Electronic Journal of Differential Equations, Vol. 2003(2003), No. 86, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

ESTIMATES FOR DERIVATIVES OF THE GREEN FUNCTIONS FOR THE NONCOERCIVE DIFFERENTIAL OPERATORS ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE, II

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Abstract. We consider the Green functions for second order non-coercive differential operators on homogeneous manifolds of negative curvature, being a semi-direct product of a nilpotent Lie group N and $A = \mathbb{R}^+$. We obtain estimates for the mixed derivatives of the Green functions that complements a previous work by the same author [17].

1. Introduction

Let M be a connected and simply connected homogeneous manifold of negative curvature. Such a manifold is a solvable Lie group $S = NA$, a semi-direct product of a nilpotent Lie group N and an Abelian group $A = \mathbb{R}^+$. Moreover, for an H belonging to the Lie algebra $\mathfrak{a}$ of A, the real parts of the eigenvalues of $\mathrm{Ad}_{\exp H}|_{\mathfrak{n}}$, where $\mathfrak{n}$ is the Lie algebra of N, are all greater than 0. Conversely, every such a group equipped with a suitable left-invariant metric becomes a homogeneous Riemannian manifold with negative curvature (see [8]).

On S we consider a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y.$$

We assume that $Y_0, Y_1, \dots, Y_m$ generate the Lie algebra $\mathfrak{s}$ of S and $Y \in \mathfrak{s}$. We can always make $Y_0, \dots, Y_m$ linearly independent and moreover, we can choose $Y_0, Y_1, \dots, Y_m$ so that $Y_1(e), \dots, Y_m(e)$ belong to $\mathfrak{n}$. (Write $\mathcal{L}$ as $\sum_{i,j=0}^{\dim\mathfrak{n}} \alpha_{i,j} E_i E_j + \sum_{j=0}^{\dim\mathfrak{n}} \beta_j E_j$, $E_0 \in \mathfrak{a}$, $\{E_j\}$ is a basis of $\mathfrak{n}$, $\alpha_{i,j}, \beta_j \in \mathbb{R}$ and then rewrite $\mathcal{L}$ as a sum of squares). Let $\pi : S \to A = S/N$ be the canonical homomorphism. Then the image of $\mathcal{L}$ under π is a second order left-invariant operator on $\mathbb{R}^+$,

$$(a\partial_a)^2 - \gamma_{\mathcal{L}} a\partial_a,$$

2000 *Mathematics Subject Classification.* 11E25, 43A85, 53C30, 31B25.
Key words and phrases. Green function, homogeneous manifolds of negative curvature, NA groups, evolutions on nilpotent Lie groups.

Submitted February 18, 2003. Published August 15, 2003.
Partly supported by grant 5P03A02821 from KBN, and by contract HPRN-CT-2001-00273-HARP from RTN Harmonic Analysis and Related Problems.

where $\gamma_{\mathcal{L}} \in \mathbb{R}$. We say that a second order differential operator $\mathcal{L}$ on a Riemannian manifold is *noncoercive* (*coercive* resp.) if there is no $\varepsilon > 0$ such that $\mathcal{L} + \varepsilon \mathrm{Id}$ admits the Green function (if such an ε exists resp.). It is worth noting that our definition of coercivity is a little bit different than that used e.g. in [1]. Namely, for us, $\mathcal{L}$ is coercive if it is weakly coercive in Ancona's terminology. There is a relation between the notion of coercivity property in the sense used in the theory of partial differential equations (i.e., that an appropriate bilinear form is coercive, [9]) and weak coercivity. For this the reader is referred to [1].

In this paper we shall study the operators $\mathcal{L}$ that are noncoercive ($\gamma_{\mathcal{L}} = 0$). In this case $\mathcal{L}$ can be written as

$$\mathcal{L} = \sum_{j=1}^{m} \Phi_a(X_j)^2 + \Phi_a(X) + a^2\partial_a^2 + \partial_a, \tag{1.1}$$

and $X, X_1, \dots, X_m$ are left-invariant vector fields on N, moreover, the vector fields $X_1, \dots, X_m$ are linearly independent and generate $\mathfrak{n}$,

$$\Phi_a = \mathrm{Ad}_{\exp(\log a)Y_0} = e^{(\log a)\,\mathrm{ad}_{Y_0}} = e^{(\log a)D},$$

where $D = \mathrm{ad}_{Y_0}$ is a derivation of the Lie algebra $\mathfrak{n}$ of the Lie group N such that the real parts d_j of the eigenvalues λ_j of D are positive. By multiplying $\mathcal{L}$ by a constant, i.e., changing Y_0, we can make d_j arbitrarily large (see [5]).

