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QUALITATIVE PROPERTIES OF SOLUTIONS FOR QUASI-LINEAR ELLIPTIC EQUATIONS

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Abstract. For several classes of functions including the special case $f(u) = u^{p-1} - u^m$, $m > p-1 > 0$, we obtain Liouville type, boundedness and symmetry results for solutions of the non-linear p-Laplacian problem $-\Delta_p u = f(u)$ defined on the whole space $\mathbb{R}^n$. Suppose $u \in C^2(\mathbb{R}^n)$ is a solution. We have that either (1) if u doesn't change sign, then u is a constant (hence, $u \equiv 1$ or $u \equiv 0$ or $u \equiv -1$); or (2) if u changes sign, then $u \in L^\infty(\mathbb{R}^n)$, moreover $|u| < 1$ on $\mathbb{R}^n$; or (3) if $|Du| > 0$ on $\mathbb{R}^n$ and the level set $u^{-1}(0)$ lies on one side of a hyperplane and touches that hyperplane, i.e., there exists $\nu \in S^{n-1}$ and $x_0 \in u^{-1}(0)$ such that $\nu \cdot (x - x_0) \geq 0$ for all $x \in u^{-1}(0)$, then u depends on one variable only (in the direction of ν).

1. Introduction

In this paper we consider the problem

$$\begin{gathered} -\Delta_p u = f(u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Δ_p denotes the p-Laplacian operator $\Delta_p = \operatorname{div}(|Du|^{p-2}Du)$, $p > 1$, $\Omega = \mathbb{R}^N, N \geq 2$, and $f(u)$ is locally Lipschitz continuous.

In the case $p = 2$, several results have been obtained starting with the famous paper by Gidas, Ni and Nirenberg [28] where, among other things, it is proved that, if Ω is a ball and $p = 2$, solutions of (1.1) are radially symmetric and strictly radially decreasing. This paper had a big impact not only in virtue of the several monotonicity and symmetry results that it contains, but also because it brought to attention the moving plane method which, since then, has been largely used in many different problems. This method, which is essentially based on maximum principles, goes back to Alexandrov [1] and was first used by Serrin in [34]. The moving plane method has been improved and simplified by Berestycki and Nirenberg in [11] with the aid of the maximum principle in small domains. Recently, In a series papers, Berestycki, Caffarelli and Nirenberg [6, 7, 8] began to study the qualitative properties of solutions when Ω is unbounded, for example slab, half plane, and $\mathbb{R}^n$.

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When $\Omega = \mathbb{R}^n$, it is related to the following conjecture of De Giorgi [19]: If u is a solution of the scalar Ginzburg-Landau equation

$$\Delta u + u(1-u^2) = 0 \quad \text{on } \mathbb{R}^n$$

such that $|u| \le 1$ and $\partial_n u > 0$ on $\mathbb{R}^n$, and

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \forall x \in \mathbb{R}^{n-1}$$

then all level sets of u are hyperplanes, at least for $n \le 8$. Here $\partial_n u$ denotes the partial derivative of u with respect to x_n, the last component of x, and x' denotes the first $n-1$ components of x. When $n = 2$, this conjecture was completely resolved by Ghoussoub and Gui [27]. When $n = 3$, it was very recently proved by Ambrosio and Cabre [3]. Both solutions of the conjecture are based on a Liouville-type theorem due to Berestycki,Caffarelli and Nirenberg [8]. The first partial answer to the De Giorgi conjecture is from the work of 1980 by Modica and Mortola [31]. In 1985, Modica [30] found a pointwise gradient bound for all bounded solutions. This estimate was further generalized by Caffarelli, Garofalo and Segala [13] to more general nonlinear partial differential equations which include the p-Laplacian. Under more assumptions on the solutions, for example, if $u(x) = u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$ holds uniformly for $x' \in \mathbb{R}^{n-1}$, the conclusion of this conjecture was confirmed in [5, 9, 26] for any $n \ge 2$. The conjecture in its original form however, remains open for $n > 3$. We refer to [2] for a fuller account of the history and progress about this conjecture. Du and Ma [23] recently removed the boundedness condition $|u| \le 1$ in De Giorgi's conjecture. This point has already been observed by Farina [26], but his conclusion does not seem to include those nonlinearities covered by Du and Ma's result.

Very little is known about the monotonicity and symmetry of solutions of (1.1) when $p \ne 2$. In this case the solutions can only be considered in a weak sense since, generally they belong to the space $C^{1,\alpha}(\Omega)$(See [21] and [36]). Anyway this is not a difficulty because the moving plane method method can be adapted to weak solutions of strictly elliptic problems in divergence form(See [14] and [16]).

The real difficulty with problem (1.1), for $p \ne 2$, is that the p-Laplacian operator is degenerate in critical points of the solutions, so that comparison principle(which could substitute the maximum principles in order to use the moving plane and sliding method when the operator is not linear) are not available in the same as for $p = 2$. Actually counterexamples both to validity of comparison principles and to the symmetry results are available(see [12])for any p with different degrees of regularity of f.

A first step towards extending the moving plane method to solutions of problems involving the p-Laplacian operator has been done in [16]. In that paper the author mainly proves some weak and strong comparison principles for solutions of differential inequalities involving the p-Laplacian. Using these principles he adapts the moving plane method to solutions of (1.1) getting some monotonicity and symmetry results in the case $1 < p < 2$. The symmetry result is not complete and relies on the assumption that the set of the critical points of u does not disconnect the caps which are constructed by the moving plane method. In [17] the author got monotonicity and symmetry for solutions u of (1.1) in smooth domains in the case $1 < p < 2$ without extra assumptions on u.

We now state the main results.We restrict our attention on the following equation

$$\Delta_p u + u^{p-1} - u^m = 0, \quad \text{on } \mathbb{R}^n \tag{1.2}$$

where $m > p - 1 > 0$.

Theorem 1.1 (Liouville Type Property). *Suppose $u \in C^2(\mathbb{R}^n)$ is a solution of (1.2). Furthermore u doesn't change sign. Then u is a constant (hence,$u \equiv 1$,or $u \equiv 0$,or $u \equiv -1$).*

Theorem 1.2 (Global Boundedness). *Suppose $u \in C^2(\mathbb{R}^n)$ is a changing-sign solution of (1.2). Then $u \in L^\infty(\mathbb{R}^n)$, moreover $|u| < 1$ on $\mathbb{R}^n$.*

Theorem 1.3 (One-dimensional Property). *Suppose that $u \in C^2(\mathbb{R}^n)$ solves (1.2) on $\mathbb{R}^n$ and $|Du| > 0$. If $u^{-1}(0)$ lies on one side of a hyperplane and touches that hyperplane, i.e., there exists $\nu \in S^{n-1}$ and $x_0 \in u^{-1}(0)$ such that $\nu \cdot (x - x_0) \geq 0$ for all $x \in u^{-1}(0)$,then u depends on one variable only (in the direction of ν).*

Similar results to our Theorems 1.1 and 1.2 are obtained by Dancer and Du [15], Du and Gu [22] by different methods.

Now we compare our results with the very interesting works of P.Pucci, J.Serrin, and H.Zou [32, 33, 35]. In their papers [32, 33], the aim is to find conditions which make the Maximum Principle to be true. So they have to assume the behavior at infinity or at some point of solutions. Since one of our aim is to get Liouville type result, we need only to use the Comparison Principles (see Theorem 2.1-2.4 below). In the paper [35], the authors consider the radial symmetry of the solutions with the assumption about the behavior at infinity of the solutions. But in our Theorem 1.3, we study the one dimensional property of solutions under different conditions of the solutions.

Throughout this paper, for simplicity, we assume that $u \in C^2(\mathbb{R}^N)$. The rest of this paper is organized as follows. Some preliminary results are given in section2. In section 3, we prove Theorem 1.1. Theorem 1.2 is proved in section 4. In section 5, we prove two lemmas which are needed in the proof of Theorem 1.3. Theorem 1.3 is proved in section 6.

