}\in\partial\Omega$ satisfies $|x_{1_\delta}-\tilde x_{1_\delta}|=\delta$. Then $\operatorname{dist}(y_{1_\delta},\partial\tilde\Omega_\delta)=1$, $\operatorname{dist}(y_{2_\delta},\partial\tilde\Omega_\delta)\ge 1$ and $|y_{1_\delta}-y_{2_\delta}|\ge 1$.

After a rotation and translation we may assume without loss of generality that $y_{1_\delta}\to(0,1)\in\mathbb{R}^{n-1}\times\mathbb{R}$ as δ tends to 0 and the domain $\tilde\Omega_\delta$ becomes the half-space $\pi=\{(y',y^n)\in\mathbb{R}^{n-1}\times\mathbb{R}:y^n>0\}$. We observe that if $\tilde G_\delta$ and $\tilde H_\delta$ are the Green's function and its regular part associated to the domain $\tilde\Omega_\delta$ then

$$\tilde G_\delta(y_1,y_2)=\delta^{n-4}G(\delta y_1,\delta y_2),\quad \tilde H_\delta(y_1,y_2)=\delta^{n-4}H(\delta y_1,\delta y_2).$$

Recall that

$$\lim_\delta\tilde H_\delta(y_1,y_2)=H_\pi(y_1,y_2)=\frac{1}{|y_1-\bar y_2|^{n-4}}\quad C^1\text{-uniformly on compact sets of }\pi^2, \tag{4.5}$$

and

$$\lim_{\delta} \tilde{G}_\delta(y_1, y_2) = G_\pi(y_1, y_2) = \frac{1}{|y_1 - y_2|^{n-4}} - \frac{1}{|y_1 - \bar{y}_2|^{n-4}}, \tag{4.6}$$

C^1-uniformly on compact sets of $\{(y_1, y_2) \in \pi^2 : y_1 \neq y_2\}$. Here for $y = (y', y^n)$, we denote $\bar{y} = (y', -y^n)$. Moreover, $\tilde{\Psi}_\delta$ denotes by

$$\tilde{\Psi}_\delta(\mu, y) = \frac{1}{2}\Big(\tilde{H}_\delta(y_1, y_1)\mu_1^2 + \tilde{H}_\delta(y_2, y_2)\mu_2^2 - 2\tilde{G}_\delta(y_1, y_2)\mu_1\mu_2\Big) + \frac{1}{2}\big(\mu_1^{\frac{8}{n-4}} + \mu_2^{\frac{8}{n-4}}\big),$$

then

$$\tilde{\Psi}_\delta(\mu, y) = \delta^{\frac{-4(n-4)}{n-8}}\Psi(\Lambda, x).$$

From [22, appendix A], we have

$$\nabla\Psi(\Lambda, x) = 0 \text{ if and only if } \nabla\tilde{\Psi}_\delta(\mu, y) = 0.$$

First of all, we claim that

$$0 < c_1 \leq \Lambda_{1_\delta}, \Lambda_{2_\delta} \leq c_2 \quad \text{ as } \delta \to 0. \tag{4.7}$$

It is easy to check that $0 < c_1 \leq |\Lambda_\delta| \leq c_2$. In fact, since $\nabla_\Lambda\Psi(\Lambda_\delta, x_\delta) = 0$, we have that

$$\Psi(\Lambda_\delta, x_\delta) = \frac{n-8}{2(n-4)}\big(\Lambda_{1_\delta}^{\frac{8}{n-4}} + \Lambda_{2_\delta}^{\frac{8}{n-4}}\big) \in [a, b],$$

and so if $|\Lambda_\delta| \to +\infty$ or $|\Lambda_\delta| \to 0$, a contradiction arises.