Let $\mathcal{G}(xa, yb)$ be the *Green function* for $\mathcal{L}$. The Green function $\mathcal{G}$ is (uniquely) defined by two conditions:

i) $\mathcal{L}\mathcal{G}(\cdot, yb) = -\delta_{yb}$ as distributions (functions are identified with distributions via the right Haar measure),

ii) for every $yb \in S$, $\mathcal{G}(\cdot, yb)$ is a potential for $\mathcal{L}$, i.e., is a positive superharmonic function such that its largest harmonic minorant is equal to zero (cf. [3]).

Let

$$\mathcal{G}(x, a) := \mathcal{G}(xa; e), \tag{1.2}$$

where e is the identity element of the group S. It is easy to show that

$$\mathcal{G}(xa; yb) = \mathcal{G}((yb)^{-1}xa; e) = \mathcal{G}((yb)^{-1}xa).$$

In this paper we refer to $\mathcal{G}(x, a)$ defined in (1.2) as the Green function for $\mathcal{L}$.

The main goal of this paper is to prove the following estimates for derivatives of the Green function (1.2) for the noncoercive operator $\mathcal{L}$, i.e., with $\gamma_{\mathcal{L}} = 0$. For every neighborhood $\mathcal{U}$ of the identity e of the group NA there is a constant $C = C(\mathcal{U})$ such that we have

$$|\partial_a^l \mathcal{X}^I \mathcal{G}(x, a)| \le \begin{cases} C(|x| + a)^{-\|I\| - Q} a^{-l} & \\ \times(1 + |\log(|x| + a)^{-1}|)^{\|I\|_0}, & (x, a) \in (\mathcal{Q} \cup \mathcal{U})^c, \\ Ca^{-l}, & (x, a) \in \mathcal{Q} \setminus \mathcal{U}. \end{cases} \tag{1.3}$$

where $|\cdot|$ stands for a "homogeneous norm" on N, $\mathcal{Q} = \{|x| \le 1, a \le 1\}$, $\|I\|$ is a suitably defined length of the multi-index I and $\|I\|_0$ is a certain number depending on I and the nilpotent part of the derivation D. In particular, $\|I\|_0$ is equal to 0 if the action of $A = \mathbb{R}^+$ on N, given by Φ_a, is diagonal or, if $I = 0$. $\mathcal{X}_1, \dots, \mathcal{X}_n$ is an appropriately chosen basis of $\mathfrak{n}$. For the precise definitions of all the notions that have appeared here see Sect. 2.

Some comments should be made at this moment. First of all, it should be said that the estimates for the Green function (i.e., when I=0) with $\gamma_{\mathcal{L}} > 0$, also from below, was proved by E. Damek in [4] and then by the author for $\gamma_{\mathcal{L}} = 0$ in [15] but at that time it was impossible to prove analogous estimate for derivatives. The reason was that we did not have sufficient estimates for the derivatives of the transition probabilities of the evolution on N generated by an appropriate operator which appears as the "horizontal" component of the diffusion on $N \times \mathbb{R}^+$ generated by $a^{-2}\mathcal{L}_{-\gamma}$ (cf. [5]). These estimates have been obtained by the author in [16] and eventually led up to the estimates for derivatives of the Green function $\mathcal{G}$ with respect to the N and A-variables (see [17]). Here we are going to present how to use the results from [17] in order to get the estimates for the mixed derivatives, i.e., not only for N and $A = \mathbb{R}^+$-variables separately.

The proof of (1.3) requires both analytic and probabilistic techniques. Some of them have been introduced in [6, 5] and [15]. This paper also heavily depends on some results from [17].

Guide to the paper. The structure of the paper is as follows. In Sect. 2 we set the notation and give all the necessary definitions. In particular, we recall a definition of the Bessel process which appears as the "vertical" component of the diffusion generated by $a^{-2}\mathcal{L}_{-\gamma}$ on $N\times\mathbb{R}^+$ as well as the notion of the evolution on N generated by an appropriate operator which appears as the "horizontal" component of the diffusion on $N \times \mathbb{R}^+$ mentioned in the Introduction above (cf. [5, 17]).

