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# POSITIVE SOLUTIONS OF A THREE-POINT BOUNDARY-VALUE PROBLEM ON A TIME SCALE

ERIC R. KAUFMANN

Abstract. Let $\mathbb{T}$ be a time scale such that $0, T \in \mathbb{T}$. We consider the second order dynamic equation on a time scale

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in (0,T) \cap \mathbb{T},$$
$$u(0) = 0, \quad \alpha u(\eta) = u(T),$$

where $\eta \in (0, \rho(T)) \cap \mathbb{T}$, and $0 < \alpha < T/\eta$. We apply a cone theoretic fixed point theorem to show the existence of positive solutions.

## 1. Introduction

The theory of time scales and measure chains was initiated by Stefan Hilger [6] as a means of unifying and extending theories from differential and difference equations. We begin by presenting some basic definitions which can be found in Atici and Guseinov [3] and Bohner and Peterson [4]. Another excellent source on dynamical systems on measure chains is the book [9].

A *time scale* $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, we define the *forward jump operator*, $\sigma$, and the *backward jump operator*, $\rho$, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T},$$
$$\rho(r) = \sup\{\tau \in \mathbb{T} \mid \tau < r\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$. If $\sigma(t) > t$, $t$ is said to be *right scattered*, and if $\sigma(t) = t$, $t$ is said to be *right dense* (rd). If $\rho(t) < t$, $t$ is said to be *left scattered*, and if $\rho(t) = t$, $t$ is said to be *left dense* (ld). A function $f$ is left-dense continuous, ld-continuous, $f$ is continuous at each left dense point in $\mathbb{T}$ and its right-sided limits exist at each right dense points in $\mathbb{T}$.

For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (assume $t$ is not left scattered if $t = \sup \mathbb{T}$), we define the *delta derivative* of $x(t)$, $x^\Delta(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, $U$, of $t$ such that

$$\big|x(\sigma(t)) - x(s) - x^\Delta(t)(\sigma(t) - s)\big| \le \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

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For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (assume $t$ is not right scattered if $t = \inf \mathbb{T}$), we define the *nabla derivative* of $x(t)$, $x^{\nabla}(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, $U$, of $t$ such that

$$\left|x(\rho(t)) - x(s) - x^{\nabla}(t)(\rho(t) - s)\right| \leq \varepsilon|\rho(t) - s|,$$

for all $s \in U$.

If $\mathbb{T} = \mathbb{R}$ then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$. If $\mathbb{T} = \mathbb{Z}$ then $f^{\Delta}(t) = f(t+1) - f(t)$ is the forward difference operator while $f^{\nabla}(t) = f(t) - f(t-1)$ is the backward difference operator.

In 1998, Ma [15] showed the existence of a positive solution to the second order three-point boundary-value problem

$$\begin{gathered} u'' + a(t)f(u) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad \alpha u(\eta) = u(1), \end{gathered}$$

where $0 < \eta < 1, 0 < \alpha < 1/\eta$ and $f$ was either superlinear or sublinear. Later Cao and Ma [5] extended these results to the $m$-point eigenvalue problem $u'' + \lambda a(t)f(u, u') = 0, u(0) = 0, \sum_{i=1}^{m-2} a_i u(\xi_i) = u(1)$. Ma and Raffoul [18] showed the existence of positive solutions for a three-point boundary-value problems for difference equation. Recently, Anderson [2] showed the existence of at least one positive solution (using the Krasnosel'skiĭ fixed point theorem) and the existence of at least three positive solutions (using the Leggett-Williams fixed point theorem) for the three-point boundary-value problem on a time scale. For other references on multi-point boundary-value problems we refer the reader to the papers [16, 17, 19] and references therein.

Many authors have studied the existence of multiple positive solutions for boundary value problems for differential and difference equations, see [7, 8, 10, 11, 13] and references therein. The book [1] is an excellent source for information on the theory of positive solutions for differential, difference and integral equations. One of the first papers to consider countably many positive solutions for boundary-value problems on a time scale is [12].