2. Preliminary Results

In this section, we collect the related weak and strong comparison principles. Let Ω be a domain in $\mathbb{R}^N, N \geq 2$, and let $u, v \in C^2(\Omega)$ be solutions of

$$\begin{aligned} -\Delta_p u &\leq f(u) \quad \text{in } \Omega \\ -\Delta_p v &\geq f(v) \quad \text{in } \Omega \end{aligned} \tag{2.1}$$

For a set $A \subseteq \Omega$ we define

$$\begin{aligned} M_A = M_A(u, v) &= \sup_A(|Du| + |Dv|) \\ m_A = m_A(u, v) &= \inf_A(|Du| + |Dv|) \end{aligned} \tag{2.2}$$

Firstly we state the weak maximum principles.

Theorem 2.1 (Weak Comparison Principle). *Let u, v be solutions of (2.1) in a bounded domain Ω and $f \in C[0, \infty), f(0) = 0$ and f is non-decreasing on some interval $[0, \delta]$. Suppose also that u and v are continuous in D, with $v < \delta$ in Ω and $u \geq v$ on $\partial\Omega$. Then $u \geq v$ in Ω.*

Theorem 2.2 (Weak Comparison Principle)**.** *Let u, v be respective solutions of (2.1) in D(maybe unbounded). Suppose that u and v are continuous in D,that $m_D > 0$,and that $u \geq v$ on ∂D. Then $u \geq v$ in D*

Theorem 2.3 (Weak Comparison Principle)**.** *Suppose that $1 < p < 2$, then there exist $\alpha, M > 0$, depending on p, $|\Omega|$, M_Ω and the L^∞ norms of u and v such that: if an open set $\Omega' \subseteq \Omega$ satisfies $\Omega' = A_1 \cup A_2, |A_1 \cap A_2| = 0, |A_1 < \alpha|, M_{A_2} < M$ then $u \leq v$ on $\partial\Omega'$ implies $u \leq v$ in Ω'*

For a proof,see [18]

Theorem 2.4 (Weak comparison Principle)**.** *Suppose that $p > 2$ and $m_\Omega > 0$,there exist $\delta, m > 0$ depending on $p, |\Omega|, m_\Omega$ such that the following holds: if $\Omega' = A_1 \cup A_2$ with $|A_1 \cap A_2| = 0, |A_1 < \delta|$ and $m_{A_2} > m$ then $u \leq v$ on $\partial\Omega'$ implies $u \leq v$ in Ω'*

For a proof, see [16] Now, we state a comparison principle in slab.

Lemma 2.5. *Let w be a function satisfying $Lw \leq 0$ in $\Omega = \mathbb{R}^{n-1} \times (b, c)$, where $b, c \in R$ and where*

$$Lw = \alpha_{ij}(x)\partial_{ij} w + \beta_j \partial_j u + \gamma(x) u \,.$$

Assume that the coefficients $\alpha_{ij}(x), \beta_j(x)$ are uniformly continuous in $\overline{\Omega}$ and that the α_{ij} satisfy

$$\exists c_0' \geq c_0 > 0, \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n, c_0|\xi|^2 \leq \alpha_{ij}(x)\xi_i\xi_j \leq c_0'|\xi|^2.$$

Furthermore, assume that

$$-C \leq \gamma(x) \leq 0 \quad \text{for all } x \in \Omega$$

for some positive real number C. The function w is required to be continuous in $\overline{\Omega}$ and to satisfy $Lw \in L^\infty(\Omega)$ and $m \leq w \leq M$ in Ω for some $m, M \in R$. If $w \geq 0$ on $\partial\Omega$, then $w \geq 0$ in Ω

For the proof of this lemma, we can refer to Lemma 3.1 of [9]. From the maximum principle, we can get the following comparison result.

Theorem 2.6. *Let f be a Lipschitz-continuous function, non-increasing on the intervals $[-1, -1+\delta]$ and $[1-\delta, 1]$ for some $\delta > 0$. Assume that u_1, u_2 are solutions of*

$$\Delta_p u_i + f(u_i) = 0 \quad \text{in } \Omega$$

and are such that $|u_i| \leq 1 (i = 1, 2)$. Furthermore, assume that

$$u_2 \geq u_1 \quad \text{on } \partial\Omega$$

and that either $u_2 \geq 1 - \delta$ in Ω or $u_1 \leq -1 + \delta$ in Ω, Where $\Omega = \mathbb{R}^{n-1} \times (b, c)$. Then $u_2 \geq u_1$.

Next we deal with a form of a strong comparison theorem.First we prove the following Harnack type comparison inequality

Lemma 2.7 (Harnack type comparison inequality)**.** *Suppose u, v satisfy*

$$-\operatorname{div} A(x, Du) + \Lambda u \leq -\operatorname{div}(x, Dv) + \Lambda v, u \leq v \quad \text{in } \Omega \tag{2.3}$$

where $\Lambda \in \mathbb{R}$ and $u, v \in W^{1,\infty}_{\rm loc}(\Omega)$ if $p \neq 2$;$u, v \in W^{1,2}_{\rm loc}(\Omega)$ if $p = 2$. Suppose $B(x, 5\delta) \subseteq \Omega$ for some $\delta > 0$ and,if $p \neq 2, \inf_D(|Du| + |Dv|) > 0$. Then, for any positive number $s < \frac{n}{n-2}$ we have

$$\|v - u\|_{L^s(B(x,2\delta))} \leq c\delta^{N/2} \inf_{B(x,\delta)} (v - u)$$

where c is a constant depending on N, p, s, c_2, δ and, if $p \neq 2$, also on $m = \inf_{B(x,5\delta)}(|Du| + |Dv|)$ and $M_{B(x,5\delta)}$

This lemma implies the following strong comparison principle:

Theorem 2.8. *Let $u, v \in C^2(\Omega)$ be solutions of (2.1)with $1 < p < \infty, 0 < u \leq v in$ Ω and f be locally Lipschitz-continuous in $(0, \infty)$. Define*

$$Z_v^u = \{x \in \Omega : |Du| = |Dv| = 0\} \quad (Z_v^u = \emptyset \quad \text{for } p = 2)$$

If there exists $x_0 \in \Omega \setminus Z_v^u$ such that $u(x_0) = v(x_0)$, then $u \equiv v$ in the connected component of $\Omega \setminus Z_v^u$ containing x_0.

For a proof, see [16]

3. Liouville Type Property

In this section, we prove a generalization of Theorem 1.1.

Theorem 3.1 (Liouville Type Property)**.** *Let $u \in C^2(\mathbb{R}^n)$ be a nonnegative solution of*

$$\Delta_p u + \lambda u^{p-1} - u^m = 0 \quad \text{on } \mathbb{R}^n$$

where λ is positive, $p > 1$ is a constant and $m > p - 1$. Then u must be a constant.

The basic ingredients in the proof consist of the following three lemmas. For use in later sections and possible future applications,these lemmas are given in much more general form than what is required in the proof of Theorem 3.1.

We consider the problem

$$\Delta_p u + \alpha(x)u^{p-1} - \beta(x)u^m = 0 \quad \text{on } \mathbb{R}^n \tag{3.1}$$

Here $p > 1$ is a constant and $m > p - 1$.

Lemma 3.2 (Comparison Principle)**.** *Suppose that Ω is a bounded domain in $\mathbb{R}^N$, $\alpha(x)$ and $\beta(x)$ are continuous functions on Ω with $\|\alpha\|_\infty < \infty$ and $\beta(x)$ positive, $p > 2$. Let $u_1, u_2 \in C^2(\Omega)$ be positive in Ω and satisfy*

$$\Delta_p u_1 + \alpha(x)u_1^{p-1} - \beta(x)u_1^m \leq 0 \leq \Delta_p u_2 + \alpha(x)u_2^{p-1} - \beta(x)u_2^m, x \in \Omega \tag{3.2}$$

and $\limsup_{x \to \partial\Omega}(u_2 - u_1) \leq 0$, and $\alpha(x) \leq \beta(x)$. Then $u_2 \leq u_1$ in Ω

Proof. Let $\varepsilon_1 > \varepsilon_2 > 0$ and denote $w_i = (u_i+\varepsilon_i)^{-1}((u_2+\varepsilon_2)^2-(u_1+\varepsilon_1)^2)_+ (i = 1, 2)$. Observe w_i be C^2 nonnegative functions on Ω and vanishing near $\partial\Omega$. Using (3.2), applying integration by parts and subtracting, we obtain

$$\begin{aligned} &-\int_\Omega [|\nabla u_2|^{p-2}\nabla u_2 \nabla w_2 - |\nabla u_1|^{p-2}\nabla u_1 \nabla w_1] dx \\ &\geq \int_\Omega \beta(x)[u_2^m w_2 - u_1^m w_2] + \int_\Omega \alpha(x)(u_1^{p-1}w_1 - u_2^{p-1}w_2) \end{aligned} \tag{3.3}$$

