THE EDDY CURRENT PROBLEM WITH TEMPERATURE DEPENDENT PERMEABILITY

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ABSTRACT. We prove the uniqueness and existence of a local solution in time for a system of PDE's modelling the eddy currents and the heating of a cylindrical conductor.

1. INTRODUCTION

Electric currents induced in a massive conductor by an external varying magnetic field are known as eddy currents. They in turn heat the body by Joule effect. The problem of predicting the magnetic field and the temperature in the conductor is modelled by the non linear system

$$\frac{\partial}{\partial t}(\mu \mathbf{H}) = -\nabla \times (\rho \nabla \times \mathbf{H}) \tag{1.1}$$

$$u_t - \nabla \cdot (\kappa \nabla u) = \rho |\nabla \times \mathbf{H}|^2, \tag{1.2}$$

where ρ is the electric resistivity, κ the thermal conductivity and μ the permeability. In the special situation of a very long cylindrical conductor of cross-section Ω (an open, bounded and connected subset of $\mathbb{R}^2$ with a regular boundary Γ) immersed in an insulating medium and with a time dependent magnetic field $\mathbf{H}$ given at infinity by $\mathbf{H}_\infty = h_b(t)\mathbf{i}_3$, $\mathbf{i}_3$ unit vector parallel to the axis of the cylinder. In view of the geometry, we assume $\mathbf{H} = h\mathbf{i}_3$; this simplifies the equations and we are led (see [3]and [11]), to the following initial-boundary value problem, closely related to the thermistor problem:

$$(\mu h)_t = \nabla \cdot (\rho \nabla h) \tag{1.3}$$

$$u_t - \nabla \cdot (\kappa \nabla u) = \rho |\nabla h|^2, \ x = (x_1, x_2) \tag{1.4}$$

$$h(x,t) = h_b(t) \quad \text{on } \Gamma \times (0,T) \tag{1.5}$$

$$h(x,0) = 0 \quad \text{in } \Omega \tag{1.6}$$

$$u(x,t) = 0 \quad \text{on } \Gamma \times (0,T) \tag{1.7}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega. \tag{1.8}$$

The system (1.1)–(1.2) has been the subject of a variety of investigations in the past decade. They all assume μ to be a constant and ρ a given function of the

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temperature u. We refer, in particular, to the work of Hong-Ming Yin [13], [14] and also to [8] and references therein. The more special problem (1.3)–(1.8) has been studied in [3] and [11], again with constant permeability. Now, in real conductors not only ρ and κ depend on u, but also μ, in certain cases dramatically [10]; this makes the problem considerably more difficult.

2. Results

In the stationary thermistor problem a suitable transformation linearizes the system (see [4], [5] and [6]) and this permits, in particular, a precise estimate of the maximum of the temperature u. Unfortunately, there does not seem to be a correspondent transformation for the parabolic system (1.3)–(1.8). However, when ρ, κ and μ are constants the transformation useful in the thermistor problem can be applied to (1.3)–(1.8). Let

$$a = \frac{\mu}{\rho}, \quad b = \frac{1}{\rho 2}, \quad c = \frac{\kappa}{\rho 2}$$

and define $\mathcal{H}(x_1, x_2, t) = h(x_1, x_2, at)$, $\mathcal{U}(x_1, x_2, t) = u(x_1, x_2, bt)$ and

$$\theta(x_1, x_2, t) = \frac{\mathcal{H}^2}{2}(x_1, x_2, t) + c \quad \mathcal{U}(x_1, x_2, t).$$

