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AVERAGING FOR NON-PERIODIC FULLY NONLINEAR EQUATIONS

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Abstract. This paper studies the averaging problem for some fully nonlinear equations of degenerate parabolic type with a Hamiltonian not necessarily periodic in the fast variable. Our aim is to point out a sufficient condition on the Hamiltonian to pass to the limit in the starting equation. Also, we investigate when this condition is not completely fulfilled and discuss some examples concerning deterministic and stochastic optimal control problems.

1. Introduction

In this paper we consider the family of Cauchy problems

$$\partial_t u_\varepsilon + H\big(x,t,t/\varepsilon,u_\varepsilon,Du_\varepsilon,D^2u_\varepsilon\big) = 0, \quad \text{in } (0,T)\times\mathbb{R}^n \tag{1.1}$$

$$u_\varepsilon(0,x) = h(x) \quad \text{on } \mathbb{R}^n, \tag{1.2}$$

where $\partial_t \equiv \partial/\partial_t$, the function u_ε is scalar, Du_ε and D^2u_ε stand respectively for the gradient and the Hessian matrix of u_ε with respect to the variable x. It is clear that, as $\varepsilon \to 0$, the nonlinearity in (1.1) oscillates more and more rapidly; the *theory of averaging* (*homogenization* if the Hamiltonian H depends on x/ε) investigates whether the solutions u_ε of (1.1)-(1.2) converge "in some sense" as $\varepsilon \to 0$ to the solution u of

$$\partial_t u + \bar{H}(x,t,u,Du,D^2u) = 0 \tag{1.3}$$

for an *effective* Hamiltonian $\bar{H}$ to be founded somehow (see the books [24, p. 323-ff] and [10, p. 233-ff and 516-ff] for solutions in Sobolev spaces).

In this paper for a solution we shall mean a *viscosity* solution (this notion was introduced by Crandall and Lions [18] for first-order equations and extended to second-order equations by Lions [25, 26]; see the paper [17] for an overview). The pioneering results in homogenization for viscosity solution are due to Lions, Papanicolaou and Varadhan [27], who faced the problem

$$\begin{aligned} \partial_t v_\varepsilon + H(x/\varepsilon, Dv_\varepsilon) = 0 \quad &\text{in } (0,T)\times\mathbb{R}^n, \\ v_\varepsilon(0,x) = v_0(x) \quad &\text{on } \mathbb{R}^n, \end{aligned} \tag{1.4}$$

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where the Hamiltonian $H = H(x,p)$ is periodic in x and coercive in p. They showed that v_ε converges uniformly to the solution v of the problem

$$\begin{gathered}\partial_t v + \bar{H}(Dv) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n,\\ v(0,x) = v_0(x) \quad \text{on } \mathbb{R}^n,\end{gathered}$$

where the effective Hamiltonian $\bar{H}$ is obtained as follows. Plugging in (1.4) the (early known and formal) expansion

$$v_\varepsilon(t,x) = v^0(t,x) + \varepsilon v^1(t,x/\varepsilon) + \varepsilon^2 v^2 \dots, \tag{1.5}$$

with $v^i(t,y)$ a periodic functions in y, they deduced the *cell problem*: for each $p \in \mathbb{R}^n$, find $\lambda \in \mathbb{R}$ such that there exists a solution $v = v(y)$ to

$$H(y, p + Dv) = \lambda \quad \text{in } \mathbb{R}^n, \quad v \text{ periodic in } y\,.$$

They proved that, for each $p \in \mathbb{R}^n$, there exists exactly one $\lambda(p)$ which solves the cell problem; moreover, the effective Hamiltonian $\bar{H}$ can be conveniently defined by $\bar{H}(p) := \lambda(p)$ (see also [13, 14] for a variational approach to $\bar{H}$). The result of [27] was generalized by Evans [19, 20] to first-order equations of the form

$$H(x, x/\varepsilon, v_\varepsilon, Dv_\varepsilon) = 0,$$

and to second-order equations of the form

$$F(x, x/\varepsilon, v_\varepsilon, Dv_\varepsilon, D^2 v_\varepsilon) = 0,$$

under the principal assumptions that $H(x,y,r,p)$ is periodic in y and coercive in p, that

$$r \to H(x,y,r,p) - \mu r \quad \text{is nondecreasing for some } \mu > 0, \forall (x,y,p), \tag{1.6}$$

and, respectively, that $F(x,y,r,p,X)$ is uniformly elliptic, periodic in y and satisfies a condition similar to (1.6). In these works, Evans introduced the *perturbed test-function* method, where the expansion (1.5) was replaced by the same expansion for the (smooth) test-function: $\phi(t,x) = \phi^0(t,x) + \varepsilon\phi^1(t,x/\varepsilon) + \varepsilon^2\phi^2 \dots$. Lions and Souganidis [28] investigated the existence or non-existence of a solution to the cell problem when the periodicity assumption is not accomplished.