Denote $\Omega_+(\varepsilon_1, \varepsilon_2) = \{x \in \Omega : u_2(x)+\varepsilon_2 > u_1(x)+\varepsilon_1\}$ and note that the integrands in (3.3) vanishing outside this set. The left side of (3.3) equals

$$\begin{aligned} &-\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} [|\nabla u_1|^{p-2}|\nabla u_2 - \frac{u_2+\epsilon_2}{u_1+\varepsilon_1}\nabla u_1|^2 + |\nabla u_2|^{p-2}|\nabla u_1 - \frac{u_2+\epsilon_2}{u_1+\varepsilon_1}\nabla u_2|^2] \\ &-\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (|\nabla u_2|^{p-2} - |\nabla u_1|^{p-2})(\nabla u_2 \nabla u_2 - \nabla u_1 \nabla u_1) dx \end{aligned} \tag{3.4}$$

Noting that $w_1 > w_2$ in $\Omega_+(\varepsilon_1, \varepsilon_2)$. We conclude that the left side of (3.4) is not positive. On the other hand as $\varepsilon_1 \to 0$ the right side of (3.3) converges to

$$\int_{\Omega_+(0,0)} [\beta(x)(u_2^{m-1} - u_1^{m-1}) - \alpha(x)(u_2^{p-1} - u_1^{p-1})](u_2^2 - u_1^2)$$

while last term in (3.3) converge to 0. Unless $\Omega_+(0,0)$ is empty, the limiting value of the right side of (3.3) is positive. Since this leads to a contradiction we conclude that $u_2 \le u_1$in Ω $\square$

Lemma 3.3 (Locally uniformly Boundedness)**.** *$u \in C^2$ is a positive solution of (3.1). Then we have the bound*

$$\max_G u(x) \le c_0$$

For every compact subset $G \subset \mathbb{R}^n$ and c_0 is a constant.

Proof. Suppose that $max_G u(x) = u_{x_0}$ for some $x_0 \in G$. If $|Du(x_0)| = 0$, Then $u \le \max_G(\alpha(x)/\beta(x))^{m-p+1}$. Otherwise, we may assume that there is a ball $B_{2r} := B_{2r}(x_0) \subset \mathbb{R}^n$with center $x_0 \subset G$ such that

$$\max_{\bar{G}} = \max_{B_r} u(x) := M(r) \quad \text{and} \quad \min_{B_{2r}} |Du| > 0$$

Since,on B_{2r}, we have

$$\Delta_p u \ge -\alpha(x) u$$

Then as pointed out in [23], u locally uniformly bounded. $\square$

Lemma 3.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Suppose α and β are smooth positive functions on $\bar{\Omega}$, and let μ_1 denote the first eigenvalue of $-\Delta_p u = \mu\alpha(x)u^{p-1}$ on Ω under Dirichlet boundary conditions on $\partial\Omega$. Then the problem*

$$-\Delta_p u = \mu u[\alpha(x)u^{p-2} - \beta(x)u^{m-1}], u|_{\partial\Omega} = 0$$

has a unique positive solution for every $\mu > \mu_1$,and the unique positive solution u_μ satisfies $u_\mu \to [\alpha(x)/\beta(x)]^{1/(m-p+1)}$

Proof. The existence from a simple upper and lower solution argument. clearly any constant greater that or equal to $M = \max_{\bar{\Omega}}[\alpha(x)/\beta(x)]^{1/(m-p+1)}$ is an upper solution. Let ϕ be a positive eigenfunction corresponding to μ_1, then for each fixed $\mu > \mu_1$ and all small positive $\epsilon, \epsilon\phi < M$ and is a lower solution. Thus there is at least one positive solution. If u_1 and u_2 are two positive solutions, we apply comparison principle to conclude that $u_1 \le u_2$ and $u_2 \le u_1$ both hold on Ω. Hence $u_1 = u_2$. This proves the uniqueness.

Given any compact subset K of Ω and any small $\epsilon > 0$ such that $\epsilon < \upsilon_0 = [\alpha(x)/\beta(x)]^{1/(m-p+1)}$ on Ω, we let $\upsilon_\epsilon = \upsilon_0 + \epsilon$, and find that $\upsilon_\epsilon(\alpha(x)\upsilon_\epsilon^{p-2} - \beta(x)\upsilon_\epsilon^{m-1}) \le -\delta$ on Ω for some positive constant $\delta = \delta(\epsilon)$ and $-\Delta_p \upsilon_\epsilon \ge -c$ on Ω for some positive constant $c = c(\epsilon)$. It follows that for all large μ, υ_ϵ is an upper solution of our problem.

On the other hand, Let ϕ be the positive eigenfunction corresponding to μ_1 with $\|\phi\|_\infty = 1$. then we can find a small neighborhood of $\partial\Omega$ in Ω, say U, such that ϕ is very small in U so that for all $\mu > \mu_1 + 1, -\Delta_p\phi = \mu_1\alpha(x)\phi^{p-1} \le \mu\phi(\alpha(x)\phi^{p-2} - \beta(x)\phi^{m-1})$ on U. By shrinking U further if necessary, we can assume that $\bar{U} \cap K = \emptyset$ and $\phi < \upsilon_0 - \epsilon$ on U. Now we can choose a smooth function w_ϵ on Ω such that $w_\epsilon = \phi$ on U, $w_\epsilon = \upsilon_0 - \epsilon$ on K and $\upsilon_0 - \epsilon/2 > w_\epsilon > 0$ on the rest of Ω. It is easily seen that such w_ϵ is a lower solution of our problem for all large μ. since $w_\epsilon < \mu_\epsilon$, we deduce $w_\epsilon \le u_\mu < v_\epsilon$ on Ω for all large μ. In particular,

$$[\alpha(x)/\beta(x)]^{1/(m-p+1)} + \epsilon \ge u_\mu \ge [\alpha(x)/\beta(x)]^{1/(m-p+1)} - \epsilon$$

on K for all large μ. this is to say that $u_\mu \to (\alpha/\beta)^{1/(m-p+1)}$ as $\mu \to \infty$ uniformly on K, as required. $\square$

Lemma 3.5. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ and suppose that there exists a large solution of the equation $\Delta_p u = u^m$ in Ω. Let Ξ be a compact subset of $\partial\Omega$ and let $P \in \Xi$. Suppose that,for every $\delta > 0$,there exists an open,connected neighborhood of P,say Q_P with C^2 boundary, such that,*

- *$\Omega_P = Q_P \cap \Omega$ is a simply connected domain.*
- *$Q_P \subset \Xi_\sigma = \{x : dist(x, \Xi) < \sigma\}$ and $\partial\Omega \cap \bar{\Omega}_P = \overline{\partial\Omega \cap Q_P}$*

Then there exists $\delta_0 > 0$(which depends on Ξ but not on P) such that,if Ω_P is contained in Ξ_{δ_0},the following statements hold.