The main tool in the proof of (1.3) is Proposition 3.1. Its proof is given in Sect. 3.

Finally, in Sect. 4 we state the estimates (1.3) precisely (see Theorem (4.1)) and we give a sketch of its proof.

2. Preliminaries

2.1. NA **groups.** Good references for this topic are [6, 5] and [7]. Let N be a connected and simply connected nilpotent Lie group. Let D be a derivation of the Lie algebra $\mathfrak{n}$ of N. For every $a \in \mathbb{R}^+$ we define an automorphism Φ_a of $\mathfrak{n}$ by the formula

$$\Phi_a = e^{(\log a)D}.$$

Writing $x = \exp X$ we put

$$\Phi_a(x) := \exp \Phi_a(X).$$

It is clear that $\Phi_a : N \to N$ defines an automorphism. Let $\mathfrak{n}^{\mathbb{C}}$ be the complexification of $\mathfrak{n}$. Define

$$\mathfrak{n}^{\mathbb{C}}_\lambda = \{X \in \mathfrak{n}^{\mathbb{C}} : \exists k > 0 \text{ such that } (D - \lambda I)^k = 0\}.$$

Then

$$\mathfrak{n} = \bigoplus_{\operatorname{Im}\lambda \geq 0} V_\lambda, \tag{2.1}$$

where

$$V_\lambda = \begin{cases} V_{\bar\lambda} = (\mathfrak{n}^{\mathbb{C}} \oplus \mathfrak{n}^{\mathbb{C}}_{\bar\lambda}) \cap \mathfrak{n} & \text{if } \operatorname{Im}\lambda \neq 0, \\ \mathfrak{n}^{\mathbb{C}}_\lambda \cap \mathfrak{n} & \text{if } \operatorname{Im}\lambda = 0. \end{cases}$$

We assume that the real parts d_j of the eigenvalues λ_j of the matrix D are strictly greater than 0. We define the number

$$Q = \sum_j \operatorname{Re}\lambda_j = \sum_j d_j \tag{2.2}$$

and we refer to this as a *"homogeneous dimension"* of N. In this paper $D = \mathrm{ad}_{Y_0}$ (see Introduction). Under the assumption on positivity of d_j, (2.1) is a gradation of $\mathfrak{n}$.

We consider a group S which is a *semi-direct* product of N and the multiplicative group $A = \mathbb{R}^+ = \{\exp tY_0 : t \in \mathbb{R}\}$:

$$S = NA = \{xa : x \in N, a \in A\}$$

with multiplication given by the formula

$$(xa)(yb) = (x\Phi_a(y)\ ab).$$

On N we define a *"homogeneous norm"*, $|\cdot|$ (cf. [6, 5]) as follows. Let $(\cdot,\cdot)$ be a fixed inner product in $\mathfrak{n}$. We define a new inner product

$$\langle X, Y\rangle = \int_0^1 \Big(\Phi_a(X), \Phi_a(Y)\Big)\frac{da}{a} \tag{2.3}$$

and the corresponding norm

$$\|X\| = \langle X, X\rangle^{1/2}.$$

We put

$$|X| = \left(\inf\{a > 0 : \|\Phi_a(X)\| \geq 1\}\right)^{-1}.$$

One can easily show that for every $Y \neq 0$ there exists precisely one $a > 0$ such that $Y = \Phi_a(X)$ with $|X| = 1$. Then we have $|Y| = a$. Finally, we define the homogeneous norm on N. For $x = \exp X$ we put

$$|x| = |X|.$$

Note that if the action of $A = \mathbb{R}^+$ on N (given by Φ_a) is diagonal the norm we have just defined is the usual homogeneous norm on N and the number Q in (2.2) is simply the homogeneous dimension of N (see [7]).