Barron [8] faced the averaging problem for the equation

$$\partial_t v_\varepsilon + H(x,t,t/\varepsilon, Dv_\varepsilon) = 0,$$

for Hamiltonian $H(x,t,\tau,p)$ periodic in τ (among other conditions). Using some properties of an underlying deterministic optimal control problem (however, an argument analogous to Evans' one could be used as well), he proved that the effective Hamiltonian $\bar{H}$ is given by

$$\bar{H}(x,t,p) := \int_0^1 H(x,t,\xi,p)\,d\xi.$$

Actually, a few years before, Chaplais [12] faced a similar problem with a non-periodic $H(x,t,\tau,p)$: he showed that also the effective problem can be written as a deterministic optimal control with the Hamiltonian given by

$$\bar{H}(x,t,p) := \lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau H(x,t,\xi,p)\,d\xi, \tag{1.7}$$

provided that the above limit exists for every (x, t, p). He proved also that the solutions of

$$\begin{gathered}\partial_t v_\varepsilon + H(x, t, t/\varepsilon, Dv_\varepsilon) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n,\\ v_\varepsilon(0, x) = v_0(x) \quad \text{on } \mathbb{R}^n,\end{gathered}$$

converge uniformly on each compact subset of $[0, T] \times \mathbb{R}^n$ to the solution of

$$\begin{gathered}\partial_t v + \bar{H}(x, t, Dv) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n,\\ v(0, x) = v_0(x) \quad \text{on } \mathbb{R}^n,\end{gathered}$$

if the following condition is fulfilled: for each (x, t, p) there exists a value $\bar{H}(x, t, p)$ such that

$$\lim_{\tau\to+\infty} \sup_{\tau_1\geq 0} \Big| \frac{1}{\tau} \int_{\tau_1}^{\tau_1+\tau} H(x, t, \xi, p)\, d\xi - \bar{H}(x, t, p) \Big| = 0.$$

Let us mention that Alvarez and Bardi [1, 2] faced the homogenization of

$$\partial_t v_\varepsilon + H(x, y, D_x v_\varepsilon, D_y v_\varepsilon/\varepsilon, D_{xx} v_\varepsilon, D_{yy} v_\varepsilon/\varepsilon, D_{xy} v_\varepsilon/\sqrt{\varepsilon}) = 0, \tag{1.8}$$

where the state variable (x, y) splits into the slow variable x and in the fast variable y. As a particular case of (1.8), they considered the equation

$$\partial_t v_\varepsilon + H(x, t/\varepsilon, Dv_\varepsilon, D^2 v_\varepsilon) = 0,$$

with $H(x, \tau, p, X)$ periodic in τ, and obtained an effective Hamiltonian as in (1.7). We recall that the case of linear uniformly parabolic equations was solved in the book [10, p. 516-ff] using different techniques. It is the purpose of this paper to extend the result by [10] to fully nonlinear degenerate equations, fulfilling the "averaging property" stated in assumption (A2) below. To this end we shall use the perturbed test-function method by Evans, the weak semi-limits and some ideas of [8].

Let us emphasize that, besides [12], the principal result known for a non-periodic Hamiltonian, is due to Ishii [22], who considered the equation

$$v_\varepsilon(x) + H(x, x/\varepsilon, Dv_\varepsilon) = 0 \quad \text{in } \mathbb{R}^n,$$

under the primary assumption that $H(x, y, p)$ is *almost periodic* in y and coercive in p. See also [5] for certain cases of second-order *quasi-periodic* Hamiltonians.

Finally, it is of some interest to recall that the homogenization simultaneous in x and in t was addressed in [3] and in [23], still under a periodicity assumption.

This paper is organized as follows: In Section 2 we give some notations and we state our main results (proved in Section 3). Section 4 gives some examples and compares our results with previously known results.

2. Mathematical framework and main results

We denote by $\mathbb{M}^{n,m}$ and $\mathbb{S}^n$ respectively the set of $n \times m$ real matrices and the space of $n \times n$ symmetric matrices. The latter is endowed with the usual order: for $X, Y \in \mathbb{S}^n$ we shall write "$X \geq Y$", if $X - Y$ is a semi-definite positive matrix. I will stand for the identity matrix in $\mathbb{S}^n$.

We denote the strip $(0, T) \times \mathbb{R}^n$ and the semi-space $(0, +\infty) \times \mathbb{R}^n$ respectively by S_T and S_∞. Given any point $x \in \mathbb{R}^n$ and any value $r \in \mathbb{R}^+$, $B_r(x)$ is the open ball centered in x of radius r. Given any point $(t, x) \in S_\infty$ and any constant $r \in \mathbb{R}^+$, $Q_r(t, x)$ will stand for the *parabolic neighborhood around* (t, x), *i.e.* $Q_r(t, x) := (t - r, t + r) \times B_r(x)$.