Let $\lim_\delta \Lambda_{1_\delta} = \Lambda_1 \in \mathbb{R}_+$ and $\lim_\delta \Lambda_{2_\delta} = \Lambda_2 \in \mathbb{R}_+$. Since $\rho(x_\delta) < 0$, there exists a positive constant C such that $|x_{1_\delta} - x_{2_\delta}| \leq C\delta$. We obtain $|y_{2_\delta}| \leq C$ and then $\lim_\delta y_{2_\delta} = \hat{y}_2$. Using the fact that $\nabla_\Lambda\Psi(\Lambda_\delta, x_\delta) = 0$, we have

$$\begin{aligned}
0 &= \delta^{n-4}\Lambda_{1_\delta}\nabla_{\Lambda_1}\Psi(\Lambda_\delta, x_\delta) \\
&= \tilde{H}_\delta(y_{1_\delta}, y_{1_\delta})\Lambda_{1_\delta}^2 - \tilde{G}_\delta(y_{1_\delta}, y_{2_\delta})\Lambda_{1_\delta}\Lambda_{2_\delta} + \frac{4}{n-4}\delta^{n-4}\Lambda_{1_\delta}^{8/(n-4)} \\
0 &= \delta^{n-4}\Lambda_{2_\delta}\nabla_{\Lambda_2}\Psi(\Lambda_\delta, x_\delta) \\
&= \tilde{H}_\delta(y_{2_\delta}, y_{2_\delta})\Lambda_{2_\delta}^2 - \tilde{G}_\delta(y_{1_\delta}, y_{2_\delta})\Lambda_{1_\delta}\Lambda_{2_\delta} + \frac{4}{n-4}\delta^{n-4}\Lambda_{2_\delta}^{8/(n-4)}.
\end{aligned}$$

Passing to the limit we deduce that

$$\lim_{\delta} \tilde{G}_\delta(y_{1_\delta}, y_{2_\delta})\Lambda_{1_\delta}\Lambda_{2_\delta} = H_\pi((0,1),(0,1))\Lambda_1^2 = H_\pi(\hat{y}_2, \hat{y}_2)\Lambda_2^2. \tag{4.8}$$

Since $|\Lambda_\delta|$ does not tend to 0 then $\Lambda_1, \Lambda_2 \in \mathbb{R}_+^*$, and (4.7) follows. Second we prove that

There exist $\hat{y} = ((0,1);(\hat{y}_2', \beta))$ with $(0,1) \neq (\hat{y}_2', \beta)$, $0, \hat{y}_2' \in \mathbb{R}^{n-1}$, $(1, \beta) \in \mathbb{R}^2$ and $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2) \in (\mathbb{R}_+^*)^2 : M_\pi(\hat{y})\hat{\mu} = 0$, $T.\nabla_y\Psi_\pi(\hat{\mu}, \hat{y}) = 0$ for all $T \in \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}^n$. (4.9)

Let $\lim_\delta y_{2_\delta} = \hat{y}_2 = (\hat{y}_2', \beta)$ and $\lim_\delta y_{1_\delta} = \hat{y}_1 = (0,1)$. Moreover, from (4.7) it follows that $\lim_\delta |\mu_\delta| = +\infty$, then up to a subsequence we can assume that $\hat{\mu} = \lim_\delta \frac{\mu_\delta}{|\mu_\delta|}$. It holds $|\hat{\mu}| = 1$. Now, since $\delta^{\frac{(n-4)(n-12)}{2(n-8)}}\nabla_\Lambda\Psi(\Lambda_\delta, x_\delta) = 0$, we have

$$\tilde{M}_\delta(y_\delta)\frac{\mu_\delta}{|\mu_\delta|} + \frac{4}{n-4}\delta^{n-4}\Big(\frac{\Lambda_{1_\delta}^{(12-n)/(n-4)}}{|\Lambda_\delta|} + \frac{\Lambda_{2_\delta}^{(12-n)/(n-4)}}{|\Lambda_\delta|}\Big) = 0,$$

and by passing to the limit we get $M_\pi(\hat{y})\hat{\mu} = 0$. Therefore 0 is the first eigenvalue of the matrix $M_\pi(\hat{y})$ and $\hat{\mu}$ is the eigenvector associated to 0 and by [3] it follows that