Having all that in mind we define appropriate derivatives (see also [6]). We fix an inner product (2.3) in $\mathfrak{n}$ so that V_{λ_j}, $j = 1,\dots,k$ are mutually orthogonal and an orthonormal basis $\mathcal{X}_1,\dots,\mathcal{X}_n$ of $\mathfrak{n}$. The enveloping algebra $\mathfrak{U}(\mathfrak{n})$ of $\mathfrak{n}$ is identified with the polynomials in $\mathcal{X}_1,\dots,\mathcal{X}_n$. In $\mathfrak{U}(\mathfrak{n})$ we define $\langle \mathcal{X}_1\otimes\dots\otimes\mathcal{X}_r, \mathcal{Y}_1\otimes\dots\otimes\mathcal{Y}_r\rangle = \prod_{j=1}^r\langle\mathcal{X}_j, \mathcal{Y}_j\rangle$. Let V_j^r be the symmetric tensor product of r copies of V_{λ_j}. For $I = (i_1,\dots,i_k) \in (\mathbb{N}\cup\{0\})^k$ let

$$\mathcal{X}^I = \mathcal{X}_1^{(i_1)}\dots\mathcal{X}_k^{(i_k)}, \quad \text{where } \mathcal{X}_j^{(i_j)} \in V_j^{i_j}.$$

Then for $\mathcal{X} \in V_{\lambda_j}$,

$$\|\Phi_a(\mathcal{X})\| \leq c\exp(d_j\log a + D_j\log(1 + |\log a|)),$$

where $d_j = \mathrm{Re}\lambda_j$ and $D_j = \dim V_{\lambda_j} - 1$, and so

$$\|\Phi_a(\mathcal{X}^I)\| \leq \exp\Big(\sum_{j=1}^k i_j(d_j\log a + D_j\log(1 + |\log a|))\Big)\prod_{j=1}^k\|\mathcal{X}_j^{(i_j)}\|. \tag{2.4}$$

2.2. **Bessel process.** Let b_t denote the Bessel process with a parameter $\alpha \geq 0$ (cf. [11]), i.e., a continuous Markov process with the state space $[0, +\infty)$ generated by $\partial_a^2 + \frac{2\alpha+1}{a}\partial_a$. The transition function with respect to the measure $y^{2\alpha+1}dy$ is given (cf. [2, 11]) by:

$$p_t(x,y) = \begin{cases} \frac{1}{2t}\exp\left(\frac{-x^2-y^2}{4t}\right) I_\alpha\left(\frac{xy}{2t}\right)\frac{1}{(xy)^\alpha} & \text{for } x, y > 0, \\ \frac{1}{2^\alpha(2t)^{\alpha+1}\Gamma(\alpha+1)}\exp\left(\frac{-y^2}{4t}\right) & \text{for } x = 0,\ y > 0, \end{cases} \tag{2.5}$$

where

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)} \tag{2.6}$$

is *the Bessel function* (see [10]). Therefore for $x \geq 0$ and a measurable set $B \subset (0, \infty)$:

$$\mathbf{P}_x(b_t \in B) = \int_B p_t(x,y)y^{2\alpha+1}dy.$$

If b_t is the Bessel process with a parameter α starting from x, i.e., $b_0 = x$, then we will write that $b_t \in \mathrm{BESS}_x(\alpha)$ or simply $b_t \in \mathrm{BESS}(\alpha)$ if the starting point is not important or is clear from the context.

Properties of the Bessel process are very well known and their proofs are rather standard. They can be found e.g. in [11, 5, 14, 13]. However, in our paper we will not explicitly make use of any particular property of the Bessel process. What we only need is the possibility of generalization of some lemmas from Section 5 in [17].

2.3. **Evolutions.** Let $X, X_1, \dots, X_m$ be as in (1.1). Let $\sigma : [0,\infty) \longrightarrow [0,\infty)$ be a continuous function such that $\sigma(t) > 0$ for every $t > 0$. We consider the family of evolutions operators $L_{\sigma(t)} - \partial_t$, where

$$L_{\sigma(t)} = \sigma(t)^{-2}\Big(\sum_j \Phi_{\sigma(t)}(X_j)^2 + \Phi_{\sigma(t)}(X)\Big). \tag{2.7}$$

