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SOLUTIONS FOR A HYPERBOLIC SYSTEM WITH BOUNDARY DIFFERENTIAL INCLUSION AND NONLINEAR SECOND-ORDER BOUNDARY DAMPING

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Abstract. In this paper we study the existence of generalized solutions for a hyperbolic system with a discontinuous multi-valued term and nonlinear second-order damping terms on the boundary.

1. Introduction

The main purpose of this paper is to investigate the initial boundary value problem for a hyperbolic system with differential inclusion on the boundary

$$\begin{gathered}
u'' - \Delta u' - M(\|\nabla u\|^2)\Delta u = f \quad \text{in } (x,t) \in Q = \Omega \times (0,T), \\
u(x,0) = u'(x,0) = 0 \quad \text{in } x \in \Omega, \\
u = 0 \quad \text{on } \Sigma_0 = \Gamma_0 \times (0,T), \\
\frac{\partial u'}{\partial \nu} + M(\|\nabla u\|^2)\frac{\partial u}{\partial \nu} + K(u)u'' + |u'|^\rho u' + \Xi = 0 \quad \text{on } \Sigma_1 = \Gamma_1 \times (0,T), \\
\Xi(x,t) \in \varphi(u'(x,t)) \quad \text{a.e. } (x,t) \in \Sigma_1 = \Gamma_1 \times (0,T),
\end{gathered} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n (n \geq 3)$ with sufficiently smooth boundary $\Gamma = \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma_0} \cap \bar{\Gamma_1} = \emptyset$ and Γ_0, Γ_1 have positive measures, $\rho \in (1,\infty)$, $M(s)$ is a C^1 class function such that $M(s) > m_0 > 0$ for some constant m_0, $K(s)$ is a continuously differentiable positive function, $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$, $\|\nabla u\|^2 = \sum_{i=1}^n \int_\Omega |\frac{\partial u}{\partial x_i}|^2 dx$, ν is the outward unit normal vector on Γ, φ is a discontinuous and nonlinear set valued mapping and T is a positive real number. The precise hypothesis on the above system will be given in the next section.

The background of these problems is in physics, especially in solid mechanics, where non-monotone and multi-valued constitutive laws lead to differential inclusion. For a brief account of the works on such variational inequalities we refer the reader to [3,4,5]. Doronin et al. [1] investigated the existence of generalized

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solutions for the hyperbolic equation of the form

$$\begin{gathered} u'' - \Delta u = f \quad \text{in } (x,t) \in Q, \\ \frac{\partial u}{\partial \nu} + K(u)u'' + |u'|^{\rho} u' = 0 \quad \text{on } \Sigma_1 = \Gamma_1 \times (0,T), \\ u = 0 \quad \text{on } \Sigma_0 = \Gamma_0 \times (0,T), \\ u(x,0) = u'(x,0) = 0 \quad \text{on } \Omega. \end{gathered}$$

Motivated the results of [1], in this paper we study the existence of solutions of the variational inequalities (1.1). It is important to observe that as far as we are concerned it has never been considered differential inclusion acting on the boundary in the literature. The plan of this paper is as follows. In section 2, the assumptions and the main results are given. In section 3, the existence of a solution to problem (1.1) is proved.

2. Assumptions and Main results

Throughout this paper we denote

$$\begin{gathered} H_1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}, \quad (u,v) = \int_\Omega u(x)v(x)dx, \\ (u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)d\Gamma, \quad \|u\|_{p,\Gamma_1} = (\int_{\Gamma_1} |u(x)|^p d\Gamma)^{1/p}. \end{gathered}$$

For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{2,\Gamma_1}$ by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_1}$, respectively. We formulate the following assumptions:

(A1) $K(s)$ is a continuous real function satisfying the conditions

$$0 < K_0 \le K(s) \le K_1(1 + |s|^\rho), \tag{2.1}$$

$$|K'(s)|^{\frac{\rho}{\rho-1}} \le K_2(1 + K(s)), \tag{2.2}$$

for some $K_0, K_1, K_2 > 0$.

(A2) $b : \mathbb{R} \to \mathbb{R}$ is a locally bounded function satisfying

$$|b(s)| \le \mu_1(1 + |s|), \ \ \forall s \in \mathbb{R}, \tag{2.3}$$

for some $\mu_1 > 0$.