Under this transformation system (1.3)–(1.8) becomes

$$\mathcal{H}_t = \Delta \mathcal{H}, \quad \theta_t = \Delta \theta.$$

The initial and boundary conditions for θ are immediately written in terms of the same data of $\mathcal{H}$ and $\mathcal{U}$. Thus $\mathcal{H}$ and θ are estimated via the parabolic maximum principle and, correspondingly, $\mathcal{U} = \frac{1}{c}\left(\theta - \frac{\mathcal{H}^2}{2}\right)$ is also estimated. In this note we present a result of uniqueness for problem (1.3)–(1.8) assuming μ, ρ and κ to be given functions of u and a result of existence of solutions local in time. The difficulties inherent to the nonlinear problem (1.3)–(1.8) are better understood if we rewrite equations. Equations (1.3) and (1.4) in normal form, i.e. solved with respect to h_t and u_t, and with the principal part in divergence form. We obtain, after easy calculations,

$$h_t = \nabla \cdot (a(u)\nabla h) - \nabla \cdot (b(u)h\nabla u) - c(u)h|\nabla h|^2 + d(u)\nabla u \cdot \nabla h + e(u)h|\nabla u|^2 \tag{2.1}$$

$$u_t = \nabla \cdot (\kappa(u)\nabla u) + \rho(u)|\nabla h|^2, \tag{2.2}$$

where

$$a(u) = \nu(u)\rho(u), \quad \nu(u) = \frac{1}{\mu(u)}, \quad b(u) = \mu'(u)\nu(u)\kappa(u), \quad c(u) = \mu'(u)\nu(u)\rho(u),$$

$$d(u) = \mu'(u)\nu(u)\kappa(u) - \nu'(u)\rho(u), \quad e(u) = \mu''(u)\nu(u)\kappa(u) + \mu'(u)\nu'(u)\kappa(u).$$

On the right-hand side of equation (2.1) the second term clearly shows the lack of uniform ellipticity in the principal part of the system. This is the main source of mathematical difficulty in these equations. We assume $b(u)$, $c(u)$, $d(u)$ and $e(u)$ to be bounded and globally Lipschitz and $a(u)$, $\kappa(u)$ to be locally bounded and to satisfy

$$a(u) \geq a_0 > 0, \quad \kappa(u) \geq \kappa_0 > 0. \tag{2.3}$$

We denote with $\| \cdot \|$ the norm in $L2(\Omega)$.

Theorem 2.1. *Under the assumptions*

$$u_0(x) \in H_0^{1,\infty}(\Omega), \ h_b(t) \in H^{1,\infty}(0,T), \ h_b(0) = 0,$$

there exists at most one solution,

$$(h,u) \in L^\infty(0,T;H^{1,\infty}(\Omega)) \times L^\infty(0,T;H_0^{1,\infty}(\Omega)),$$
$$(h_t,u_t) \in L2(0,T;H^{-1}(\Omega)) \times L2(0,T;H^{-1}(\Omega)),$$

to problem (2.1)*,* (2.2)*,* (1.5)–(1.8)*.*

Proof. Let (h_1,u_1) and (h_2,u_2) be two solutions and let $z = h_1 - h_2$, $w = u_1 - u_2$. Subtracting equation (2.1) from (2.2), we obtain

$$
\begin{aligned}
z_t &= \nabla \cdot \left[a(u_1)\nabla h_1 - a(u_2)\nabla h_2\right] - \nabla \cdot \left[b(u_1)h_1\nabla u_1 - b(u_2)h_2\nabla u_2\right] \\
&\quad - \left[c(u_1)h_1|\nabla h_1|^2 - c(u_2)h_2|\nabla h_2|^2\right] + \left[d(u_1)\nabla u_1 \cdot \nabla h_1 - d(u_2)\nabla h_2\right] \\
&\quad + \left[e(u_1)h_1|\nabla u_1|^2 - e(u_2)h_2|\nabla u_2|^2\right] \\
&= \{1\} + \{2\} + \{3\} + \{4\} + \{5\}, \\
w_t &= \nabla \cdot \left[\kappa(u_1)\nabla u_1 - \kappa(u_2)\nabla u_2\right] + \rho(u_1)|\nabla h_1|^2 - \rho(u_2)|\nabla h_2|^2 = \{6\} + \{7\},
\end{aligned}
$$