In this paper, we show the existence of multiple positive solutions for a second order three-point boundary-value problem on a time scale. Let $\mathbb{T}$ be a time scale such that $0, T \in \mathbb{T}$ and denote the set of all left-dense continuous functions from $\mathbb{T}$ to $E \subseteq \mathbb{R}$ by $C_{ld}(\mathbb{T}, E)$. Consider the second order dynamic equation

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in (0,T) \cap \mathbb{T}, \tag{1.1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(T), \tag{1.2}$$

where $\eta \in (0, \rho(T)) \cap \mathbb{T}$, and $0 < \alpha < T/\eta$. We will assume throughout that $f : \mathbb{T} \to [0, +\infty)$ is continuous. We will also assume that $a \in C_{ld}(\mathbb{T}, [0, +\infty))$ and there exists at least one $t_0 \in [\eta, T) \cap \mathbb{T}$ such that $a(t_0) > 0$.

Define

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} \quad \text{and} \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Note that $f_0 = 0$ and $f_\infty = \infty$ correspond to $f$ being superlinear and that $f_0 = \infty$ and $f_\infty = 0$ correspond to $f$ being sublinear. We show the existence of two positive solutions for the boundary-value problem (1.1), (1.2) when $f_0 = 0$ and $f_\infty = 0$ and when $f_0 = \infty$ and $f_\infty = \infty$.

In section 2 we state some lemmas that will be needed in order to prove our main theorems. We also define an operator whose fixed points are solutions to (1.1), (1.2) and state a fixed point theorem due to Krasnosel'skiĭ, see [14]. In section 3 we state and prove two theorems for the existence of two positive solutions of (1.1), (1.2). We begin section 4 with a modification of Lemma 2.5 in section 2. This new lemma will allow us to define a sequence of cones in which we will find fixed points of our operator.

## 2. Preliminaries

We will need the following lemmas, whose proofs can be found in Anderson [2], in order to prove our main theorems. Consider the linear boundary-value problem

$$u^{\Delta\nabla}(t) + y(t) = 0, \quad t \in (0,T) \cap \mathbb{T}, \tag{2.1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(T). \tag{2.2}$$

**Lemma 2.1.** *If $\alpha\eta \neq T$ then for $y \in C_{ld}(\mathbb{T},\mathbb{R})$ the boundary-value problem (2.1), (2.2) has the unique solution*

$$u(t) = -\int_0^t (t-s)y(s)\,\nabla s - \frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta-s)y(s)\,\nabla s + \frac{t}{T-\alpha\eta}\int_0^T (T-s)y(s)\,\nabla s.$$

**Lemma 2.2.** *If $u(0) = 0$ and $u^{\Delta\nabla} \leq 0$, then $\frac{u(s)}{s} \leq \frac{u(t)}{t}$ for all $s,t \in (0,T] \cap \mathbb{T}$ with $t \leq s$.*

**Lemma 2.3.** *Let $0 < \alpha < T/\eta$. If $y \in C_{ld}(\mathbb{T},\mathbb{R})$ and $y \geq 0$ then the solution $u$ of boundary-value problem (2.1), (2.2) satisfies $u(t) \geq 0$ for all $t \in [0,T] \cap \mathbb{T}$.*

**Lemma 2.4.** *Let $\alpha\eta > T$. If $y \in C_{ld}(\mathbb{T},\mathbb{R})$ and $y \geq 0$ then the boundary-value problem (2.1), (2.2) has no nonnegative solution.*

In view of Lemma 2.4, we will assume that $\alpha\eta < T$ for the rest of the paper.

Our Banach space is $\mathcal{B} = C_{ld}(\mathbb{T},\mathbb{R})$ with norm $\|u\| = \sup_{t\in[0,T]\cap\mathbb{T}} |u(t)|$. Define the operator $A\colon \mathcal{B} \to \mathcal{B}$ by

$$\begin{aligned} Au(t) = &-\int_0^t (t-s)a(s)f(u(s))\,\nabla s - \frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta-s)a(s)f(u(s))\,\nabla s \\ &+ \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\,\nabla s. \end{aligned}$$

The function $u$ is a solution of the boundary-value problem (1.1), (1.2) if and only if $u$ is a fixed point of the operator $A$.