Let the Hamiltonian $H : \mathbb{R}^n \times [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ (eventually, $T = +\infty$) satisfy the following assumptions:

(A0) H is continuous and *proper* (*i.e.*, for all $x, p \in \mathbb{R}^n$, $t \in [0,T]$, $\tau \in \mathbb{R}^+$, $r, s \in \mathbb{R}$ and $X, Y \in \mathbb{S}^n$, it satisfies: $H(x,t,\tau,r,p,X) \leq H(x,t,\tau,s,p,Y)$ whenever $r \leq s$ and $X \geq Y$). H satisfies the usual condition for the Comparison Principle in bounded domains (see: [17, pag 48] and [16, pag 38]): there exists a function $\omega : [0, +\infty] \to [0, +\infty]$ with $\lim_{s\to 0^+} \omega(s) = 0$ such that, for all $x, y \in \mathbb{R}^n$, $t \in [0,T]$, $\alpha, \tau \in \mathbb{R}^+$ and $X, Y \in \mathbb{S}^n$, there holds

$$\begin{aligned} &H\left(y,t,\tau,r,\alpha(x-y),Y\right) - H\left(x,t,\tau,r,\alpha(x-y),X\right) \\ &\leq \omega\left(\alpha|x-y|^2 + |x-y|\right) \end{aligned} \tag{2.1}$$

whenever

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

(A1) Fix $(x,t,r,p,X) \in \mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$. For each $\varepsilon > 0$, there exists $\delta > 0$ such that, for $\max\{|x-x'|, |t-t'|, |r-r'|, |p-p'|, |X-X'|\} < \delta$, there holds

$$|H(x,t,\tau,r,p,X) - H(x',t',\tau,r',p',X')| < \varepsilon \quad \forall \tau \in \mathbb{R}^+.$$

(A2) The Hamiltonian H satisfies the *averaging property*: for each $(x,t,r,p,X) \in \mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ there exists the following limit:

$$\lim_{\tau\to+\infty} \frac{1}{\tau} \int_0^\tau H(x,t,\xi,r,p,X)\, d\xi =: \bar{H}(x,t,r,p,X).$$

Remark 2.1. *Let us observe that hypothesis (A2) is equivalent to requiring that, for each $(x,t,r,p,X) \in \mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, there exists exactly one value λ such that the ordinary differential equation*

$$\lambda + \frac{d\chi}{d\tau} + H(x,t,\tau,r,p,X) = 0 \quad \text{in } (0,+\infty) \tag{2.2}$$

admits a solution sublinear at infinity, i.e. such that $\lim_{\tau\to+\infty} \chi(\tau)/\tau = 0$. The parameter λ shall depend on (x,t,r,p,X) and it can be easily checked that $\lambda = -\bar{H}(x,t,r,p,X)$. It is of some interest to note that, plugging in (1.1) *the formal expansion (analogous to* (1.5)*), $u_\varepsilon(t,x) = u^0(t,x) + \varepsilon u^1(t/\varepsilon, x)$, we obtain exactly relation* (2.2) *as cell problem.*

Remark 2.2. Of course, any Hamiltonian $H(x,t,\tau,r,p,X)$, continuous and periodic in τ with period l, satisfies assumptions (A1)–(A2) and

$$\bar{H}(x,t,r,p,X) \equiv \frac{1}{l} \int_0^l H(x,t,\xi,r,p,X)\, d\xi$$

(see Subsection 4.2 below for other generalizations of periodicity).

Let us first state some useful properties of the effective Hamiltonian $\bar{H}$ and then state our main result.

Proposition 2.3. *Assume that the Hamiltonian H satisfies conditions (A0)–(A2); then $\bar{H}$ satisfies assumption (A0). In particular, the Comparison Principle is valid for the solutions of* (1.3).

Theorem 2.4. *Assume that the Hamiltonian H satisfies conditions (A0)–(A2) and that $\{u_\varepsilon\}_{0<\varepsilon\le 1}$ is a locally equi-bounded family of solutions to problem* (1.1)–(1.2). *Then $\{u_\varepsilon\}$ converges uniformly on each compact subsets of $[0,T]\times\mathbb{R}^n$ to the (unique) solution u to the effective Cauchy problem*

$$\begin{gathered}\partial_t u+\bar{H}\left(x,t,u,Du,D^2u\right)=0 \quad \text{in } (0,T)\times\mathbb{R}^n,\\ u(0,x)=h(x) \quad \text{on } \mathbb{R}^n.\end{gathered} \tag{2.3}$$

For investigating Hamiltonians which do not satisfy assumption (A2), let us introduce the following conditions:

(A3) For each $(x,t,r,p,X)\in\mathbb{R}^n\times[0,T]\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ there exists
$$H_S(x,t,r,p,X):=\limsup_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau H(x,t,\xi,r,p,X)\,d\xi<+\infty;$$

(A4) For each $(x,t,r,p,X)\in\mathbb{R}^n\times[0,T]\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ there exists
$$H_i(x,t,r,p,X):=\liminf_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau H(x,t,\xi,r,p,X)\,d\xi>-\infty.$$

It is clear that assumption (A2) entails (A3) and (A4) with $\bar{H}\equiv H_S\equiv H_i$.

Theorem 2.5. *Assume that the Hamiltonian H fulfills conditions (A0)–(A1) and that the family $\{u_\varepsilon\}_{0<\varepsilon\le 1}$ of solutions to problem* (1.1)–(1.2) *converges to a function u locally uniformly in $\bar{S}_T$. We have also the following:*

(a) *If the Hamiltonian H satisfies assumption (A3), then the function u is a subsolution to the Cauchy problem*
$$\begin{gathered}\partial_t u+H_S\left(x,t,u,Du,D^2u\right)=0 \quad \text{in } (0,T)\times\mathbb{R}^n,\\ u(0,x)=h(x) \quad \text{on } \mathbb{R}^n.\end{gathered} \tag{2.4}$$