Since we can assume that $X_1, \dots, X_m$ are linearly independent, we can select $X_{m+1}, \dots, X_n$ so that $X_1, \dots, X_n$ form a basis of $\mathfrak{n}$. For a multi-index $I = (i_1, \dots, i_n)$, $i_j \in \mathbb{Z}^+$ and the basis $X_1, \dots, X_n$ of the Lie algebra $\mathfrak{n}$ of N we write: $X^I = X_1^{i_1} \dots X_n^{i_n}$ and $|I| = i_1 + \dots + i_n$. For $k = 0, 1, \dots, \infty$ we define:

$$C^k = \{f : X^I f \in C(N),\ \text{for } |I| < k+1\}$$

and

$$C^k_\infty = \{f \in C^k : \lim_{x\to\infty} X^I f(x) \text{ exists for } |I| < k+1\}.$$

For $k < \infty$ the space C^k_∞ is a Banach space with the norm

$$\|f\|_{C^k_\infty} = \sum_{|I|\leq k} \|X^I f\|_{C(N)}.$$

Let $\{U^\sigma(s,t) : 0 \leq s \leq t\}$ be the unique family of bounded operators on $C_\infty = C^0_\infty$ which satisfy

(i) $U^\sigma(s,s) = I$,
(ii) $U^\sigma(s,r)U^\sigma(r,t) = U^\sigma(s,t)$, $s < r < t$,
(iii) $\partial_s U^\sigma(s,t)f = -L_{\sigma(s)}U^\sigma(s,t)f$ for every $f \in C_\infty$,
(iv) $\partial_t U^\sigma(s,t)f = U^\sigma(s,t)L_{\sigma(t)}f$ for every $f \in C_\infty$,
(v) $U^\sigma(s,t) : C^2_\infty \longrightarrow C^2_\infty$.

The operator $U^\sigma(s,t)$ is a convolution operator. Namely, $U^\sigma(s,t)f = f * p^\sigma(t,s)$, where $p^\sigma(t,s)$ is a smooth density of a probability measure. By ii) we have $p^\sigma(t,r)* p^\sigma(r,s) = p^\sigma(t,s)$ for $t > r > s$. The existence of the family $U^\sigma(s,t)$ follows from [12].

3. Probabilistic Lemma

For the rest of this article, as in [17], we consider a Bessel process σ_t with a parameter 0, i.e., $\sigma_t \in \mathrm{BESS}(0)$. In other words $\sigma_t = \|w_t\|$, where w_t is a standard Brownian motion on $\mathbb{R}^2$. In the whole section we use the following notation. For every $\eta > 0$ and $a > 0$, $I_{a,\eta} = [a-\eta, a+\eta]$. The kernel p^σ is the evolution kernel defined in the previous section and "directed" by the Bessel process $\sigma_t \in \mathrm{BESS}(0)$ and is fixed for the whole section. Moreover, for every measurable subset B of $\mathbb{R}^+$, $m(B) = \int_B y dy$. For a positive $\delta < 1/2$, let

$$\begin{aligned} T_\delta &= \{(x,a) \in N \times \mathbb{R}^+ : 1-\delta < a < 1+\delta, |x| < \delta\}, \\ \mathcal{Q} &= \{(x,a) \in N \times \mathbb{R}^+ : |x| \leq 1, a \leq 1\}. \end{aligned}$$

The main probabilistic fact is the following proposition which gives estimates on the set $\mathcal{Q} \setminus T_\delta$.

Proposition 3.1. *Let $k \geq 1$ be fixed. For every $1 > \delta > 1/2$, for every $0 < \chi_0 \leq 1$, $0 < r_0 \leq 1$ and for every multi-index I such that $|I| > 0$ there exists a constant C such that for every $(x,a) \in \mathcal{Q} \setminus T_\delta$ and for every $0 \leq l \leq k-1$,*

$$\sup_{0<\eta<\delta/2} \Big| \int_1^\infty \mathbf{E}_\chi X^I p^\sigma(t,0)(x) \partial_a^l m(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma_t) dt \Big| \leq C a^{-k}.$$

Sketch of the proof. To prove Proposition 3.1, we divide the set $\mathcal{Q} \setminus T_\delta$ into five subsets as in Lemmas 5.1-5.5 in [17] (in the case $\chi = 1$). The case $0 < \chi \leq \chi_0$ is done similarly. What we have to notice is the fact that the only difference between the above proposition and corresponding Lemmas 5.1-5.5 in [17] is the appearance of derivatives with respect to a variable and in some sense uniform estimate with respect to the starting point $\chi \leq \chi_0$. To overcame this difficulty we notice the following.