**Lemma 2.5.** *Let $0 < \alpha\eta < T$. If $y \in C_{ld}(\mathbb{T},[0,\infty))$, then the unique solution $u$ of (2.1), (2.2) satisfies*

$$\min_{t\in[\eta,T]\cap\mathbb{T}} u(t) \geq \gamma\|u\| \tag{2.3}$$

*where*

$$\gamma = \min\Big\{\frac{\alpha\eta}{T}, \frac{\alpha(T-\eta)}{T-\alpha\eta}, \frac{\eta}{T}\Big\}.$$

**Remark:** Since $\alpha\eta < T$ and since $\eta < T$, it follows that $0 < \gamma < 1$.

**Definition 2.6.** Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be closed and nonempty. Then $\mathcal{P}$ is said to be a cone if

(1) $\alpha u + \beta v \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and for all $\alpha, \beta \geq 0$, and
(2) $u, -u \in \mathcal{P}$ implies $u = 0$.

As in [2] we define the cone

$$\mathcal{P} = \{u \in \mathcal{B} \colon u(t) \geq 0, t \in \mathbb{T} \text{ and } \min_{t \in [\eta, T] \cap \mathbb{T}} u(t) \geq \gamma \|u\|\}.$$

From Lemma 2.5 we have $A \colon \mathcal{P} \to \mathcal{P}$. Standard arguments show that the operator $A$ is completely continuous.

Before we state the fixed point theorem, we establish some inequalities. Since both $a$ and $f$ are nonnegative then for all $u \in \mathcal{B}$

$$Au(t) \leq \frac{t}{T - \alpha\eta} \int_0^T (T - s) a(s) f(u(s)) \nabla s. \tag{2.4}$$

Furthermore,

$$\begin{aligned}
Au(\eta) &= -\int_0^\eta (\eta - s) a(s) f(u(s)) \nabla s - \frac{\alpha\eta}{T - \alpha\eta} \int_0^\eta (\eta - s) a(s) f(u(s)) \nabla s \\
&\quad + \frac{\eta}{T - \alpha\eta} \int_0^T (T - s) a(s) f(u(s)) \nabla s \\
&= -\frac{T}{T - \alpha\eta} \int_0^\eta (\eta - s) a(s) f(u(s)) \nabla s + \frac{\eta}{T - \alpha\eta} \int_0^T (T - s) a(s) f(u(s)) \nabla s \\
&= \frac{\eta T}{T - \alpha\eta} \int_\eta^T a(s) f(u(s)) \nabla s + \frac{T}{T - \alpha\eta} \int_0^\eta s a(s) f(u(s)) \nabla s \\
&\quad - \frac{\eta}{T - \alpha\eta} \int_0^T s a(s) f(u(s)) \nabla s \\
&\geq \frac{\eta}{T - \alpha\eta} \int_\eta^T (T - s) a(s) f(u(s)) \nabla s.
\end{aligned}$$

That is,

$$Au(\eta) \geq \frac{\eta}{T - \alpha\eta} \int_\eta^T (T - s) a(s) f(u(s)) \nabla s. \tag{2.5}$$

The inequalities (2.4) and (2.5), which are also found in [2] and [19], will play critical roles in the proofs of our main theorems. We will also need the following fixed point theorem found in [14]

**Theorem 2.7** (Krasnosel'skiĭ)**.** *Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Assume $\Omega_1$, $\Omega_2$ are bounded open balls of $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that*

$$A \colon \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

*is a completely continuous operator such that, either*

(1) $\|Au\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, *or*
(2) $\|Au\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

*Then $A$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

## 3. Two Positive Solutions

In this section we use Theorem 2.7 to establish the existence of two positive solutions of the boundary-value problem (1.1), (1.2). In Theorems 3.1 and 3.2, the inequalities we derive that are based on $f_0$ and $f_\infty$ are similar to those found in [2], [15], and [19] and are included for completeness.