(b) *If the Hamiltonian H satisfies assumption (A4), then the function u is a supersolution of the Cauchy problem*
$$\begin{gathered}\partial_t u+H_i\left(x,t,u,Du,D^2u\right)=0 \quad \text{in } (0,T)\times\mathbb{R}^n,\\ u(0,x)=h(x) \quad \text{on } \mathbb{R}^n.\end{gathered} \tag{2.5}$$

Remark 2.6. It is worth noting that in the proofs of Theorems 2.4 and 2.5, the Comparison Principle is needed but not the regularity property (2.1). Henceforth, for a first-order Hamiltonian $H(x,t,\tau,r,p)$, assumption (A0) can be replaced by

(A5) H is continuous and satisfies the usual condition for the Comparison Principle in bounded domains ([7]): for each compact G in $\mathbb{R}^n\times[0,T]\times\mathbb{R}^+\times\mathbb{R}$, there exist a constant $\mu\in\mathbb{R}^+$ and a function $\omega_1:[0,+\infty)\to[0,+\infty)$, with $\lim_{s\to0^+}\omega_1(s)=0$, such that, for each $(x,t,\tau,r)\in G$ and $p\in\mathbb{R}^n$, $r\to H(x,t,\tau,r,p)+\mu r$ is non decreasing and there holds
$$|H(x,t,\tau,r,p)-H(x',t,\tau,r,p)|\le\omega_1\left(|x-x'|(1+|p|)\right).$$

3. Proofs of main results

Proof of Proposition 2.3. Fix $(x,t,r,p,X)\in\mathbb{R}^n\times[0,T]\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$. By assumptions (A1)–(A2), there holds
$$\begin{aligned}&\left|\bar{H}(x,t,r,p,X)-\bar{H}(x',t',r',p',X')\right|\\ &\le\lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau|H(x,t,\xi,r,p,X)-H(x',t',\xi,r',p',X')|\,d\xi\le\varepsilon,\end{aligned}$$

if $\max\{|x-x'|,|t-t'|,|r-r'|,|p-p'|,|X-X'|\}$ is sufficiently small; hence the effective Hamiltonian $\bar{H}$ is continuous. The proofs of the properness and of the condition for the Comparison Principle are similar so we omit them. $\square$

Proof of Theorem 2.4. Let us introduce the weak semi-limits

$$\bar{u}(t,x) := \limsup_{(\varepsilon,t',x')\to(0^+,t,x)} u_\varepsilon(t',x')$$
$$\underline{u}(t,x) := \liminf_{(\varepsilon,t',x')\to(0^+,t,x)} u_\varepsilon(t',x').$$

We shall prove that $\bar{u}$ and $\underline{u}$ are respectively a sub- and a supersolution of (2.3); the two proofs are similar so we will omit the latter. Fix $(\bar{t},\bar{x})\in S_T$ and let ϕ be a test-function such that $\bar{u}-\phi$ admits a local maximum in $(\bar{t},\bar{x})$; without any loss of generality, we can assume that the maximum is strict and that there holds

$$\bar{u}(\bar{t},\bar{x}) = \phi(\bar{t},\bar{x}). \tag{3.1}$$

We proceed by contradiction, assuming that for some $\eta>0$ there holds

$$\partial_t\phi(\bar{t},\bar{x}) + \bar{h} \equiv \partial_t\phi(\bar{t},\bar{x}) + \bar{H}\left(\bar{x},\bar{t},\phi(\bar{t},\bar{x}),D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right) > \eta. \tag{3.2}$$

By assumption (A1), there exists a constant $\delta_0\in\mathbb{R}^+$ such that

$$\begin{aligned}&\big|H\left(\bar{x},\bar{t},\tau,\phi(\bar{t},\bar{x})-\delta_0,D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right)\\ &\quad -H\left(\bar{x},\bar{t},\tau,\phi(\bar{t},\bar{x}),D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right)\big| \le \frac{\eta}{12} \quad \forall\tau\in\mathbb{R}^+.\end{aligned} \tag{3.3}$$

Let us fix $0<r\le\bar{t}/2$, sufficiently small to have:

$$\begin{aligned}&\big|H\left(x,t,\tau,\phi(t,x)-\delta_0,D\phi(t,x),D^2\phi(t,x)\right)\\ &-H\left(\bar{x},\bar{t},\tau,\phi(\bar{t},\bar{x})-\delta_0,D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right)\big| \le \eta/3, \quad \forall\tau\in\mathbb{R}^+,\ (t,x)\in Q_r,\end{aligned} \tag{3.4}$$

$$\big|\partial_t\phi(\bar{t},\bar{x})-\partial_t\phi(t,x)\big| \le \eta/3, \quad \forall(t,x)\in Q_r, \tag{3.5}$$

$$u(t,x)-\phi(t,x) \le -\mu, \quad \text{for some } \mu\in\mathbb{R}^+,\ \forall(t,x)\in\partial Q_r \tag{3.6}$$