$\hat\mu_1, \hat\mu_2 \in \mathbb{R}^*_+$. From (4.5) and (4.6) we get $\nabla_y \Psi_\pi(\hat\mu, \hat y) = \lim_\delta \frac{1}{|\mu_\delta|^2} \nabla_y \tilde\Psi_\delta(\mu_\delta, y_\delta)$ and then (4.9) follows.

Finally we prove that by (4.9) we get a contradiction. We write now the function Ψ_π explicitly:

$$\Psi_\pi(\mu, y) = \frac{1}{2}\Big(\frac{1}{(2y_1^n)^{n-4}}\mu_1^2 + \frac{1}{(2y_2^n)^{n-4}}\mu_2^2 - 2G_\pi(y_1, y_2)\mu_1\mu_2\Big) + \frac{1}{2}\big(\mu_1^{\frac{8}{n-4}} + \mu_2^{\frac{8}{n-4}}\big).$$

We have two cases:

If $\hat y_2' \neq 0$ then

$$\begin{aligned} \hat y_2'.\nabla_{y_2'}\Psi_\pi(\hat\mu, \hat y) &= -\hat y_2'.\nabla_{y_2'} G_\pi(\hat y_1, \hat y_2)\hat\mu_1\hat\mu_2 \\ &= (n-4)|\hat y_2'|^2\Big(\frac{1}{|(\hat y_2', \beta - 1)|^{n-2}} - \frac{1}{|(\hat y_2', \beta+1)|^{n-2}}\Big)\hat\mu_1\hat\mu_2 \neq 0, \end{aligned}$$

and a contradiction arises.

If $\hat y_2' = 0$ then $\beta > 1$ and

$$0 = \nabla_{y_2^n}\Psi_\pi(\hat\mu, \hat y) = (n-4)\hat\mu_2\Big(\Gamma_{n-3}(\beta)\hat\mu_1 - \frac{1}{(2\beta)^{n-3}}\hat\mu_2\Big),$$

where

$$\Gamma_{n-3}(\beta) = \frac{1}{(\beta-1)^{n-3}} - \frac{1}{(\beta+1)^{n-3}} > 0.$$

We deduce that

$$\hat\mu_2 = (2\beta)^{n-3}\Gamma_{n-3}(\beta)\hat\mu_1. \tag{4.10}$$

On the other hand, by the condition $M_\pi(\hat y)\hat\mu = 0$, we get

$$\begin{gathered} \frac{1}{2^{n-4}}\hat\mu_1 - \Gamma_{n-4}(\beta)\hat\mu_2 = 0, \\ -\Gamma_{n-4}(\beta)\hat\mu_1 + \frac{1}{(2\beta)^{n-4}}\hat\mu_2 = 0, \end{gathered} \tag{4.11}$$

where

$$\Gamma_{n-4}(\beta) = \frac{1}{(\beta-1)^{n-4}} - \frac{1}{(\beta+1)^{n-4}}.$$

Equations (4.10) and (4.11) imply

$$\left(2\beta\Gamma_{n-3}(\beta) - \Gamma_{n-4}(\beta)\right)\hat\mu_1 = 0$$

and a contradiction arises since $2\beta\Gamma_{n-3}(\beta) - \Gamma_{n-4}(\beta) > 0$.