**Theorem 3.1.** *Assume that $f$ satisfies conditions*

(A1) $f_0 = +\infty$, $f_\infty = +\infty$; *and*
(B1) *there exists a* $p > 0$ *such that if* $0 \le x \le p$ *then* $f(x) \le \mu p$ *where* $\mu = \Big(\frac{T}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s\Big)^{-1}$.

*Then the boundary-value problem (1.1), (1.2) has at least two positive solutions* $u_1, u_2 \in \mathcal{P}$ *such that*

$$0 < \|u_1\| \le p \le \|u_2\|.$$

*Proof.* Choose $m > 0$ such that

$$\frac{m\eta\gamma}{T-\alpha\eta}\int_\eta^T (T-s)a(s)\,\nabla s \ge 1. \tag{3.1}$$

By condition (A1), ($f_0 = +\infty$), there exists an $0 < r < p$ such that

$$f(u) \ge mu \tag{3.2}$$

for all $0 \le u \le r$.

Let $u \in \mathcal{P}$ with $\|u\| = r$. From (2.5), (2.3), (3.1) and (3.2) we have

$$\begin{aligned}
Au(\eta) &\ge \frac{\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)f(u(s))\,\nabla s\\
&\ge \frac{m\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)u(s)\,\nabla s\\
&\ge \Big(\frac{m\eta\gamma}{T-\alpha\eta}\int_\eta^T (T-s)a(s)\,\nabla s\Big)\|u\|\\
&\ge \|u\|.
\end{aligned}$$

Define $\Omega_1 = \{u \in \mathcal{B} \colon \|u\| < r\}$. From the above string of inequalities we have

$$\|Au\| \ge \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.3}$$

Now consider $u \in \mathcal{P}$ with $\|u\| \le p$. From (2.4) and condition (B1) we have

$$\begin{aligned}
Au(t) &\le \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\,\nabla s\\
&\le \frac{T}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\,\nabla s\\
&\le \Big(\frac{T}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s\Big)\mu p = p.
\end{aligned}$$

Define $\Omega_2 = \{u \in \mathcal{B} \colon \|u\| < p\}$. Then

$$\|Au\| \le \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.4}$$

Using the inequalities (3.3) and (3.4) there exists, by Theorem 2.7, a fixed point $u_1$ of $A$ in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$. This fixed point satisfies $r \le \|u_1\| \le p$.

Using condition (A1) again, ($f_\infty = \infty$), we know there exists an $R_1 > p$ such that

$$f(u) \ge Mu \tag{3.5}$$

for all $u \ge R_1$ where $M$ is chosen so that

$$\frac{M\eta\gamma}{T-\alpha\eta}\int_\eta^T (T-s)a(s)\,\nabla s \ge 1. \tag{3.6}$$

Set $R = R_1/\gamma$ and pick $u \in \mathcal{P}$ so that $\|u\| = R$. Since $0 < \gamma < 1$ then $R > R_1 > p$. Furthermore, $\min_{t\in[\eta,T]\cap\mathbb{T}} u(t) \geq \gamma R \geq R_1$. From (2.5), (2.3), (3.5), and (3.6) we have

$$\begin{aligned} Au(\eta) &\geq \frac{\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)f(u(s))\,\nabla s \\ &\geq \frac{M\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)u(s)\,\nabla s \\ &\geq \Big(\frac{M\eta\gamma}{T-\alpha\eta}\int_\eta^T (T-s)a(s)\,\nabla s\Big)\|u\| \\ &\geq \|u\|. \end{aligned}$$

Define $\Omega_3 = \{u \in \mathcal{B} \colon \|u\| < R\}$. Then

$$\|Au\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_3. \tag{3.7}$$

Theorem 2.7 together with (3.4) and (3.7) implies that there exists a fixed point $u_2$ of $A$ that satisfies $p \leq \|u_2\| \leq R$ and the proof is complete. □

**Theorem 3.2.** *Assume that $f$ satisfies conditions*

(A2) $f_0 = 0$, $f_\infty = 0$*; and*

(B2) *there exists a $q > 0$ such that if $\gamma q \leq x \leq q$ then $f(x) \geq \nu q$ where* $\nu = \Big(\frac{\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)\,\nabla s\Big)^{-1}$.