(Q_r is the abridged notation of $Q_r(\bar{t},\bar{x})$). We consider the perturbed test-function

$$\phi_\varepsilon(t,x) := \phi(t,x) + \varepsilon\chi(t/\varepsilon)$$

where $\chi=\chi(\tau)$ is a classical solution (not necessarily unique) of the following ordinary differential equation:

$$-\bar{h} + \frac{d\chi}{d\tau} + H\left(\bar{x},\bar{t},\tau,\phi(\bar{t},\bar{x}),D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right) = 0 \tag{3.7}$$

such that $\lim_{\tau\to+\infty}\chi(\tau)/\tau=0$. It is easily checked that the last relation implies

$$\lim_{\varepsilon\to+\infty} \varepsilon\chi(t/\varepsilon) = 0 \quad \text{uniformly in } [\bar{t}/2, 2\bar{t}],$$

and therefore,

$$\lim_{\varepsilon\to+\infty} \phi_\varepsilon(t,x) = \phi(t,x) \quad \text{uniformly in } Q_r. \tag{3.8}$$

Now we claim that ϕ_ε is a supersolution to (1.1). Let us first observe that, by assumption (A0), the function χ exists and is C^1; hence, ϕ_ε can solve (1.1) in the classical sense. Testing ϕ_ε in (1.1), for each $(t,x) \in Q_r$ we obtain (for ε sufficiently small):

$$
\begin{aligned}
&\partial_t\phi_\varepsilon(t,x) + H\left(x,t,t/\varepsilon,\phi_\varepsilon(t,x),D\phi_\varepsilon(t,x),D^2\phi_\varepsilon(t,x)\right)\\
&\geq \partial_t\phi(t,x) + \frac{d\chi}{d\tau}(t/\varepsilon) + H\left(x,t,t/\varepsilon,\phi(t,x)-\delta_0,D\phi(t,x),D^2\phi(t,x)\right)\\
&= \partial_t\phi(t,x) + \frac{d\chi}{d\tau}(t/\varepsilon) + H\left(\bar{x},\bar{t},t/\varepsilon,\phi(\bar{t},\bar{x}),D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right)\\
&\quad - \Big[H\left(\bar{x},\bar{t},t/\varepsilon,\phi(\bar{t},\bar{x}),D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right)\\
&\quad - H\left(\bar{x},\bar{t},t/\varepsilon,\phi(\bar{t},\bar{x})-\delta_0,D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right)\Big]\\
&\quad + \Big[H\left(x,t,t/\varepsilon,\phi(t,x)-\delta_0,D\phi(t,x),D^2\phi(t,x)\right)\\
&\quad - H\left(\bar{x},\bar{t},t/\varepsilon,\phi(\bar{t},\bar{x})-\delta_0,D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right)\Big]\\
&\geq \partial_t\phi(t,x) + \frac{d\chi}{d\tau}(t/\varepsilon) + H\left(\bar{x},\bar{t},t/\varepsilon,\phi(\bar{t},\bar{x}),D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right) - \frac{5}{12}\eta,
\end{aligned}
$$

where the latter inequality is due to relations (3.3)–(3.4) while the former one to the properness of H and to relation (3.8). By relations (3.5) and (3.7), we deduce

$$
\partial_t\phi_\varepsilon(t,x) + H\left(x,t,t/\varepsilon,\phi_\varepsilon(t,x),D\phi_\varepsilon(t,x),D^2\phi_\varepsilon(t,x)\right) \geq \partial_t\phi(\bar{t},\bar{x}) + \bar{h} - \frac{3}{4}\eta.
$$

By assumption (3.2), for each $(t,x) \in Q_r$ there holds

$$
\partial_t\phi_\varepsilon(t,x) + H\left(x,t,t/\varepsilon,\phi_\varepsilon(t,x),D\phi_\varepsilon(t,x),D^2\phi_\varepsilon(t,x)\right) \geq 0 \tag{3.9}
$$

hence, the function ϕ_ε is a classical supersolution of (1.1).

Let us prove that for some $\varepsilon_0 \in (0,1]$ there holds

$$
u_\varepsilon(t,x) - \phi_\varepsilon(t,x) < -\mu/2 \quad \forall (t,x) \in \partial Q_r, \quad \varepsilon < \varepsilon_0. \tag{3.10}
$$

In fact, if the previous relation fails, then there exist sequences ε_n and $(t_n,x_n) \in \partial Q_r$, such that

$$
\varepsilon_n \to 0 \text{ as } n \to +\infty \quad \text{and} \quad u_{\varepsilon_n}(t_n,x_n) - \phi_{\varepsilon_n}(t_n,x_n) \geq -\mu/2. \tag{3.11}
$$

Being ∂Q_r compact, it is possible to extract a subsequence (which we still denote by (t_n,x_n)) converging to a point $(\tilde{t},\tilde{x}) \in \partial Q_r$. By equality (3.8) and by definition of $\bar{u}$, we have the relation

$$
\bar{u}(\tilde{t},\tilde{x}) - \phi(\tilde{t},\tilde{x}) \geq \varliminf_{n\to+\infty} [u_{\varepsilon_n}(t_n,x_n) - \phi_{\varepsilon_n}(t_n,x_n)] \geq -\mu/2,
$$

which contradicts (3.6); hence claim (3.10) is proved.