For every $\eta < a$ we have

$$\partial_a^l m(I_{a,\eta})^{-1} = (-1)^l l! m(I_{a,\eta})^{-(l+1)} C^l, \quad l \geq 0 \tag{3.1}$$

and for every $\chi > 0$,

$$\begin{aligned} \mathbf{E}_\chi \partial_a 1_{I_{a,\eta}}(\sigma_t) = &\lim_{h\to 0} h^{-1}(\mathbf{P}_\chi(a+\eta \leq \sigma_t \leq a+\eta+h) \\ &- \mathbf{P}_\chi(a-\eta+h \leq \sigma_t \leq a-\eta)) \\ =& p_t(\chi, a+\eta)(a+\eta) - p_t(\chi, a-\eta)(a-\eta), \end{aligned} \tag{3.2}$$

where p_t is the transition function (2.5). Formula (3.2) together with (2.5) allow us to calculate $\mathbf{E}_\chi \partial_a^l 1_{I_{a,\eta}}(\sigma_t)$ for $l \geq 2$,

$$\begin{aligned} \mathbf{E}_\chi \partial_a^l 1_{I_{a,\eta}}(\sigma_t) =& (2t)^{-1} e^{-\chi^2/4t} \partial_a^{l-1} (e^{-(a+\eta)^2/4t} I_0(\chi(a+\eta)/2t)(a+\eta) \\ &- e^{-(a-\eta)^2/4t} I_1(\chi(a-\eta)/2t)(a-\eta)). \end{aligned} \tag{3.3}$$

Since we get $\lim_{\eta\to 0}\frac{(C\eta)^l}{m(I_{a,\eta})^l} = \partial_a^l m(I_{a,\eta})$. Using (3.2), (3.3) and the following formulae (cf. [10]):

$$\frac{d}{dx}I_\alpha(x) = I_{\alpha-1}(x) - \frac{\alpha}{x}I_\alpha(x) = I_{\alpha+1}(x) + \frac{\alpha}{x}I_\alpha(x)$$

we get, after not difficult but a little tedious computation, that

$$\begin{aligned}\lim_{\eta\to 0}|\mathbf{E}_\chi m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(b_{t/2})| &\le Ct^{-k+l-2}a^{-k+l}e^{-\chi^2/4t}e^{-a^2/4t}\\ &\times \sum_{(w_1,w_2,w_3,w_4)\in W} c_{w_1,w_2,w_3,w_4}\chi^{w_1}a^{w_2}t^{w_3}I_{w_4}(\chi a/2t), \end{aligned}\tag{3.4}$$

where

$$\begin{aligned}W = \big\{(w_1,w_2,w_3,w_4) : 0\le w_1\le k-l+1, 0\le w_2\le 2(k-l)+1,\\ 0\le w_3\le k-l, w_4\in\{0,1\} \text{ and } w_1/2+w_3<k-l+1\big\}\end{aligned}$$

and c_{w_1,w_2,w_3,w_4} are nonnegative real numbers. Assume now that $a\le 1$. Then, by $w_1/2+w_3<k-l+1$, using asymptotic behavior of the Bessel function (2.6), (see [10]):

$$I_\alpha(x) \asymp \begin{cases}\frac{x^\alpha}{2^\alpha\Gamma(1+\alpha)}, & x\to 0,\\ \frac{\exp(x)}{(2\pi x)^{1/2}}, & x\to\infty,\end{cases}$$

we can estimate (3.4), independently of χ, as follows:

$$\begin{aligned}&\lim_{\eta\to 0}|\mathbf{E}_\chi m(I_{a,\eta})^{-1}\partial_a^{k-l}1_{I_{a,\eta}}(b_{t/2})|\\ &\le Ct^{-k+l-2}a^{-k+l}\sum_{(w_1,w_2,w_3,w_4)\in A} c_{w_1,w_2,w_3,w_4}a^{w_2}t^{w_1/2+w_3} \le Ct^{-1}a^{-k+l}.\end{aligned}\tag{3.5}$$

Having the above we proceed exactly like in the paper [17]. $\square$

4. The main result and its proof

In this section we obtain pointwise estimates for derivatives of the Green function (1.2). Recall that for a positive $\delta<1/2$ we have defined,

$$\begin{aligned}T_\delta &= \{(x,a)\in N\times\mathbb{R}^+ : 1-\delta<a<1+\delta, |x|<\delta\},\\ \mathcal{Q} &= \{(x,a)\in N\times\mathbb{R}^+ : |x|\le 1, a\le 1\}.\end{aligned}$$