*Then the boundary-value problem (1.1), (1.2) has at least two positive solutions $u_1, u_2 \in \mathcal{P}$ such that*

$$0 < \|u_1\| \leq q \leq \|u_2\|.$$

*Proof.* Choose $m > 0$ such that

$$\frac{Tm}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s \leq 1. \tag{3.8}$$

By condition (A2), ($f_0 = 0$), there exists an $0 < r < q$ such that

$$f(u) \leq mu \tag{3.9}$$

for all $0 \leq u \leq r$. Define $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r\}$ and let $u \in \mathcal{P} \cap \partial\Omega_1$. Then,

$$\begin{aligned} Au(t) &\leq \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\,\nabla s \\ &\leq \frac{Tm}{T-\alpha\eta}\int_0^T (T-s)a(s)u(s)\,\nabla s \\ &\leq \Big(\frac{Tm}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s\Big)\|u\| \\ &\leq \|u\|. \end{aligned}$$

And so,

$$\|Au\| \leq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.10}$$

Now define $\Omega_2 = \{u \in \mathcal{B} : \|u\| < q\}$. Notice that if $u \in \mathcal{P} \cap \partial\Omega_2$ then

$$\min_{t\in[\eta,T]\cap\mathbb{T}} u(t) \geq \gamma\|u\| \geq \gamma q.$$

By condition (B2) we have

$$\begin{aligned}Au(t) &\geq \frac{\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)f(u(s))\,\nabla s\\ &\geq \Big(\frac{\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)\,\nabla s\Big)\nu q\\ &= q \;=\; \|u\|.\end{aligned}$$

Hence,

$$\|Au\| \geq \|u\|, \quad u\in\mathcal{P}\cap\partial\Omega_2. \tag{3.11}$$

Consider the second condition in (A2), $f_\infty = 0$. There exists an $R_1 > q$ such that $f(u) \leq Mu$ for all $u \geq R_1$ where $M$ is chosen so that

$$\frac{TM}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s \leq 1.$$

There are two cases to consider: $f$ is bounded or $f$ is unbounded.

Suppose that $f$ is bounded. Let $K$ be such that $f(u) \leq K$ for all $u$ and choose

$$R = \max\Big\{2q, \frac{TK}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s\Big\}.$$

Let $u\in\mathcal{P}$ be such that $\|u\| = R$. Then

$$\begin{aligned}Au(t) &\leq \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\,\nabla s\\ &\leq \frac{TK}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s\\ &\leq R.\end{aligned}$$

Hence $\|Au\| \leq \|u\|$.

Now suppose that $f$ is unbounded. From condition (A2) there exists an $R \geq \frac{R_1}{\gamma}$ such that $f(u) \leq f(R)$ for all $0 < u \leq R$. Since $\gamma < 1$ then $q < R_1 < \frac{R_1}{\gamma} = R$. Let $u\in\mathcal{P}$ be such that $\|u\| \leq R$. Then

$$\begin{aligned}Au(t) &\leq \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\,\nabla s\\ &\leq \frac{T}{T-\alpha\eta}\int_0^T (T-s)a(s)f(R)\,\nabla s\\ &\leq \frac{TMR}{T-\alpha\eta}\int_0^T (T-s)a(s)u\,\nabla s\\ &\leq R.\end{aligned}$$

Hence $\|Au\| \leq \|u\|$.

In either case, if we define $\Omega_3 = \{\mathcal{B} : \|u\| < R\}$ then $\Omega_2 \subsetneq \Omega_3$ and

$$\|Au\| \geq \|u\|, \quad u\in\mathcal{P}\cap\partial\Omega_3. \tag{3.12}$$

The conclusion of the theorem follows by applying Theorem 2.7 to the inequalities (3.10), (3.11), (3.12). $\square$

## 4. Countably Many Positive Solutions

In this section we show the existence of countably many positive solutions when $f$ satisfies an oscillatory like growth about a wedge. We begin with a modification of Lemma 2.5.