Now we claim that, for $0 < \alpha < \min\{\delta_0, -\mu/2\}$, the function $\phi_\varepsilon - \alpha$ is a classical supersolution to the initial-boundary value problem

$$
\begin{gathered}
\partial_t v + H\left(x,t,t/\varepsilon,v,Dv,D^2v\right) = 0 \quad \forall (t,x) \in Q_r,\\
v(\bar{t}-r,x) = u_\varepsilon(\bar{t}-r,x) \quad \forall x \in B_r(\bar{x}),\\
v(t,x) = u_\varepsilon(t,x) \quad \forall (t,x) \in (\bar{t}-r,\bar{t}+r) \times \partial B_r(\bar{x}),
\end{gathered} \tag{3.12}
$$

for $\varepsilon \leq \varepsilon_1$, with ε_1 sufficiently small. To this end, it suffices to replace ϕ_ε by $\phi_\varepsilon - \alpha$ in the calculations above and to require that for $\varepsilon < \varepsilon_1$ there holds:

$$
|\varepsilon\chi(t/\varepsilon)| < \delta_0 - \alpha \quad \text{uniformly in } [\bar{t}/2, 2\bar{t}].
$$

On the other hand, it is straightforward to recognize that u_ε is a subsolution to the problem (3.12). By the Comparison Principle, we have

$$u_\varepsilon(t,x) \le \phi_\varepsilon(t,x) - \alpha \quad \forall (t,x) \in Q_r,\ \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}.$$

Passing to the lim sup in the previous relation, we obtain:

$$\bar{u}(\bar{t},\bar{x}) = \limsup_{(\varepsilon,t,x)\to(0^+,\bar{t},\bar{x})} u_\varepsilon(t,x) \le \limsup_{(\varepsilon,t,x)\to(0^+,\bar{t},\bar{x})} \phi_\varepsilon(t,x) - \alpha = \phi(\bar{t},\bar{x}) - \alpha,$$

which contradicts assumption (3.1).

Finally by Proposition 2.3, the Comparison Principle is valid for sub- and super-solutions to problem (2.3) and it yields: $\bar{u} \le \underline{u}$. By the definition of weak semi-limit, the opposite inequality is obvious so $\bar{u} \equiv \underline{u} =: u$. It is well known (for instance, see: [7, p. 290]) that the above relation is equivalent to the following statement: the sequence $\{u_\varepsilon\}$ converges uniformly to u on each compact subset of S_T. □

Remark 3.1. By the same argument as in the proof above, one can prove that $\bar{u}$ and $\underline{u}$ are respectively a sub- and a super solution to (1.3) also if $[0,T]$ is replaced by $(0,T)$ in (A0)–(A2).

Proof of Theorem 2.5. We prove only part (a); the proof of part (b) is similar and we shall omit it. We fix $(\bar{t},\bar{x}) \in S_\infty$ and a test-function ϕ as in the proof of Theorem 2.4. We proceed by contradiction, assuming that for some $\eta > 0$ there holds

$$\partial_t\phi(\bar{t},\bar{x}) + h_S \equiv \partial_t\phi(\bar{t},\bar{x}) + H_S\left(\bar{x},\bar{t},\phi(\bar{t},\bar{x}), D\phi(\bar{t},\bar{x}), D^2\phi(\bar{t},\bar{x})\right) > \eta. \tag{3.13}$$

Let us choose $0 < r \le \bar{t}/2$ sufficiently small to accomplish relations (3.4)–(3.6) and consider the perturbed test-function $\phi_\varepsilon(t,x) := \phi(t,x) + \varepsilon\chi(t/\varepsilon)$, where $\chi = \chi(\tau)$ is a classical solution of the following ordinary differential equation

$$-h_S + \frac{d\chi}{d\tau} + H\left(\bar{x},\bar{t},\tau,\phi(\bar{t},\bar{x}), D\phi(\bar{t},\bar{x}), D^2\phi(\bar{t},\bar{x})\right) = 0 \tag{3.14}$$

such that $\liminf_{\tau\to+\infty} \chi(\tau)/\tau = 0$. Consequently, we have

$$\liminf_{\varepsilon\to+\infty} \phi_\varepsilon(t,x) = \phi(t,x) \quad \text{uniformly in } Q_r. \tag{3.15}$$

As in the proof of Theorem 2.4, it can be showed that, for $\alpha \in \mathbb{R}$ sufficiently small, the functions $\phi_\varepsilon - \alpha$ and u_ε are respectively a super- an a subsolution of

$$\begin{gathered}\partial_t v + H\left(x,t,t/\varepsilon,v,Dv,D^2v\right) = 0 \quad \forall (t,x) \in Q_r,\\ v(\bar{t}-r,x) = u_\varepsilon(\bar{t}-r,x) \quad \forall x \in B_r(\bar{x}),\\ v(t,x) = u_\varepsilon(t,x) \quad \forall (t,x) \in (\bar{t}-r,\bar{t}+r) \times B_r(\bar{x}),\end{gathered}$$

for $\varepsilon \le \varepsilon_0$ (let us recall that the constant ε_0 was introduced in (3.10)). Taking into account the Comparison Principle, we have

$$u_\varepsilon(t,x) \le \phi_\varepsilon(t,x) - \alpha \quad \forall (t,x) \in Q_r,\ \varepsilon < \varepsilon_0.$$

Taking the lim inf in the previous relation, by relation (3.15) we obtain

$$u(\bar{t},\bar{x}) = \liminf_{(\varepsilon,t,x)\to(0^+,\bar{t},\bar{x})} u_\varepsilon(t,x) \le \liminf_{(\varepsilon,t,x)\to(0^+,\bar{t},\bar{x})} \phi_\varepsilon(t,x) - \alpha = \phi(\bar{t},\bar{x}) - \alpha$$

which contradicts assumption (3.1). □

4. Examples

4.1. **Convergence of functions u_ε.** Here we show two cases where the hypothesis on the convergence of u_ε stated respectively in Theorems 2.4 and 2.5 are fulfilled.