Theorem 4.1. *For a multi-index $I=(i_1,\dots,i_k)$, for every integer $l\ge 0$ and all operators $\mathcal{X}^I = \mathcal{X}_1^{(i_1)}\dots\mathcal{X}_k^{(i_k)}$, where $\mathcal{X}_j^{(i_j)}\in V_j^{i_j}$, with $\|\mathcal{X}^I\|\le 1$, there are constants C such that*

$$|\partial_a^l\mathcal{X}^I\mathcal{G}(x,a)| \le \begin{cases} C(|x|+a)^{-\|I\|-Q}a^{-l} & \\ \times(1+|\log(|x|+a)^{-1}|)^{\|I\|_0}, & (x,a)\in(\mathcal{Q}\cup T_\delta)^c,\\ Ca^{-l}, & (x,a)\in\mathcal{Q}\setminus T_\delta,\end{cases}$$

where $\|I\| = \sum_{j=1}^k i_jd_j$, $d_j = Re\lambda_j$, and $\|I\|_0 = \sum_{j=1}^k i_jD_j$, $D_j = dimV_{\lambda_j}-1$.

Proof. Having all the preparatory material from Sect. 3 we proceed exactly as in the proof of Theorem 6.1 in [17]. $\square$

Acknowledgements. The author wishes to thank Ewa Damek for helpful discussions concerning the content of this paper. The author also thank the referee for his/her suggestions that improved the presentation of this paper.

References

[1] A. Ancona. Negatively curved manifolds, elliptic operators, and the Martin boundary. *Ann. of Math.*, 125:495–536, 1987.

[2] A.N. Borodin and P. Salminen. *Handbook of Brownian motion - facts and formulae*. Birkhäuser Verlag, 1996.

[3] C. Constantinescu and A. Cornea. *Potential theory on harmonic spaces*. Springer-Verlag, New York, 1972. With a preface by H. Bauer, Die Grundlehren der mathematischen Wissenschaften, Band 158.

[4] E. Damek. Fundamental solutions of differential operators on homogeneous manifolds of negative curvature and related Riesz transforms. *Coll. Math*, 73(2):229–249, 1997.

[5] E. Damek, A. Hulanicki, and R. Urban. Martin boundary for homogeneous Riemannian manifolds of negative curvature at the bottom of the spectrum. *Rev. Mat. Iberoamericana*, 17(2):257–293, 2001.

[6] E. Damek, A. Hulanicki, and J. Zienkiewicz. Estimates for the Poisson kernels and their derivatives on rank one NA groups. *Studia Math.*, 126(2):115–148, 1997.

[7] G.B. Folland and E. Stein. *Hardy spaces on homogeneous groups*. Princeton Univ. Press, 1982.

[8] E. Heintze. On homogeneous manifolds of negative curvature. *Math. Ann.*, 211:23–34, 1974.

[9] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*. Academic Press, 1980.

[10] N.N. Lebedev. *Special functions and their applications*. Dover Publications, Inc., 1972.

[11] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Springer-Verlag, 1991.

[12] H. Tanabe. *Equations of evolutions*. London, Pitman, 1979.

[13] R. Urban. *Estimates for the Poisson kernels on NA groups. A probabilistic method*. PhD thesis, Wroclaw University, 1999.

[14] R. Urban. Estimates for the Poisson kernels on homogeneous manifolds of negative curvature and the boundary Harnack inequality in the noncoercive case. *Probab. Math. Statist.*, 21(1, Acta Univ. Wratislav. No. 2298):213–229, 2001.

[15] R. Urban. Noncoercive differential operators on homogeneous manifolds of negative curvature and their Green functions. *Colloq. Math.*, 88(1):121–134, 2001.

[16] R. Urban. Estimates for derivatives of the Poisson kernels on homogeneous manifolds of negative curvature. *Math. Z.*, 240(4):745–765, 2002.

[17] R. Urban. Estimates for derivatives of the Green functions for the noncoercive differential operators on homogeneous manifolds of negative curvature. *Potential Analysis*, 19(4):317–339, 2003.

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