**Lemma 4.1.** *Let $0 < \alpha\eta < T$. Let $\tau_k \in \mathbb{T} \cap (\eta, \rho(T))$. If $y \in C_{ld}(\mathbb{T}, [0, \infty))$, then the unique solution $u$ of (2.1), (2.2) satisfies*

$$\min_{t \in [\tau_k, T] \cap \mathbb{T}} u(t) \geq \gamma_k \|u\|$$

*where*

$$\gamma_k = \min\Big\{\frac{\alpha\eta\tau_k}{T^2}, \frac{\alpha\eta(T - \tau_k)}{\tau_k(T - \alpha\eta)}, \frac{\eta\tau_k}{T^2}\Big\}$$

*Proof.* Suppose that $0 \leq \alpha < 1$. Let $t_0 \in (0, T) \cap \mathbb{T}$ be such that $u(t_0) = \|u\|$. By the second boundary condition in (1.2) we have $u(\eta) \geq u(T)$. There are two cases to consider. Suppose $t_0 \leq \eta < \rho(T)$ then $\min_{t \in [\tau_k, T] \cap \mathbb{T}} u(t) = u(T)$ and

$$\begin{aligned} u(t_0) &\leq u(T) + \frac{u(T) - u(\tau_k)}{T - \tau_k}(0 - T) \\ &= \frac{-\alpha\tau_k u(\eta) + Tu(\tau_k)}{T - \tau_k}. \end{aligned}$$

By Lemma 2.2, we know that $\frac{u(\tau_k)}{\tau_k} \leq \frac{u(\eta)}{\eta}$. Hence

$$\begin{aligned} u(t_0) &\leq \frac{\frac{\tau_k}{\eta} Tu(\eta) - \alpha\tau_k u(\eta)}{T - \tau_k} \\ &\leq \frac{\tau_k(T - \alpha\eta)}{\eta(T - \tau_k)} u(\eta) \\ &\leq \frac{\tau_k(T - \alpha\eta)}{\alpha\eta(T - \tau_k)} u(T). \end{aligned}$$

Consequently, $\min_{t \in [\tau_k, T] \cap \mathbb{T}} u(t) = u(T) \geq \frac{\alpha\eta(T - \tau_k)}{\tau_k(T - \alpha\eta)} \|u\|$.

Now suppose that $\eta \leq t_0 \leq T$. Again we have $\min_{t \in [\tau_k, T] \cap \mathbb{T}} u(t) = u(T)$. By Lemma 2.2 we know $\frac{u(\eta)}{\eta} \geq \frac{u(t_0)}{t_0}$. Hence, $u(\eta) \geq \eta\frac{u(t_0)}{t_0}$ and so, $u(T) = \alpha u(\eta) \geq \alpha\eta\frac{u(t_0)}{t_0} \geq \frac{\alpha\eta}{T} u(t_0)$. Since $\tau_k < T$, then $\min_{t \in [\tau_k, T] \cap \mathbb{T}} u(t) = u(T) \geq \frac{\alpha\eta}{T^2} \|u\|$.

If $1 \leq \alpha < T/\eta$. Then $u(\eta) \leq u(T)$. Let $t_0$ be such that $u(t_0) = \|u\|$. In this case $t_0 \in [\eta, T] \cap \mathbb{T}$ and $\min_{t \in [\eta, T] \cap \mathbb{T}} u(t) = u(\eta)$. From Lemma 2.2 we have $u(\eta) \geq \eta\frac{u(t_0)}{t_0}$. Consequently,

$$\min_{t \in [\eta, T] \cap \mathbb{T}} u(t) = u(\eta) \geq \frac{\eta}{t_0}\|u\| \geq \frac{\eta}{T}\|u\| \geq \frac{\eta\tau_k}{T^2}\|u\|,$$

Since $\tau_k \geq \eta$, then $\min_{t \in [\tau_k, T] \cap \mathbb{T}} u(t) \geq \frac{\eta\tau_k}{T^2}\|u\|$ and the proof is complete. □