Step2. as in step 1, we prove the following: for any $(\Lambda, x) \in \mathbb{R}^2_+ \times \Omega^2$ with $\Psi(\Lambda, x) \in [a, b]$, $\nabla_\Lambda\Psi(\Lambda, x) = 0$, $\rho(x) < -\rho$, $\operatorname{dist}(x_1, \partial\Omega) \geq \delta$, $\operatorname{dist}(x_2, \partial\Omega) \geq \delta$ and $|x_1 - x_2| = \delta$, then there exist a vector T tangent to $\mathbb{R}^2_+ \times \partial V_\delta$ at the point (Λ, x) such that

$$\nabla\Psi(\Lambda, x).T \neq 0.$$

The lemma follows. $\square$

Lemma 4.6. *There exist $\delta_0' > 0$ and $\rho_0' > 0$ such that for any $\delta \in (0, \delta_0')$ and $\rho \in (0, \rho_0')$ the function Ψ satisfies the following property: For any sequence $(\Lambda_n, x_n) \in \mathbb{R}^2_+ \times (W^\delta_\rho \cap V_\delta)$ such that $\lim_n(\Lambda_n, x_n) = (\Lambda, x) \in \partial(\mathbb{R}^2_+ \times (W^\delta_\rho \cap V_\delta))$ and $\Psi(\Lambda_n, x_n) \in [a, b]$ there exists a vector T tangent to $\partial(\mathbb{R}^2_+ \times (W^\delta_\rho \cap V_\delta))$ at the point (Λ, x) such that*

$$\nabla\Psi(\Lambda, x).T \neq 0.$$

Proof. First, it is easy to check that $0 < c \le |\Lambda_n| \le c'$. In fact we have that $|\Lambda_n| \to +\infty$ and $|\Lambda_n| \to 0$ yield respectively to $|\Psi(\Lambda_n, x_n)| \to +\infty$ and $|\Psi(\Lambda_n, x_n)| \to 0$, which is impossible.

Let $\Lambda = \lim_n \Lambda_n$ and $x = \lim_n x_n$. If $\nabla_\Lambda \Psi(\Lambda, x) \neq 0$, then T can be chosen parallel to $\nabla_\Lambda \Psi(\Lambda, x)$. In the other case we have $\Lambda \in (\mathbb{R}_+^*)^2$. In fact if $\Lambda_2 = 0$, by

$$0 = \nabla_{\Lambda_1} \Psi(\Lambda, x) = H(x_1, x_1)\Lambda_1 + \frac{4}{n-4} \Lambda_1^{\frac{12-n}{n-4}},$$

we get a contradiction. Analogously $\Lambda_1 \neq 0$. Thus $x \in \partial(W_\rho^\delta \cap V_\delta)$. Now we claim that there exists $\rho_0' > 0$ such that

$$\rho(x) < -\rho_0'. \tag{4.12}$$

In fact, since $\nabla_\Lambda \Psi(\Lambda, x) = 0$, we have

$$\Psi(\Lambda, x) = \frac{n-8}{2(n-4)} \big(\Lambda_1^{\frac{8}{n-4}} + \Lambda_2^{\frac{8}{n-4}} \big) = \frac{8-n}{4} (M(x)\Lambda, \Lambda),$$

and since $\Psi(\Lambda, x) \in [a, b]$ we deduce that

$$|\Lambda|^2 \le \big(\frac{2(n-4)}{n-8} \big)^{\frac{n-4}{4}} b^{\frac{n-4}{4}} \text{ and } (M(x)\Lambda, \Lambda) \le \frac{4}{8-n} a,$$

which implies (4.12) because $(M(x)\Lambda, \Lambda) \ge \rho(x)|\Lambda|^2$. Therefore we have that $x \in \partial V_\delta$ (if we choose $\rho < \rho_0'$) and we can apply Lemma 4.5 to conclude the proof. $\square$

Lemma 4.7. *The function Ψ constrained to $\mathbb{R}_+^2 \times (W_\rho^\delta \cap V_\delta)$ satisfies the Palais-Smal condition in $[a, b]$.*