The multi-valued function $\varphi : \mathbb{R} \to \mathbb{R}$ is obtained by filling in jumps of a function $b : \mathbb{R} \to \mathbb{R}$ by means of the functions $\underline{b}_\epsilon, \overline{b}_\epsilon, \underline{b}, \overline{b} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{gathered} \underline{b}_\epsilon(t) = \operatorname{ess\,inf}_{|s-t|\le\epsilon} b(s), \quad \overline{b}_\epsilon(t) = \operatorname{ess\,sup}_{|s-t|\le\epsilon} b(s), \\ \underline{b}(t) = \lim_{\epsilon\to 0^+} \underline{b}_\epsilon(t), \quad \overline{b}(t) = \lim_{\epsilon\to 0^+} \overline{b}_\epsilon(t), \\ \varphi(t) = [\underline{b}(t), \overline{b}(t)]. \end{gathered}$$

We shall use the regularization of b defined by

$$b^m(t) = m\int_{-\infty}^{\infty} b(t-\tau)\rho(m\tau)d\tau,$$

where $\rho \in C_0^\infty((-1,1)), \rho \ge 0$ and $\int_{-1}^1 \rho(\tau)d\tau = 1$.

Remark 2.1. It is easy to show that b^m is continuous for all $m \in \mathbb{N}$ and that $\underline{b}_\epsilon$, $\overline{b}_\epsilon$, $\underline{b}$, $\overline{b}$, b^m satisfy condition (A2) with a possibly different constant when b satisfies (A2).

Definition A function $u(x,t)$ such that

$$
\begin{gathered}
u \in L^\infty(0,T;H_1(\Omega)),\\
u' \in L^2(0,T;H_1(\Omega)) \cap L^\infty(0,T;L^{\rho+2}(\Gamma_1)),\\
u'' \in L^2(0,T;L^2(\Omega)\cap L^2(\Gamma_1)),\\
u(x,0)=u'(x,0)=0
\end{gathered}
$$

is a generalized solution to (1.1) if there exists $\Xi \in L^2(0,T;L^2(\Gamma_1))$ and for any functions $v \in W = H_1(\Omega)\cap L^{\rho+2}(\Gamma_1)$ and $\psi \in C^1(0,T)$ with $\psi(T)=0$ the relations hold:

$$
\begin{aligned}
&\int_0^T \Big\{(u'',v) + (\nabla u',\nabla v) + M(\|\nabla u\|^2)(\nabla u,\nabla v)\\
&+ (|u'|^\rho u' - K'(u)(u')^2 + \Xi, v)_{\Gamma_1}\Big\}\psi(t)dt - \int_0^T (K(u)u',v)_{\Gamma_1}\psi'(t)dt \qquad (2.4)\\
&= \int_0^T (f,v)\psi(t)dt,
\end{aligned}
$$

$$
\Xi(x,t) \in \varphi(u'(x,t)) \quad \text{a.e. } (x,t)\in\Sigma_1. \tag{2.5}
$$

Now we are in position to state our existence result.

Theorem 2.2. *Assume that (A1) and (A2) hold and $f \in L^2(0,T;L^2(\Omega))$. Then, for all $T>0$ there exists a generalized solution to the problem (1.1).*

3. Proof of main theorem

In this section we are going to show the existence of solution for problem (1.1) using the Faedo-Galerkin's approximation. For this end we represent by $\{w_j\}_{j\ge1}$ a basis in $W = H_1(\Omega)\cap L^{\rho+2}(\Gamma_1)$. Let $W_m = \operatorname{span}\{w_1,w_2,\dots,w_m\}$. Next we define the approximations $u_m(t) = \sum_{j=1}^m g_{jm}(t)w_j$, where $g_{jm}(t)$ are solutions to the Cauchy problem

$$
\begin{aligned}
&(u_m'',w_j) + (\nabla u_m',\nabla w_j) + M(\|\nabla u_m\|^2)(\nabla u_m,\nabla w_j)\\
&\quad +(K(u_m)u_m'' + |u_m'|^\rho u_m' + b^m(u_m'), w_j)_{\Gamma_1} = (f,w_j),
\end{aligned} \tag{3.1}
$$

$$
u_m(0)=u_m'(0)=0. \tag{3.2}
$$

By the same argument as in [1], the approximate system (3.1) and (3.2) has solutions $u_m(t)$ in $[0,t_m)$. The extension of these solutions to the whole interval $[0,T]$ is a consequence of the priori estimate which we are going to prove below.