In Theorem 4.2 we show the existence of a countably infinite number of solutions. We will require that there exists at least one right dense point $\tau^* \in \mathbb{T} \cap (\eta, \rho(T))$. In addition, we will need a countable collection of cones. For each $k \in \mathbb{N}$ define the cone

$$\mathcal{P}_k = \{u \in \mathcal{B} : u(t) \geq 0, t \in \mathbb{T} \text{ and } \min_{t \in [\tau_k, T] \cap \mathbb{T}} u(t) \geq \gamma_k \|u\|\}.$$

**Theorem 4.2.** *Let $\tau^* \in \mathbb{T}$ be r.d. and suppose that $\tau^* > \eta$. Let $\{\tau_k\} \subset \mathbb{T}$ be such that $\eta < \tau_1 < \rho(T)$ and $\tau_k \downarrow \tau^*$. Let $t_0$ be such that $t_0 \in [\tau_1, T) \cap \mathbb{T}$ and $a(t_0) > 0$. Let $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ be such that*

$$A_{k+1} < \gamma_k B_k < B_k < CB_k < A_k, \quad k \in \mathbb{N}$$

*where*

$$C = \max\Big\{\Big(\frac{\eta}{T-\alpha\eta}\int_{\tau_1}^{T}(T-s)a(s)\,\nabla s\Big)^{-1}, 1\Big\}.$$

*Assume*

(A3) $f(x) \le MA_k$ *for all* $x \in [0, A_k], k \in \mathbb{N}$ *where*

$$M < \Big(\frac{T}{T-\alpha\eta}\int_{0}^{T}(T-s)a(s)\,\nabla s\Big)^{-1};$$

*and*

(B3) $f(x) \ge CB_k$ *for all* $x \in [\gamma_k B_k, B_k]$.

*Then the boundary-value problem (1.1), (1.2) has infinitely many positive solutions $\{u_k\}_{k=1}^{\infty}$ such that $B_k \le \|u_k\| \le A_k$ for all $k \in \mathbb{N}$.*

*Proof.* First note that since $\frac{T}{T-\alpha\eta}\int_0^T(T-s)a(s)\,\nabla s \ge \frac{\eta}{T-\alpha\eta}\int_{\tau_1}^T(T-s)a(s)\,\nabla s$, it follows that $M < C$ (otherwise the theorem is vacuously true).

Fix $k \in \mathbb{N}$. Define $\Omega_{1k} = \{u \in \mathcal{B} : \|u\| < A_k\}$. Let $u \in \mathcal{P}_k \cap \partial\Omega_{1k}$. Then $u(t) \le A_k = \|u\|$ for all $t \in [0, T] \cap \mathbb{T}$. So,

$$\begin{aligned} Au(t) &\le \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\,\nabla s \\ &\le \frac{TM}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s A_k \\ &\le A_k. \end{aligned}$$

Thus $\|Au\| \le A_k = \|u\|$ for $u \in \mathcal{P}_k \cap \partial\Omega_{1k}$.

Now define $\Omega_{2k} = \{u \in \mathcal{B} \colon \|u\| < B_k\}$ and let $u \in \mathcal{P}_k \cap \partial\Omega_{2k}$. Let $t \in [\tau_k, T] \cap \mathbb{T}$. Then

$$B_k = \|u\| \ge u(t) \ge \min_{t\in[\tau_k,T]\cap\mathbb{T}} u(t) =\ge \gamma_k\|u\| = \gamma_k B_k.$$

So,

$$\begin{aligned} Au(\eta) &\ge \frac{\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)f(u(s))\,\nabla s \\ &\ge \Big(\frac{\eta}{T-\alpha\eta}\int_{\tau_1}^T (T-s)a(s)\,\nabla s\Big)CB_k \\ &\ge B_k. \end{aligned}$$

Thus $\|Au\| \ge B_k = \|u\|$ for all $u \in \mathcal{P}_k \cap \partial\Omega_{2k}$.