Proof. Let $(\Lambda_n, x_n) \in \mathbb{R}_+^2 \times (W_\rho^\delta \cap V_\delta)$ be such that $\lim_n \Psi(\Lambda_n, x_n) = c$ and $\lim_n \nabla\Psi(\Lambda_n, x_n) = 0$. Arguing as in the proof of Lemma 4.5 it can be shown that Λ_n remains bounded component-wise from above and below by a positive constant. As in Lemma 4.6, $\Lambda \in (\mathbb{R}_+^*)^2$ and by Lemma 4.5, $x \in (W_\rho^\delta \cap V_\delta)$. $\square$

Proposition 4.8. *There exists a critical level for Ψ between a and b.*

Proof. Assume by contradiction that there are no critical levels in the interval $[a, b]$. By Lemmas 4.5 and 4.6, We can define an appropriate negative flow that will remain in $\mathcal{A} := \mathbb{R}_+^2 \times (W_\rho^\delta \cap V_\delta)$ at any level $c \in [a, b]$. Moreover the Palais-Smale condition holds for $\Psi_{|\mathcal{A}}$ in $[a, b]$ (see Lemma 4.7). Hence there exists a continuous deformation

$$\eta : [0, 1] \times \Psi_{|\mathcal{A}}^b \to \Psi_{|\mathcal{A}}^b,$$

such that for some $a' \in (0, a)$

$$\begin{gathered} \eta(0, u) = u \quad \forall u \in \Psi_{|\mathcal{A}}^b \\ \eta(t, u) = u \quad \forall u \in \Psi_{|\mathcal{A}}^{a'} \\ \eta(1, u) \in \Psi_{|\mathcal{A}}^{a'}. \end{gathered}$$

Then there exist a continuous function $\gamma \in \Gamma$ such that

$$\gamma_{/[0, R_0] \times (\mathcal{S}^2 \setminus T_\delta)} = \eta_{/[0, R_0] \times (\mathcal{S}^2 \setminus T_\delta)}$$

and using (4.4), we obtain $\Psi(\gamma(R, x, 1)) < a$ for all $(R, x) \in [0, R_0] \times \mathcal{S}^2$, this condition provides a contradiction with Corollary 4.4 $\square$

Proof of Theorem 1.2. Arguing as in [22] and using Proposition 4.8 and Lemma 4.1, it is possible to construct a critical point $(\lambda^\varepsilon, x^\varepsilon)$ of the function K_ε^2 for ε small enough. We only need to prove that $(\alpha^\varepsilon_{(\lambda^\varepsilon,x^\varepsilon)}, \lambda^\varepsilon, x^\varepsilon, v^\varepsilon_{(\alpha^\varepsilon,\lambda^\varepsilon,x^\varepsilon)})$ satisfies (2.5) and (2.4). Indeed, we have by easy computation

$$
\begin{aligned}
0 &= \frac{\partial K_\varepsilon}{\partial x_i} + \big(\frac{\partial K_\varepsilon}{\partial v}, \frac{\partial v}{\partial x_i}\big) + \big(\frac{\partial K_\varepsilon}{\partial \alpha_i}, \frac{\partial \alpha_i}{\partial x_i}\big) \\
&= \frac{\partial K_\varepsilon}{\partial x_i} + \Big(\sum_{i=1}^{k} \Big(A_i P\delta_{x_i,\lambda_i} + B_i \frac{\partial P\delta_{x_i,\lambda_i}}{\partial \lambda_i} + \sum_{j=1}^{n} C_{ij} \frac{\partial P\delta_{x_i,\lambda_i}}{\partial (x_i)_j}\Big), \frac{\partial v}{\partial x_i}\Big).
\end{aligned}
$$

Using the fact that $v \in E$, then (2.5) is satisfied, in the same way we proof that (2.4) is satisfied. $\square$

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Mokhless Hammami

Département de Mathématiques, Faculté des Sciences de Sfax, Route Soukra, 3018, Sfax, Tunisia

E-mail address: `Mokhless.Hammami@fss.rnu.tn`