(2.4)

(2.5)

$$z(x,0) = 0, \quad w(x,0) = 0, \quad x \in \Omega, \tag{2.6}$$
$$z(x,t) = 0, \quad w(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T). \tag{2.7}$$

We multiply (2.4) by z and (2.5) by Kw, where K is a positive constant to be defined later. Integrating by parts over Ω and using the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (z2 + Kw2)dx + a_0\|\nabla z\|^2 + K\kappa_0\|\nabla w\|^2 \\
&\leq C_{11}\|w\|\|\nabla z\| + \left(C_{21}\|w\|\|\nabla z\| + C_{22}\|z\|\|\nabla z\| + C_{23}\|\nabla w\|\|\nabla z\|\right) \\
&\quad + \left(C_{31}\|w\|\|z\| + C_{32}\|z\|^2 + C_{33}\|z\|\|\nabla z\|\right) \\
&\quad + \left(C_{41}\|w\|\|z\| + C_{42}\|z\|\|\nabla w\| + C_{43}\|z\|\|\nabla z\|\right) \\
&\quad + \left(C_{51}\|w\|^2 + C_{52}\|z\|^2 + C_{53}\|z\|\|\nabla w\|\right) \\
&\quad + C_{61}K\|w\|\|\nabla w\| + KC_{71}\|w\|^2 + KC_{72}\|w\|\|\nabla z\|
\end{aligned}
$$

(2.8)

where the first index refers, in the various C_{ij}, to the grouping given in (2.4) and (2.5). We apply the elementary inequality $2ab \leq \gamma a2 + \frac{1}{\gamma}b2$ with various choices of γ and with the goal of absorbing in the left-hand side of (2.8) all the terms containing $\|\nabla z\|$ and $\|\nabla w\|$. Clearly, in this respect, the most difficult term is $C_{23}\|\nabla w\|\|\nabla z\|$. With $\epsilon > 0$ we have

$$C_{11}\|w\|\|\nabla z\| \leq C_{11}\frac{\epsilon}{2}\|\nabla z\|^2 + C_{11}\frac{1}{2\epsilon}\|w\|^2$$

and

$$KC_{72}\|w\|\|\nabla z\| \leq KC_{72}\frac{\epsilon}{2K}\|\nabla z\|^2 + \frac{K}{2\epsilon}C_{72}\|w\|^2.$$

Treating the other groups of terms similarly, we obtain in the end

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega (z2 + Kw2)dx + \big(a_0 - A\epsilon - KC_{72}\frac{\epsilon}{2K}\big)\|\nabla z\|^2
$$
$$
+ \big(\frac{K\kappa_0}{2} - B\epsilon - \frac{C_{23}}{2\epsilon}\big)\|\nabla w\|^2
$$
$$
\leq L(\epsilon, K)\|w\|^2 + M(\epsilon, K)\|z\|^2
$$

where $L(\epsilon, K)$ and $M(\epsilon, K)$ are positive functions defined for $\epsilon > 0$ and $K > 0$, and A, B constants easily computed. With the choice

$$
\epsilon = \bar\epsilon = \frac{a_0}{A + C_{72}}, \quad K = \bar K = \max\Big\{1, \frac{4}{\kappa_0}\Big[\frac{Ba_0}{(A + C_{72})} + \frac{2C_{23}(A + C_{72})}{a_0}\Big]\Big\},
$$

we have

$$
\frac{d}{dt}\int_\Omega (z2 + \bar Kw2)dx + a_0\|\nabla z\|^2 + \frac{\bar K\kappa_0}{2}\|\nabla w\|^2 \leq C\int_\Omega (z2 + \bar Kw2)dx
$$

where $C = 2\max\{L(\bar\epsilon, \bar K), M(\bar\epsilon, \bar K)\}$. Hence the Gronwall lemma, gives

$$
\|u_1(t) - u_2(t)\|^2 + \bar K\|h_1(t) - h_2(t)\|^2 \leq e^{Ct}(\|u_1(0) - u_2(0)\|^2 + \bar K\|h_1(0) - h_2(0)\|^2
$$

and uniqueness follows. $\square$

To prove a result of existence local in time, we apply a theorem by Sobolewskii in [12], which we quote below.