(a) *There exists a large solution of (3.1) in Ω_P;*
(b) *There exists a positive solution v of (3.1) in Ω_P such that*

$$v(x) \to \infty \quad \text{locally uniformly as } x \to \Gamma_1 = \partial\Omega \cap Q_P \tag{3.5}$$

$$v \in C(\Omega_P \cup \Gamma_2) \quad \text{and } v = 0 \text{ on } \Gamma_2 = \Omega \cap \partial Q_P \tag{3.6}$$

Proof. (a) Let $b = 2\sup_\Omega \beta(x)$ and let $c = \sup\{-\alpha(x)t^{p-1} - \frac{1}{2}bt^m : t > 0, x \in \Omega\}$. Then, every positive solution u of (3.1) satisfies

$$\Delta_p u \le bu^m + c$$

Let U be a large solution of $\Delta_p u = 2bu^m$ in Ω. This means that u is a solution of $\Delta_p u = 2bu^m$ with boundary value $u = +\infty$ on $\partial\Omega$. Let $M = \inf\{U(x) : x \in \Omega \cap \Xi\}$ and choose δ_0 sufficiently small so that $bM^m \ge c$. Then

$$\Delta_p U \ge bU^m + c \quad \text{in } \Omega_P \tag{3.7}$$

Let $\{\Theta_n\}$ be an increasing sequence of domains with C^2 boundary such that

$$\bar{\Theta}_n \subset \Theta_{n+1} \subset \Omega_P \quad \text{and} \quad \Theta_n \uparrow \Omega_P. \tag{3.8}$$

Let u_n and V be large solutions of (3.1) in Θ_n and Q_P respectively. By comparison principle $\{u_n\}$ is monotone decreasing and $u_n \ge V$ in Θ_n. By the comparison principle, (3.7) and (3.8) $u_n \ge U$ in Θ_n. Hence $\lim u_n$ is a large solution of (3.1) in Ω_P

(b) For the proof of the second statement we may assume (in view of (a)) that there exists a large solution of (3.1) in Ω. Now, Let $\{\Theta_n\}$ be an increasing sequence of domain with C^2 boundary such that,

$$\Theta_n \subset \Omega_P, \Theta_n \uparrow \Omega_P \quad \text{and} \quad \Omega_P \setminus \Theta_n \subset K_n = \{x : dist(x, \Gamma_1) < 2^{-n}\}.$$

Denote $\Gamma_{1,n} = \partial\Theta_n \cap K_n, \Gamma_{2,n} = \partial\Theta_n \cap (\bar{K_n})^c$. Thus $\Gamma_{2,n} \subset \Gamma_{2,n+1} \subset \Gamma_2$. We shall also assume that the sets $\Gamma_{1,n}$ are disjoint.

For each n,consider a sequence of functions $\{\varphi_{n,k}\}_{k=1}^\infty$ on $\partial\Theta_n$ satisfying the following properties.

- $\varphi_{n,k} = k$ on $\Gamma_{1,n}; \varphi_{n,k} = 0$ for $x \in \Gamma_{2,n}$ such that $dist(x, \Gamma_{1,n}) > 2^{-n}$;
- $0 \le \varphi_{n,k} \le k$ everywhere; $\varphi_{n,k} \in C^2(\partial\Theta_n)$;
- $\varphi_{n,k} \ge \varphi_{n-1,k}$ on $\Gamma_{2,n}$ and $\varphi_{n,k} \le \varphi_{n,k-1}$ on $\partial\Theta_n$

Let $v_{n,k}$ be a solution of (3.1) in Θ_n in Θ_n such $v_{n,k} = \varphi_{n,k}$ on $\partial\Theta$. By comparison principle $\{v_{n,k}\}_{k=1}^{\infty}$ is monotone increasing and by Lemma 3.2 the sequence is locally bounded. Hence $v_n = \lim_{k\to\infty} v_{n,k}$ is a solution of (3.1) in Θ_n such that

$$\begin{gathered} v_n \to \infty \quad \text{as } x \to \Gamma_{1,n}; v_n \in C(\Theta_n \cup \Gamma_{2,n}) \\ v_n = 0 \quad \text{on } \Gamma_{2,n} \end{gathered} \tag{3.9}$$

Furthermore,by their construction,$v_{n,k} \geq v_{n+1,k}$ so that $\{v_n\}$ is monotone decreasing. Consequently $v = \lim_{n\to\infty} v_n$ is a solution of (3.1) in Ω_P. If V is a large solution of (3.1) in $Q_P, v_n + V$ is a supersolution of (3.1) in Θ_n which blows up on $\partial\Theta_n$. Hence $v_n + V \geq U$, where U is a large solution of (3.1) in Ω. Thus $v + V \geq U$ and this implies (3.5) Finally by (3.9), v satisfies (3.6) □

Lemma 3.6. *The problem*

$$-\Delta_p u = \mu u[\alpha(x)^{p-2} - \beta(x)u^{m-1}], u|_{\partial\Omega} = \infty \tag{3.10}$$

has a unique positive solution for each $\mu > 0$, and the unique positive solution u_μ satisfies $u_\mu \to (\alpha/\beta)^{1/(m-p+1)}$ uniformly on any compact subset of Ω as $\mu \to \infty$

Here and throughout this paper, by $u|_{\partial\Omega}$, we mean $u(x) \to \infty$ as $d(x, \partial\Omega) \to 0$. We also write $x \to \partial\Omega$ when $d(x, \partial\Omega) \to 0$.

Proof. 1. Existence: The existence follows from a simple upper and lower solution argument. Suppose $\mu > 0$. For any positive integer $n > M = \max_{\bar{\Omega}}(\alpha/\beta)^{1/(m-p+1)}$, the problem

$$-\Delta_p u = \mu u(\alpha u^{p-2} - \beta u^{m-1}), u|_{\partial\Omega} = n$$

has a unique positive solution. Indeed $u \equiv 0$ and $u \equiv n$ are lower and upper solution to this problem,and hence there is at least one positive solution. By comparison principle,there is at most one positive solution. Therefore there is a unique positive solution. Denoting this solution by u_n, we find, by comparison principle, that u_n increases with n. By Lemma 3.3 we can find a uniform upper bound for u_n on any compact subset of Ω, then by a standard regularity argument,$u_\mu = lim_{n\to\infty} u_n$ would be a positive solution of (3.10).

2. Uniqueness: Suppose that u is a large solution of (3.10). Note that for every $\epsilon > 0$ there exists $\beta_\epsilon > 0$ such that

$$k(1-\epsilon)u^m \leq \Delta_p u \leq k(1+\epsilon)u^m \quad \text{in } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \beta_\epsilon\}$$

Let $P \in \partial\Omega$ and assume (as we may) that the set Q_p mentioned above is an open,bounded spherical cylinder centered at P, with axis parallel to the ξ_n axis. Thus,

$$Q_p = \{\eta : |\eta'| < \rho_P, |\eta_N| < \tau_P\}$$

where $\eta = \xi - P$ and $\eta' = (\eta_1, \dots, \eta_{N-1})$. By appropriately choosing σ_P and τ_P we may also assume that $\partial\Omega$ is bounded away from the 'top' and 'bottom' of the cylinder Q_P and that $\partial\Omega \cup \bar{Q_P} = \overline{\partial\Omega \bigcap Q_P}$. finally we assume that ρ_P and τ_P are sufficiently small so that Lemma 3.5 can be applied to Q_P and so that

$$\begin{gathered} k(P)(1-\epsilon)u^m(x) \leq \Delta_p u \leq k(P)(1+\epsilon)u^m(x) \\ \forall x \in \Theta = Q_P \cap \Omega. \end{gathered} \tag{3.11}$$

Therefore there exists a solution v of the problem

$$\begin{gathered}\Delta_p v = v^m \quad \text{and } v > 0 \text{ in } \Theta = Q_P \cap \Omega \\ v(x) \to \infty \quad \text{locally uniformly as } x \to Q_P \cap \partial\Omega \\ v(x) \to 0 \quad \text{locally uniformly as } x \to \partial Q_P \cap \Omega\end{gathered}$$

Next denote

$$\begin{gathered}v_1 = (k(P)(1-\epsilon))^{-1/(m-1)} v \\ v_2 = (k(P)(1+\epsilon))^{-1/(m-1)} v\end{gathered}$$

and let w be the large solution of (3.10) in Q_P. We claim that

$$v_2 < u < v_1 + w \quad \text{in } \Theta \tag{3.12}$$

To verify this claim,let ξ denote the unit vector parallel to the axis of Q_P such that $P+\xi$ is outside Ω and set $\Theta_\sigma = \{x - \sigma\xi : x \in \Theta, \sigma > 0\}$. If f is a function defined in Θ, set $f_\sigma(x) = f(x+\sigma\xi)$ for $x \in \Theta_\sigma$. Assume that σ is a sufficiently small positive number so that $\Theta_\sigma \subset\subset \Omega$. Then $v_{1,\sigma} + w_\sigma$ is a supersolution in Θ_σ and hence $v_{1,\sigma} + w_\sigma > u$ there. On the other hand, by(3.11), $v_{2,-\sigma} < u$ on $\partial(\Theta_{-\sigma} \cap \Omega)$ and hence $v_{2,-\sigma} < u$ in $\Theta_{-\sigma} \cap \Omega$. Thus,for $0 < \sigma$ sufficiently small, $v_{2,-\sigma} < u < u_{1,\sigma} + w_\sigma$ in $\Theta_{-\sigma} \cap \sigma$ and hence,letting σ tend to zero, we obtain .Finally,since w is bounded in every compact subset of Q_P, it follows that

$$u(x)/(k(x)^{-1/(m-1)} v(x)) \to 1 \quad \text{locally uniformly as } x \to Q_P \cap \partial\Omega \tag{3.13}$$

Therefore if u_1 and u_2 are tow positive solutions of (3.10) ,then

$$\lim_{x \to \partial\Omega} u_1(x)/u_2(x) = 1$$

It follows that for any $\epsilon > 0$,

$$\lim_{x \to \partial\Omega} [(1+\epsilon)u_1 - u_2] = \infty$$

As $(1+\epsilon)u_1$ is an upper solution to (3.10), we can apply comparison principle to conclude that $(1+\epsilon)u_1 \geq u_2$ on Ω. As $\epsilon > 0$ is arbitrary, we deduce $u_2 \geq u_1$. Thus $u_1 = u_2$ on Ω. This proves the uniqueness.