Proposition 4.1. *Let the Hamiltonian H fulfills (A0) and the following condition*

$$|H(x,t,\tau,0,0,0)| \le M \quad \forall (x,t,\tau) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^+, \tag{4.1}$$

for some $M > 0$. For each $h \in BUC(\mathbb{R}^n)$ (i.e., it is bounded and uniformly continuous), there exists a unique solution u_ε to problem (1.1)–(1.2)*. Moreover, the functions u_ε ($\varepsilon > 0$) are uniformly bounded.*

Using the Perron method [17] and the Comparison Principle, one can easily obtain the proof of Proposition 4.1 so we omit it. Taking into account Theorem 2.4 and the previous Proposition, we have the following statement.

Corollary 4.2. *Let the Hamiltonian H satisfy (A0)–(A2) and* (4.1)*. Then the solutions u_ε to* (1.1)–(1.2) *converge locally uniformly to the solution of* (2.3)*.*

Remark 4.3. *This result applies to the Hamilton-Jacobi-Bellman-Isaacs operators*

$$H(x,t,\tau,r,p,X) := \sup_\alpha \inf_\beta L_{\alpha,\beta}(x,t,\tau,r,p,X),$$

$$\begin{aligned} &L_{\alpha,\beta}(x,t,\tau,r,p,X) \\ &:= -\operatorname{tr}\big(a_{\alpha,\beta}(x,t,\tau)X\big) - g_{\alpha,\beta}(x,t,\tau)\cdot p - l_{\alpha,\beta}(x,t,\tau)r - f_{\alpha,\beta}(x,t,\tau) \end{aligned}$$

where "tr" denotes the trace, and $a_{\alpha,\beta} = \sigma_{\alpha,\beta}\sigma^T_{\alpha,\beta}$. It is well known [2] *that, if $\sigma_{\alpha,\beta}$, $g_{\alpha,\beta}$ and $l_{\alpha,\beta}$ are Lipschitz continuous functions of $\mathbb{R}^n \times [0,T] \times \mathbb{R}^+$ respectively in $\mathbb{M}^{n,n}$, $\mathbb{R}^n$, and $\mathbb{R}^+$ and if $f_{\alpha,\beta} \in C(\mathbb{R}^n \times [0,T] \times \mathbb{R}^+)$, then the Hamiltonian H satisfies assumption (A0) and* (4.1)*, provided that all local bounds and all moduli of continuity of $\sigma_{\alpha,\beta}$, $g_{\alpha,\beta}$, $l_{\alpha,\beta}$ and $f_{\alpha,\beta}$ are uniform in α and in β. In this case, for each $\varepsilon > 0$, there exists a solution u_ε to* (1.1)–(1.2)*, bounded independently of ε.*

Now, let us consider a free endpoint problem of Lagrange in optimal control. The *dynamics* are given by

$$\frac{d\xi}{d\eta} = f\left(\xi(\eta), \eta, \eta/\varepsilon, \zeta(\eta)\right), \quad t \le \eta \le T,$$
$$\xi(t) = x \in \mathbb{R}^n,$$

for a *control function* ζ in the class $\mathcal{Z}[t,T] := \{\zeta : [t,T] \to Z;\ \zeta \text{ is measurable}\}$, with Z compact in $\mathbb{R}^m$, $m \in \mathbb{N}$. The player's objective is to minimize the *pay-off*

$$P_\varepsilon(t,x,\zeta) := e^{\int_t^T g(\xi(s),s,s/\varepsilon,\zeta(s))\,ds} u_0(\xi(T)) + \int_t^T h\left(\xi(s), s, s/\varepsilon, \zeta(s)\right) ds.$$

Assume that the functions f, g, u_0, and h satisfy the following assumptions:

(B1) If ϕ is any of the functions f, g or h, then ϕ is defined in $\mathbb{R}^n \times [0,T] \times \mathbb{R}^+ \times Z$, is continuous and, for some $K > 0$, satisfies

$$|\phi(x,t,\tau,z) - \phi(x',t',\tau',z)| \le K\left(|t-t'| + |\tau-\tau'| + |x-x'|\right), \tag{4.2}$$

for every $(x,t,\tau,z), (x',t',\tau',z) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^+ \times Z$; f satisfies also

$$|f(x,t,\tau,z)| \le K(1+|x|) \quad \forall (x,t,\tau,z) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^+ \times Z. \tag{4.3}$$

The functions h and u_0 are bounded and the latter is Lipschitz continuous.