Theorem 2.2. *For $\alpha, \beta, i, j = 1, 2$, let the functions*

$$
a_{ij\alpha\beta}(t, v_1, v_2, v_{11}, v_{12}, v_{21}, v_{22}), \quad f_\alpha(t, t, v_1, v_2, v_{11}, v_{12}, v_{21}, v_{22})
$$

be of class $C2(\mathbb{R}^7)$ and $\tilde v_1(x), \tilde v_2(x)$ be in $C^2(\bar\Omega)$, with $\tilde v_1, \tilde v_2 = 0$ on Γ. Define

$$
\tilde a_{ij\alpha\beta}(x) = a_{ij\alpha\beta}\Big(0, \tilde v_1(x), \tilde v_2(x), \frac{\partial \tilde v_1}{\partial x_1}(x), \frac{\partial \tilde v_1}{\partial x_2}(x), \frac{\partial \tilde v_2}{\partial x_1}(x), \frac{\partial \tilde v_2}{\partial x_2}(x)\Big).
$$

Suppose that

$$
\sum_{ij\alpha\beta=1}^2 \tilde a_{ij\alpha\beta}(x)\xi_{i\alpha}\xi_{j\beta} \geq k\sum_{i\alpha=1}^2 \xi_{i\alpha}^2, \quad k > 0 \tag{2.9}
$$

for all $x \in \Omega$. Then the problem

$$
\frac{\partial v_\alpha}{\partial t} - \sum_{ij\alpha\beta}^2 \frac{\partial}{\partial x_i}\Big(a_{ij\alpha\beta}\Big(t, v_1, v_2, \frac{\partial v_1}{\partial x_1}, \frac{\partial v_1}{\partial x_2}, \frac{\partial v_2}{\partial x_1}, \frac{\partial v_2}{\partial x_2}\Big)\frac{\partial v_\beta}{x_j}\Big)
$$
$$
= f_\alpha\Big(t, v_1, v_2, \frac{\partial v_1}{\partial x_1}, \frac{\partial v_1}{\partial x_2}, \frac{\partial v_2}{\partial x_1}, \frac{\partial v_2}{\partial x_2}\Big) \tag{2.10}
$$
$$
v_\alpha(0, x) = \tilde v_\alpha(x) \quad x \in \Omega
$$
$$
v_\alpha(t, x) = 0, \quad (t, x) \in \Gamma \times [0, T), \quad \alpha = 1, 2
$$

has one and only one solution defined in a suitably small interval $[0, t_0), t_0 < T$.

To apply this theorem to our case we assume $a(u), b(u), c(u), d(u), e(u), \kappa(u)$ and $\rho(u)$ to be $C2(\mathbb{R})$. We define as new unknown function $w(t, x) = h(t, x) - h_b(t)$ to obtain the homogeneous boundary condition required in (2.10) and suppose $h_b(t) \in C2(\mathbb{R})$ and $u_0(x) \in C2(\bar\Omega)$. In view of (6) and (11) the crucial condition (2.9) is satisfied: indeed we have

$$
a(u_0(x))(\xi_{11}^2 + \xi_{12}^2) + \kappa(u_0(x))(\xi_{21}^2 + \xi_{22}^2) \geq k(\xi_{11}^2 + \xi_{12}^2 + \xi_{21}^2 + \xi_{22}^2)
$$

with $k = \min(a_0, \kappa_0)$. Hence the initial boundary value problem (2.1), (2.2), (1.5)—
(1.8) has a unique classical solution local in time.

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