3. Asymptotic behavior. Now we know that the positive solution u_μ constructed above is the unique positive solution. Let K be an arbitrary compact subset of $\Omega, v_0 = (\alpha/\beta)^{1/(m-p+1)}$ and $\epsilon > 0$ any small positive number satisfying $\epsilon < v_0$ on Ω. It is easily seen that ,for all large u,$w_\epsilon = v_0 - \epsilon$ is a lower solution for the problem satisfied by u_n with $u_n > w_\epsilon$.

On the other hand,fix a $\mu_0 > 0$ then we can find a small neighborhood U of $\partial\Omega$ in Ω such that $u_0 = u_{\mu_0} > v_0 + \epsilon$ on U. Therefore,

$$-\Delta_p u_0 = \mu_0 u_0(\alpha u_0^{p-2} - \beta u_0^{m-1}) \geq \mu u_0(\alpha u_0^{p-2} - \beta u_0^{m-1})$$

on U for all $\mu > \mu_0$. Now let us choose a smooth function v_ϵ satisfying $v_\epsilon = u_0$ on U,$v_\epsilon = v_0 + \epsilon$ on K and $v_\epsilon = v_0 + \epsilon/2$ on the rest of Ω. Then it is easily checked that v_ϵ is an upper solution for the equation of u_n provided that μ is large enough. As $w_\epsilon > v_\epsilon$ on Ω,we must have $w_\epsilon \leq u_n \leq v_\epsilon$ on Ω for all large μ and every large n. It follows that $w_\epsilon \leq u_\mu \leq v_\epsilon$ on Ω. This implies that $u_\mu \to v_0$ on K as $\mu \to \infty$, as required. The proof of the lemma is now complete. $\square$

Remark. In the above argument, we used the idea of [10]. However, our case is more complicated. We have to overcome this difficulty.

Proof of Theorem 3.1. Let us first observe that a nonnegative entire solution of $\Delta_p(u) + \lambda u^{p-1} - u^{m-1}$ is either identically zero or positive everywhere,due to the Harnack inequality. therefore,we need only consider positive solutions.

Set $\Omega = \{x : |Du(x)| = 0\}$. if $\Omega = \mathbb{R}^n$,we are done. It is easy to see that Ω is closed. Let x_0 be an arbitrary point in $\mathbb{R}^n$, we will show that $u(x_0) = \lambda^{1/(m-p+1)}$,using only pointwise convergence of v_α and w_α. For $\alpha > 0$ let us define

$$u_\alpha(x) = u(x_0 + \alpha(x - x_0)) \tag{3.14}$$

It easily checked that u_α satisfies

$$\Delta_p u + \alpha^p(\lambda u^{p-1} - u^m)$$

Let B denote the a ball with center x_0 and $B \cap \Omega = \emptyset$. By Lemma 3.4, for large α, the problem

$$\Delta_p v + \alpha^p(v^{p-1} - v^m), v|_{\partial B} = 0$$

has a unique positive solution v_α and as $\alpha \to \infty, v_\alpha \to \lambda^{1/(m-p+1)}$ at $x = x_0 \in B$. Applying comparison principle we see that $u_\alpha \geq v_\alpha$ on B, and hence

$$u(x_0) = u_\alpha(x_0) \geq v_\alpha(x_0)$$

Letting $\alpha \to \infty$ in the above inequality we conclude that $u(x_0) \geq \lambda^{1/(m-p+1)}$.

Let w_α be the unique positive solution of

$$\Delta_p w + \alpha^p(w^{p-1} - w^m), w|_{\partial B} = \infty$$

by Lemma 3.6 we know that as $\alpha \to \infty, w_\alpha(x) \to \lambda^{1/(m-p+1)}$ at $x = x_0 \in B$. Applying comparison principle we can see that $u_\alpha \leq w_\alpha$ on B. Thus

$$u(x_0) = u_\alpha(x_0) \leq w_\alpha(x_0)$$

Letting $\alpha \to \infty$ we obtain $u(x_0) \leq \lambda^{1/(m-p+1)}$. Therefore $u(x_0) = \lambda^{1/(m-p+1)}$.As x_0 is arbitrary, we conclude that $u \equiv \lambda^{1/(m-p+1)}$. □

Remark: We believe this result is also true when p-Laplacian is replaced by the MCO $\mathrm{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})$.

4. Global Boundedness and Related Results

In this section, we prove general result which contains Theorem 1.2 as a special case.

Let us observe the following result for the ODE problem

$$u' = f(u), u(0) = u_0. \tag{4.1}$$

Lemma 4.1. *Suppose f is C^1 and satisfies*

$$f(0) = f(1) = 0, \quad f(u) > 0\ \forall u \in (0, 1), \quad f(u) < 0\ \forall u > 1$$

Then for any $u_0 > 0$, the unique solution $u(t)$ of (4.1) satisfies $\lim_{t\to\infty} u(t) = 1$.

Proof. If $u_0 = 1$, we have $u(t) \equiv 1$ and there is nothing to prove. If $0 < u_0 < 1$, then $u(t)$ is increasing and upper bounded by 1. Therefore $\lim_{t\to\infty} u(t) = u(\infty)$ exists and satisfies $u(\infty) \in (0, 1]$. But then $u(\infty)$ must be a positive root of f. Therefore $u(\infty) = 1$. The case $u_0 > 1$ follows from a similar analysis, except that now $u(t)$ is decreasing. □

Theorem 4.2. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of (1.2). Then the conclusions in Theorem 1.2 hold.*

Proof. Let us first observe that it suffices to show $|u| \le 1$ in $\mathbb{R}^n$. Indeed, if $|u(x_0)| = 1$ say $u(x_0) = -1$,then, $w := u + 1$ satisfies

$$-\operatorname{div}(|\nabla w|^{p-2}\nabla u) = f(w-1), w \ge 0, w(x_0) = 0\,.$$

Hence, it follows from the strong maximum principle that $w \equiv 0$, contradicting our assumption that u changes sign.

We now prove $|u| \le 1$ on $\mathbb{R}^n$. Set $D = \{x : |Du(x)| = 0\}$ it is easily seen that $|u| \le 1$ on D. On $\mathbb{R}^n \setminus D$, let $g(u) = -u^r, p-1 < r < m$, we can use the proof of Theorem 1 of [20] to conclude that the problem

$$\Delta_p v = g(v), v|_{\partial B} = \infty$$

has a unique positive solution v,where B stands for a ball centered at the origin with small radius. We claim that $u \le c = \min_B v(x)$ on $\mathbb{R}^n \setminus D$. Otherwise, We can find $x_0 \in \mathbb{R}^n \setminus D$ such that $u(x_0) > c$. Define $v(x) = v(x - x_0)$. We find that the set $\{x \in B(x_0) : u(x) > v(x)\}$ has a component Ω whose closure lies entirely in the open ball $B(x_0) = \{x : x - x_0 \in B\}$. On Ω, we have $u(x) > v(x) \ge c > M$ where M satisfies $-u^r(M) = u^{p-1}(M) - u^m(M)$ and $\Delta_p u + g(u) \ge 0 = \Delta_p v + g(v)$. Moreover, $u = v$ on $\partial\Omega$. As $g(u)$ is decreasing for $u > M$, from comparison principle, we deduce that $u \equiv v$ in Ω. This contradiction shows that we must have $u \le c$ on $\mathbb{R}^n$.