By Theorem 2.7 there exists a fixed point $u_k$ of $A$ such that $B_k \le \|u_k\| \le A_k$. Since $k$ was arbitrary the result follows and the proof is complete. □

The proofs for our last two theorems require slight modifications of the proofs for Theorems 3.1, 3.2 and 4.2 and as such will be omitted. Theorem 4.3 shows the existence of an odd number of solutions and Theorem 4.4 shows the existence of an even number of solutions. It is not difficult to establish other theorems of these forms stating the existence of multiple positive solutions.

**Theorem 4.3.** *Let* $m \geq 1$ *be a fixed integer. Let* $\{\tau_k\}_{k=1}^m \subset \mathbb{T}$ *be such that* $\eta < \tau_{k+1} < \tau_k < \rho(T)$. *Let* $t_0$ *be such that* $t_0 \in [\tau_1, T) \cap \mathbb{T}$ *and* $a(t_0) > 0$. *Let* $\{A_k\}_{k=1}^m$ *and* $\{B_k\}_{k=1}^m$ *be such that*

$$A_{k+1} < \gamma_k B_k < B_k < CB_k < A_k, \quad k = 1, 2, \dots m-1$$

*and* $0 < B_m < CB_m < A_m$ *where*

$$C = \max\Big\{\Big(\frac{\eta}{T-\alpha\eta}\int_{\tau_1}^T (T-s)a(s)\,\nabla s\Big)^{-1}, 1\Big\}.$$

*Assume*

(A4) $f_0 = 0$ *and* $f_\infty = +\infty$;

(B4) $f(x) \leq MA_k$ *for all* $x \in [0, A_k], k = 1, 2, \dots m$ *where*

$$M < \Big(\frac{T}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s\Big)^{-1};$$

*and*

(C4) $f(x) \geq CB_k$ *for all* $x \in [\gamma_k B_k, B_k], k = 1, 2, \dots m$.

*Then the boundary-value problem (1.1), (1.2) has at least* $2m+1$ *positive solutions* $\{u_k\}_{k=1}^{2m+1}$ *such that* $0 < \|u_{2m+1}\| \leq B_m \leq \|u_{2m}\| \leq A_m \leq \dots \leq B_1 \leq \|u_2\| \leq A_1 \leq \|u_1\| < \infty$.

**Theorem 4.4.** *Let* $m \geq 1$ *be a fixed integer. Let* $\{\tau_k\}_{k=1}^m \subset \mathbb{T}$ *be such that* $\eta < \tau_{k+1} < \tau_k < \rho(T)$. *Let* $t_0$ *be such that* $t_0 \in [\tau_1, T) \cap \mathbb{T}$ *and* $a(t_0) > 0$. *Let* $\{A_k\}_{k=1}^m$ *and* $\{B_k\}_{k=1}^{m-1}$ *be such that*

$$A_{k+1} < \gamma_k B_k < B_k < CB_k < A_k, \quad k = 1, 2, \dots m-1,$$

*and* $A_m > 0$, *where*

$$C = \max\Big\{\Big(\frac{\eta}{T-\alpha\eta}\int_{\tau_1}^T (T-s)a(s)\,\nabla s\Big)^{-1}, 1\Big\}.$$

*Assume*

(A5) $f_0 = +\infty$ *and* $f_\infty = +\infty$;

(B5) $f(x) \leq MA_k$ *for all* $x \in [0, A_k], k = 1, 2, \dots m$ *where*

$$M < \Big(\frac{T}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s\Big)^{-1};$$

*and*

(C5) $f(x) \geq CB_k$ *for all* $x \in [\gamma_k B_k, B_k]$, $k = 1, 2, \dots m-1$.

*Then the boundary-value problem (1.1), (1.2) has at least* $2m$ *positive solutions* $\{u_k\}_{k=1}^{2m}$ *such that* $0 < \|u_{2m}\| \leq A_m \leq \|u_{2m}\| \leq B_{m-1} \leq \dots \leq B_1 \leq \|u_2\| \leq A_1 \leq \|u_1\| < \infty$.

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Eric R. Kaufmann
Department of Mathematics and Statistics
University of Arkansas at Little Rock
Little Rock, Arkansas 72204-1099, USA
*E-mail address*: erkaufmann@ualr.edu