It is well known that, under assumption (B1), the *value-function*

$$V_\varepsilon(t,x) := \inf_{\zeta\in\mathcal{Z}[t,T]} P_\varepsilon(t,x,\zeta)$$

is the unique solution to the problem

$$\begin{gathered}\partial_t V_\varepsilon + H\left(x,t,t/\varepsilon,V_\varepsilon,DV_\varepsilon\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n,\\ V_\varepsilon(T,x) = u_0(x) \quad \text{on } \mathbb{R}^n,\end{gathered} \tag{4.4}$$

where the Hamiltonian H is defined by

$$H\left(x,t,t/\varepsilon,r,p\right) := \inf_{z\in Z}\left\{f(x,t,t/\varepsilon,z)\cdot p + g(x,t,t/\varepsilon,z)\,r + h(x,t,t/\varepsilon,z)\right\}$$

and satisfies assumptions (A0)–(A1) and (A5) (see: [8, 9, 29]). By small changes of the argument in [8], one can show the following statement.

Proposition 4.4. *Under assumption (B1), there exists a subsequence of $\{V_\varepsilon\}$ locally uniformly convergent to a continuous function V.*

Taking into account Theorem 2.5 and the previous Proposition, we have the following corollary which generalizes the results stated in [8, 12].

Corollary 4.5. *Let the optimal control problem satisfy assumption (B1). If assumption (A3) (respectively, (A4)) holds, then the functions V_ε converge locally uniformly to a subsolution of* (2.4) *(respectively, a supersolution to* (2.5)*).*

Remark 4.6. The results of Proposition 4.4 and of Corollary 4.5 can be extended to a *two-person zero-sum* differential game. In this case, the dynamics are given by

$$\begin{gathered}\frac{d\xi}{d\eta} = f\left(\xi(\eta),\eta,\eta/\varepsilon,\zeta(\eta),\gamma(\eta)\right) \quad t\le\eta\le T,\\ \xi(t) = x\in\mathbb{R}^n;\end{gathered}$$

where the controls ζ, γ are chosen respectively in $\mathcal{Z}[t,T] := \{\zeta : [t,T]\to Z;\ \zeta$ is measurable$\}$ and in $\mathcal{J}[t,T] := \{\gamma : [t,T]\to J;\ \gamma$ is measurable$\}$, where Z and J are compacts respectively in $\mathbb{R}^{m_1}$ and in $\mathbb{R}^{m_2}$ ($m_1,\ m_2\in\mathbb{N}$). The aim of ζ-player and of γ-player is respectively to minimize and to maximize the pay-off

$$P_\varepsilon(t,x,\zeta,\gamma) := e^{\int_t^T g(\xi(s),s,s/\varepsilon,\zeta(s),\gamma(s))\,ds}u_0(\xi(T)) + \int_t^T h\left(\xi(s),s,s/\varepsilon,\zeta(s),\gamma(s)\right)\,ds.$$

Assume that the so-called *minimax* (or *Isaacs'*) condition

$$\begin{aligned}&\inf_{z\in Z}\sup_{j\in J}\left\{f(x,t,t/\varepsilon,z,j)\cdot p + g(x,t,t/\varepsilon,z,j)\,r + h(x,t,t/\varepsilon,z,j)\right\}\\ &= \sup_{j\in J}\inf_{z\in Z}\left\{f(x,t,t/\varepsilon,z,j)\cdot p + g(x,t,t/\varepsilon,z,j)\,r + h(x,t,t/\varepsilon,z,j)\right\}.\end{aligned}$$

is fulfilled with $H_1(x,t,t/\varepsilon,r,p)$ denoting the common value. Let us recall that, under conditions similar to (B1), H_1 satisfies assumptions (A0)–(A1), (A5) and the value W_ε of the game solves (4.4) with H replaced by H_1 (see: [21]).

4.2. **Almost periodic Hamiltonians and the averaging property.** We recall that, for each almost periodic (*a. p.* in the sequel) function f, the limit $\lim_{\tau\to+\infty}\frac{1}{\tau}\int_a^{a+\tau} f(\xi)d\xi$ exists uniformly with respect to a and it is independent of a (see: [11, 15, 4]); hence, any Hamiltonian $H(x,t,\tau,r,p,X)$, a. p. in τ, satisfies assumption (A2). In particular, the Hamilton-Jacobi-Bellman-Isaacs operator introduced in Remark 4.3 fulfills (A2) provided that the functions $\sigma_{\alpha,\beta}$, $g_{\alpha,\beta}$, $l_{\alpha,\beta}$, $f_{\alpha,\beta}$ are a. p. in τ uniformly with respect to α and β. Actually, assumption (A2) still holds for *asymptotically* a. p. Hamiltonians, i. e. of the form

$$H(x,t,\tau,r,p,X) = H_1(x,t,\tau,r,p,X) + H_2(x,t,\tau,r,p,X),$$

where H_1 is a. p. in τ and $\lim_{\tau\to+\infty} H_2(x,t,\tau,r,p,X) = 0$ uniformly in (x,t,r,p,X).

Finally, regarding the generalizations of the notion of periodic functions, it is of some interest to observe that $H(x,t,\tau,r,p,X)$ satisfies assumption (A2) provided that, for each $(x,t,r,p,X) \in \mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, the function

$$f(\tau) := \begin{cases} H(x,t,\tau,r,p,X) & \tau \geq 0 \\ H(x,t,-\tau,r,p,X) & \tau < 0, \end{cases}$$

belongs to the Besicovitch space $B^1_{ap}(\mathbb{R})$ (see [6, 30] for the precise definition).

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