Applying the above argument to $w = -u$ which satisfies

$$\Delta_p w = g(w), g(w) = -f(-w),$$

we deduce that $u \ge -c$ on $\mathbb{R}^n$. Therefore,

$$-c \le u(x), \forall x \in \mathbb{R}^n \setminus D.$$

Let u_c and u_{-c} denote the unique solution of

$$u' = f(u), \quad u(0) = u_0$$

with $u_0 = c$ and $u_0 = -c$, respectively. Then it follows from Lemma 4.1 that $u_c(t) \to 1$ and $u_{-c}(t) \to -1$ as $t \to +\infty$. One the other hand, u, u_c, u_{-c} are all bounded solutions of the parabolic problem

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u)\,.$$

Since $u_c(0) \ge u(x) \ge u_{-c}(0)$ on $\mathbb{R}^n \setminus D$, by the parabolic maximum principle and the boundedness of u, u_c, u_{-c}, we conclude that $u_{-c}(t) \le u(x) \le u_c(t)$ for all $t > 0$. Letting $t \to \infty$, we obtain $-1 \le u(x) \le 1$, as required. This finishes our proof of Theorem 4.2. $\square$

5. Some Lemmas

In this section, we prove two lemma which are needed in the proof of Theorem 1.3. Consider the solutions of the problem

$$\Delta_p u + f(u) = 0 \quad \text{in } \mathbb{R}^n \tag{5.1}$$

and that satisfy $|u| \le 1$ together with the asymptotic conditions

$$u(x', x_n) \to \pm 1 \quad \text{as } x_n \to \pm\infty \text{ uniformly in } x' = (x_1, \dots, x_{n-1}), \tag{5.2}$$

$$|Du| > 0\,. \tag{5.3}$$

Assume that the function $f = f(u)$ is Lipschitz-continuous on $[-1, 1]$, and that there exists $\delta > 0$ such that

$$f \quad \text{is non-increasing on } [-1, -1+\delta] \text{ and on } [1-\delta, 1]. \tag{5.4}$$

Lemma 5.1. *Let u be a solution of (5.1), (5.2) and (5.3) such that $|u| \le 1$. Then $u(x', x_n) = u_0(x_n)$*

The proof uses a sliding method and a version of comparison principle in slab; i.e., Theorem 2.6.

Proof. Let us now consider a solution of (5.1), (5.2) and (5.3) such that $|u| \le 1$, and let f satisfy (5.4). We are first going to prove that u is increasing in any direction $v = (v_1, \dots, v_n)$ such that $v_n > 0$. In order to do so, for any $t \in R$, we define the function u^t by $u^t(x) = u(x + tv)$.

From (5.2), there exists real $a > 0$ such that $u(x', x_n) \ge 1 - \delta$ for all $x' \in \mathbb{R}^{n-1}$ and $x_n \ge a$ and $u(x', x_n) \le -1 + \delta$ for all $x' \in \mathbb{R}^{n-1}$ and $x_n \le -a$. For any $t \ge 2a/v_n$, the functions u and u^t are such that

$$\begin{gathered}
u^t(x', x_n) \ge 1 - \delta \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and for all } x_n \ge -a, \\
u(x', x_n) \le -1 + \delta \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and for all } x_n \le -a, \\
u^t(x', -a) \ge u(x', -a) \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and for all } x_n \le -a,
\end{gathered}$$

We now apply comparison principle in slabs of the type

$$\Omega_h = \mathbb{R}^{n-1} \times (-a, h)$$

with $h > -a$.

Due to (5.2), there exists a function $\varepsilon(h) \ge 0$ such that $u^t(x', h) - u(x', h) \ge -\varepsilon(h)$ for all $x' \in \mathbb{R}^{n-1}$ and $\varepsilon(h) \to 0$ as $h \to +\infty$. Choose any $h > -a$ and set

$$w = u^t(x) + \varepsilon(h)\,.$$

Then w, u fulfill the assumption of Theorem 2.6. We have $w \ge u$ in Ω_h. By passing to the limit $h \to \infty$, we conclude that

$$u^t(x', x_n) \ge u(x', x_n) \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \ge -a.$$

Similarly, we could show that

$$u^t \ge u(x', x_n) \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \le -a$$

whence $u^t \ge u$ in $\mathbb{R}^n$

Let us now decrease t. We claim that $u^t \ge u$ for all $t > 0$. Indeed, define $\tau = \inf\{t > 0, u^t \ge u \text{in } \mathbb{R}^n\}$. By continuity, we see that $u^\tau \ge u$ in $\mathbb{R}^n$. Let us now argue by contradiction and suppose that $\tau > 0$. Two cases may occur.

Case 1. Suppose that

$$\inf_{\mathbb{R}^{n-1} \times [-a,a]} (u^\tau - u) > 0. \tag{5.5}$$

From standard elliptic estimates, u is globally Lipschitz-continuous. Hence, there exists a real η_0 small enough, which can be chosen smaller than τ, such that for all $\tau \ge t > t - \eta_0$, we have

$$u^t(x', x_n) - u(x', x_n) > 0 \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \in [-a, a]$$

Since $u \ge 1 - \delta$ in $\mathbb{R}^{n-1} \times [a, +\infty)$ it follows that

$$u^t(x', x_n) - u(x', x_n) > 0 \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \in [-a, a].$$

We may now apply Theorem 2.6 in the two half-space $\Omega^+ = \{x_n > a\}$ and $\Omega^- = \{x_n < -a\}$. We then infer that, for all $\eta \in [0, \eta_0]$, $u^{\tau-\eta}(x', x_n) \geq u(x', x_n)$ for all $x' \in \mathbb{R}^{n-1}$ and for all $x_n \in (-\infty, -a) \bigcup (a, +\infty)$ and so for all $x_n \in R$ owing to (5.2). This is contradiction with the minimality of τ. Hence (5.5) is ruled out.
Case 2. Suppose

$$\inf_{\mathbb{R}^{n-1}\times[-a,a]} (u^\tau - u) = 0. \tag{5.6}$$

Then there exists a sequence $x^k_{k\in N} \in \mathbb{R}^{n-1} \times [-a, a]$ such that $u^\tau(x^k) - u(x^k) \to 0$ as $k \to +\infty$. We normalize u by translation on $\mathbb{R}^n$ by setting $u_k(x) = u(x + x_k)$. Then by standard elliptic estimate we may assume that u_k converges to a solution u_∞ of (5.1) as $k \to \infty$. We have $u^\tau_\infty(0) = u_\infty(0)$ and $u^\tau_\infty \geq u_\infty$ because $u^\tau_k \geq u_k$ for any $k \in N$. We have

$$\begin{gathered}
\Delta_p u^(\tau_\infty) + f(u^\tau_\infty) = \Delta_p u_\infty + f(u_\infty) \quad \text{in } \mathbb{R}^n \\
u^\tau_\infty \geq u_\infty \quad \text{in } \mathbb{R}^n \\
u^\tau_\infty(0) = u_\infty(0)
\end{gathered}$$

Strong Comparison Principle yields $u^\tau_\infty(x) \equiv u_\infty(x)$. This means that $u_\infty(x) \equiv u_\infty(x + \tau v)$. Letting $\xi = \tau v$, we see that u_∞ is periodic with respect to the vector ξ. Recalling that $-a \leq x^k_n \leq a$, we see that the function u_∞ also satisfies the uniform limiting conditions (5.2). hence, since $\xi_n > 0$, the function u_∞ cannot be $\xi - periodic$. So Case 2 is also ruled out.

Therefore, we have proved that $\tau = 0$. the function u is then increasing in any direction$v = (v_1, \dots, v_n)$ such that $v_n > 0$. From the continuity of ∇u, we deduce that $\partial_v u \geq 0$ for any v such that $v_n = 0$. If $v_n = 0$,by taking v and $-v$, we find that $\partial_v u = 0$. Since this is true for all v with $v_n = 0$. Since this is true for all v with $v_n = 0$, this implies that $u(x) = u(x_n)$.

Since the solutions of (5.1) are unique up to translations,it then follows that the solutions u of (5.1), (5.2) such that $|u| \leq 1$ are unique up to translations of the origin. The proof is complete. □

Lemma 5.2. *Let f be a Lipschitz continuous function which is positive over (0,1), and satisfies $f(1) = 0, f(t) \geq \delta_0 t$ on $(0, t_0)$ for some small $\delta_0 > 0$ and $t_0 > 0$. If u is C^2 on the half plane $\Sigma_M := \{x \in \mathbb{R}^n : x_n > M\}$ and satisfies*

$$\Delta_p u + f(u) \leq 0, 0 < u \leq 1 on\ \Sigma_M$$

then $u(x', x_n) \to 1$ uniformly in $x' \in \mathbb{R}^{n-1}$ as $x_n \to +\infty$

To prove this lemma, we need following lemmas.

Lemma 5.3. *Let u be a positive function in some domain (open connected set) D satisfying*

$$\Delta_p u + f(u) \leq 0 \quad in\ D$$

with f locally Lipschitz continuous. Let B be a ball with closure $\overline{B}$ in D, and suppose z is a function in $C(\overline{B})$ satisfying

$$\begin{gathered}
z \leq u \quad in\ B \\
\Delta_p z + f(z) \geq 0 \quad wherever\ z > 0\ in\ B \\
z \leq 0 \quad on\ \partial B
\end{gathered}$$

Then, for any continuous one-parameter family of Euclidean motions(i.e., translations and rotations) $A(t)$ for $0 \le t \le T$ with $A(0) = Id$ and $A(t)\overline{B} \subset D, \forall t$, we have for all $t \in [0, t]$:

$$z_t(x) := z(A(t)^{-1}x) < u(x) in\ B_t := A(t)B \tag{5.7}$$

Proof. For all $t \ge 0, z_t$ we have

$$\begin{gathered}\Delta_p z(t) + f(z(t)) \ge 0 \quad \text{wherever } z_t > 0 \text{ in } B_t \\ z_t \le 0 \quad \text{on } \partial B\end{gathered}$$

Thus in $B_t, z(t), z$ satisfies $\Delta_p z_t + f(z_t) \ge \Delta_p z + f(z)$ wherever $z_t > 0$ in B_t and

$$z_t < u \quad \text{on } \partial B_t \tag{5.8}$$

Since $z_0 \le u$ in B, it follows by the comparison principle that $z_0 < u$ in B.

To prove (5.7) we argue by contradiction. Suppose there is a first t such that the graph of z_t touches that of u in B_t at some point x_0. Then, for that t, $z_t \le u$ in $B_t, z_t(x_0) = u(x_0)$. The strong comparison principle implies that $z_t \equiv u$ in G where G is the component containing x_0 of the set of points in B_t where $z_t > 0$. Consequently, by (5.8), any $\tilde{x} \in \partial G$ lies in B_t. Hence $z_t(\tilde{x}) > 0$ and $z_t(x) > 0$ for x near $\tilde{x}$, which shows that $\tilde{x} \in G$. We have reached a contradiction. Hence, for all $t \in [0, T]$, the graph of z_t always lies below that of u in B_t. □

Lemma 5.4. *There exist $\epsilon_1, R_0 > 0$ with R_0 depending only on n and δ_0 of Lemma 5.2 such that*

$$u(x) > \epsilon_1 \quad \textit{if}\ \operatorname{dist}(x, \Gamma) > R_0$$

Proof. Let B_{R_0} be a ball with R_0 so large that the principal eigenvalue $\lambda_1 = \lambda_1(B_{R_0})$ of Δ_p in B_{R_0} under Dirichlet boundary conditions satisfies

$$\lambda_1 = \lambda_1(B_{R_0}) < \delta_0.$$

Let φ_1 be the eigenfunction of $-\Delta_p$ in B_{R_0}, i.e.,

$$\begin{gathered}\varphi_1 > 0, -\Delta_p \varphi_1 = \lambda_1 \varphi_1^{p-1} \quad \text{in } B_{R_0} \\ \varphi_1 = 0 \quad \text{on } \partial B_{R_0}\end{gathered}$$

with $\max \varphi_1 = 1$. then for $0 < \epsilon \le s_0$ the function $z = \epsilon\varphi_1$ is a subsolution of our equation, i.e.,

$$\begin{gathered}\Delta_p z + f(z) \ge 0 \quad \text{in } B_{R_0} \\ z = 0 \quad \text{on } \partial B_{R_0}\end{gathered}$$

Let us choose $a = (0, a_n)$ with a_n large enough so that $\overline{B_{R_0}(a)}$ lies in Ω. For $B = B_{R_0}(a)$, set $\epsilon_0 = \min_{\bar{B}} u$ (clearly $\epsilon_0 > 0$), and set $\epsilon_1 = \min(\epsilon_0, s_0)$. Since $\max_{\bar{B}} \varphi_1 = 1$, it follows that

$$\epsilon_1 \varphi_1(x - a) \le u(x) \quad \text{in } B_{R_0}(a).$$

In view of Lemma 5.3, we find then that $\forall y \in \Omega$ with $\operatorname{dist}(y, \Gamma) > R_0$

$$\epsilon_1 \varphi_1(x - a) < u(x) \quad \text{in } B_{R_0}(y).$$

In particular, we have $u(y) > \epsilon_1$, thereby proving Lemma 5.4 □

Using Lemma 5.4, we prove a result that implies Lemma 5.2

Lemma 5.5. *Let y be a point with $dist(y,\Gamma) > R_0$. By Lemma 5.4, $\epsilon_1 \le u(y)$. Set*
$$\delta = \delta(y) = \min\{f(s) : s \in [\epsilon_1, u(y)]\}$$
There is a constant C_1 depending only on n such that
$$C_1\delta \le [\operatorname{dist}(y,\Gamma) - R_0]^{-2} \tag{5.9}$$

Proof. We first choose C_1. Let v be the solution in $B_1(0)$ of
$$\begin{gathered}\Delta_p v = -1 \quad \text{in } B_1(0)\\ v = 0 \quad \text{on } \partial B_1(0)\end{gathered}$$
We take $C_1 = \max_{B_1(0)} v = v(x_0)$. We assume that (5.9) does not hold and argue by contradiction;i.e., we assume
$$C_1\delta > [\operatorname{dist}(y,\Gamma) - R_0]^{-2}$$
Fix $R < \operatorname{dist}(y,\Gamma) - R_0$ such that $C_1\delta > \mathbb{R}^{-1}$. Since $\Delta_p u < 0$ at y, u cannot achieve a local minimum there. choose a point y_1 close to y with $u(y_1) < u(y)$ and such that $\operatorname{dist}(y_1,\Gamma) > R_0 + R$. By Lemma 5.4, $u \ge \epsilon_1$ in $B_R(y_1) =: B$.

Let z be the solution in B of
$$\begin{gathered}\Delta_p z = -\delta \quad \text{in } B\\ z = 0 \quad \text{on } \partial B\end{gathered} \tag{5.10}$$
By scaling, one finds that
$$\max z = z(y_1) = C_1\delta\mathbb{R}^2$$
For $0 < \tau$ small, $\tau z < u$ in B. As we increase τ, there is necessarily a first value-which we call τ-for which the graph of τz touches that of u at some point x_0. Since $z = 0$ on ∂B, x_0 in B. Now
$$u(x_0) = \tau z(x_0) \le \tau z(y_1) = \tau C_1\delta\mathbb{R}^2 \le u(y_1) < u(y) < 1 \tag{5.11}$$
and hence $\tau < 1$. Thus
$$w := \tau z - u \ge 0 \text{in } B, w(x_0) = \tau z(x_0) - u(x_0) = 0 \tag{5.12}$$
By (5.10), $u(x_0) < u(y)$, and so in a neighborhood, say N, of $x_0, u < u(y)$. By the definition of δ, it follows that $\Delta_p \le -\delta$ in the neighborhood N, and since $\tau < 1, \Delta_p(\tau z) > \Delta_p u > 0$ in this neighborhood N. this contradicts the fact that $\tau z - u$ has a local maximum at x_0 □

Proof of Lemma 5.2. Since $\operatorname{dist}(x,\Gamma) \to \infty$, from (5.9) follows that $\min_{[\epsilon_1,u(x)]} f \to 0$ which implies that $u(x) \to 1$ uniformly in Ω □

6. Odd Nonlinearity and Related Results

In this section, we prove a generalization of Theorem 1.3.

Theorem 6.1. *Suppose f is Lipschitz continuous and satisfies*
$$f(-1) = f(0) = f(1) = 0, \quad tf(t) > 0 \quad \textit{when } 0 < |t| < 1,$$
and for small positive constants δ_0, t_0 and δ
$$\begin{gathered}\frac{f(t)}{t} \ge \delta_0 \quad \textit{when } 0 < |t| < t_0\\ f \textit{ is non-increasing on } [-1,-1+\delta] \cup [1-\delta, 1].\end{gathered}$$

Furthermore, assume that $f(t)$ is odd in t. Then the statement in Theorem 1.3 holds for any solution $u \in C^2(\mathbb{R}^n)$ of (5.1), (5.2), (5.3) satisfying $|u| \leq 1$

Proof. After a rotation and a translation, we may assume that the hyperplane is given by $x_n = 0, u(0) = 0$ and $u^{-1}(0) \subset \{x : x_n \leq 0\}$. We may assume that $u(x', x_n) > 0, \forall x' \in \mathbb{R}^{n-1}, \forall x_n > 0$; the other possibility that $u(x', x_n) < 0, \forall x^{'} \in \mathbb{R}^{n-1}, \forall x_n > 0$ can be handled analogously.

For $\tau \geq 0$, let us define

$$u_\tau(x', x_n) = -u(x', 2\tau - x_n).$$

Since f is odd, we easily see that

$$-\Delta_p u_\tau = f(u_\tau)$$

clearly $u|_{x_n=\tau} \geq 0 \geq u_\tau|_{x_n=\tau}$.

We want to show that for every $\tau \geq 0$, $u \geq u_\tau$ on the half space$\{x : x_n \geq \tau\}$. Since $u(x) > 0$ when $x_n > 0$, it follows from Lemma 5.2 that $u(x^{'}, x_n) \to 1$ as $x_n \to \infty$ uniformly in $x^{'} \in \mathbb{R}^{n-1}$. Therefore, for large τ we can apply comparison principle to

$$\Omega := \{x : x_n > \tau\}$$

to conclude that $u \geq u_\tau$ on Ω. Now define

$$\tau_0 = \inf\{\tau \in [0, \infty) : u(x', x_n) \geq u_\tau(x^{'}, x_n), \forall x^{'} \in \mathbb{R}^{n-1}, \forall x_n \geq \tau\}.$$

Claim: $\tau_0 = 0$. Otherwise,$\tau_0 > 0$ and $u(x) \geq u_{\tau_0}(x)$ on the set $\Omega_0 := \{x : x_n \geq \tau_0\}$. Clearly u, u_{τ_0} satisfied

$$\Delta_p u + f(u) = \Delta_p u_\tau + f(u_\tau)$$

Since $u > 0 > u_{\tau_0}$ on $\partial\Omega_0$, by the definition of τ_0,we have two possibilities.

(a) $u(x_0) = u_\tau(x_0)$ for some $x_0 \in \Omega_0$, or

(b) $u(x) > u_\tau(x) > 0$ in Ω_0 and $u(z_k) - u_\tau(z_k) \to 0$ for some $z_k \in \bar{\Omega}_0$ with $|z_k| \to \infty$.

If case(a) occurs, then the Harnack inequality forces $w \equiv 0$ on Ω_0, which is impossible as $w > 0$ on $\partial\Omega_0$. If (b) occurs, we set $u_k(x) = u(x + z_k)$. By standard elliptic estimates, up to extraction of a subsequence, u_k converges in $C^2_{\rm loc}(\mathbb{R}^n)$ to a solution u^* of (5.1) as $k \to \infty$. Moreover,

$$v := u^* - u^*_{\tau_0}$$

satisfies $v(0) = 0$ and

$$\Delta_p u^* + f(u^*) = \Delta_p u^*_{\tau_0} + f(u^*_{\tau_0}), v \geq 0, \forall x \in \Omega^*$$

where $\Omega^* = \{x : x_n > \tau^*\}$ with $\tau^* \in [-\infty, 0]$ determined by (passing to a subsequence when necessary)

$$\tau^* = -\lim_{k\to\infty} d(z_k, \partial\Omega_0).$$

If $0 \in \Omega^*$ then we obtain from the Harnack inequality that $v \equiv 0$ on Ω^*, i.e.,

$$u^*(x^{'}, x_n) = -u^*(x', 2\tau_0 - x_n), \forall x^{'} \in \mathbb{R}^{n-1}, \forall x_n > \tau^* .$$

Taking $x_n = \tau_0$ we deduce $u^*(x', \tau_0) = 0$. This implies that $d(z_k, \partial\Omega_0)$ is bounded, for otherwise, due to $u(x', x_n) \to 1$ uniformly in $x' \in \mathbb{R}^{n-1}$ as $x_n \to +\infty$, we would have $u^* \equiv 1$. The boundedness of $\{d(z_k, \partial\Omega_0)\}$ and the fact that $u(x', x_n) \to 1$ uniformly in $x' \in \mathbb{R}^{n-1}$ as $x_n \to +\infty$ imply $u^*(x', x_n) \to 1$ uniformly in $x' \in \mathbb{R}^{n-1}$ as $x_n \to +\infty$. This together with comparison principle implies that $u^*(x', x_n) \to -1$

uniformly in $x' \in \mathbb{R}^{n-1}$ as $x_n \to -\infty$. Hence we can use Lemma 5.1 to conclude that $u^*(x) = u^*(x_n)$ and is increasing in x_n. On the other hand,since $u_k(0) = u(z_k) > 0$, we have $u^*(0) \geq 0$, a contradiction to the monotonicity of $u^*(x_0)$ and $u^*(\tau_0) = 0$

If $0 \in \partial\Omega^*$, we necessarily have $\{d(z_k, \partial\Omega_0)\} \to 0$ and hence $\tau^* = 0, \Omega^* = \{x : x_n > 0\}$. As before,this implies $u^*(x', x_n) \to 1$ uniformly in x' as $x_n \to \infty$. Moreover for any$\eta \geq -\tau_0$, since $u_k(x', \eta) = u((x', \eta) + z_k) \geq 0$, we deduce

$$u^*(x', \eta) \geq 0, \forall x' \in \mathbb{R}^{n-1}.$$

In particular,

$$u^*(0, x_n) \geq 0, \forall x_n \geq \tau_0 \tag{6.1}$$

As $v(0) = 0$, we have $u^*(0) = -u^*(0, 2\tau_0)$. Therefore we necessarily have $u^*(0) = u^*(0, 2\tau_0) = 0$. In view of (6.1),the function $g(t) := u^*(0, t)$ has a local minimum at $t = 0$ and at$t = 2\tau_0$.Therefore,$g'(0) = g'(2\tau_0) = 0$.This implies that$\partial_n v(0) = 0$.Since v satisfies

$$\Delta_p u^* + f(u^*) = \Delta_p u^*_\tau + f(u^*_\tau) \geq 0, \forall x \in \Omega^*, v(0) = 0, 0 \in \partial\Omega*$$

an application of the strong comparison principle gives $v \equiv 0$, i.e., $u^*(x', x_n) = -u^*(x', 2\tau_0 - x_n)$ for all $x' \in \mathbb{R}^{n-1}$ and all $x_n \geq 0$. We can now argue as in the case that $0 \in \Omega^*$ to conclude that $u^*(x) = u^*(x_n)$ and is increasing in x_n. But this is in contradiction with our earlier observation that $u^*(0) = u^*(2\tau_0)$.This proves our claim.

From $\tau_0 = 0$ we obtain $u(x^{'}, x_n) \geq -u(x^{'}, -x_n)$ for all $x^{'} \in \mathbb{R}^{n-1}$and $x_n \geq 0$. Hence we must have $u(x^{'}, x_n) = -u(x^{'}, -x_n)$ for all$x^{'} \in \mathbb{R}^{n-1}$ and all $x_n > 0$. Recall that we have $u(x^{'}, x_n) \to -1$ uniformly in $x^{'}$ as $x_n \to -\infty$. Therefore we can use Lemma 5.1 and conclude. The proof of Theorem 1.3 is complete. □

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