Step 1 : A priori estimate. Multiplying (3.1) by $g_{jm}'(t)$ and summing from $j=1$ to $j=m$, we get

$$
\begin{aligned}
&\frac12\frac{d}{dt}\Big\{\|u_m'(t)\|^2 + \bar M(\|\nabla u_m(t)\|^2) + \int_{\Gamma_1} K(u_m(t))(u_m'(t))^2 d\Gamma\Big\}\\
&+ (b^m(u_m'(t)),u_m'(t))_{\Gamma_1} + \|\nabla u_m'(t)\|^2 + \|u_m'(t)\|_{\rho+2,\Gamma_1}^{\rho+2}\\
&- \frac12\int_{\Gamma_1} K'(u_m(t))(u_m'(t))^3 d\Gamma\\
&= (f(t),u_m'(t)),
\end{aligned} \tag{3.3}
$$

where $\bar{M}(s)=\int_0^s M(r)dr$. By the condition (A2), we have

$$\begin{aligned}
\|b^m(u_m'(t))\|_{\Gamma_1}^2 &= \int_{\Gamma_1}\big(b^m(u_m'(x,t))\big)^2 d\Gamma \\
&\le \int_{\Gamma_1} c_1(1+|u_m'(x,t)|)^2 d\Gamma \\
&\le 2c_1\int_{\Gamma_1}(1+|u_m'(x,t)|^2)d\Gamma = c_2+2c_1\|u_m'(t)\|_{\Gamma_1}^2,
\end{aligned} \tag{3.4}$$

where c_1, c_2 are positive constants(dependent on the geometry of Γ but independent of m). In what follows $c_i (i\ge 3)$ denote generic constants independent of m. Inequality (3.4) and Hȯlder's inequality imply that

$$\begin{aligned}
&\Big|\int_0^t (b^m(u_m'(s)),u_m'(s))_{\Gamma_1}ds\Big| \\
&\le \Big(\int_0^t \|b^m(u_m'(s))\|_{\Gamma_1}^2 ds\Big)^{1/2}\Big(\int_0^t \|u_m'(s)\|_{\Gamma_1}^2 ds\Big)^{1/2} \\
&\le \Big(\int_0^t (c_2+2c_1\|u_m'(s)\|_{\Gamma_1}^2)ds\Big)^{1/2}\Big(\int_0^t \|u_m'(s)\|_{\Gamma_1}^2 ds\Big)^{1/2} \\
&\le c_3\Big(1+\int_0^t \|u_m'(s)\|_{\Gamma_1}^2 ds\Big).
\end{aligned}$$

Note that, by Young's inequality,

$$\begin{aligned}
&\int_0^t\{\|u_m'(s)\|_{\rho+2,\Gamma_1}^{\rho+2}-\frac{1}{2}\int_{\Gamma_1}K'(u_m(s))(u_m'(s))^3 d\Gamma\}ds \\
&\ge \int_0^t\int_{\Gamma_1}|u_m'(s)|^2\{|u_m'(s)|^\rho-\epsilon|u_m'(s)|^\rho - C(\epsilon)|K'(u_m(s))|^{\frac{\rho}{\rho-1}}\}d\Gamma ds,
\end{aligned} \tag{3.5}$$

where ϵ is an arbitrary positive number. Therefore, integrating (3.3) over $(0,t)$ and taking $\epsilon=\frac{1}{2}$ in (3.6), from (2.2), (3.5) and (3.6) we obtain

$$\begin{aligned}
&\frac{1}{2}\{\|u_m'(t)\|^2+\bar{M}(\|\nabla u_m(t)\|^2)+\int_{\Gamma_1}K(u_m(t))(u_m'(t))^2 d\Gamma\} \\
&+\int_0^t\|\nabla u_m'(s)\|^2 ds+\frac{1}{2}\int_0^t\|u_m'(s)\|_{\rho+2,\Gamma_1}^{\rho+2}ds \\
&\le c_3\Big(1+\int_0^t\|u_m'(s)\|_{\Gamma_1}^2 ds\Big)+\int_0^t\|f(s)\|^2 ds+\int_0^t\|u_m'(s)\|^2 ds \\
&\quad + c_4\int_0^t\int_{\Gamma_1}|u_m'(s)|^2(1+K(u_m(s)))d\Gamma ds.
\end{aligned} \tag{3.6}$$

On the other hand, note that $K(u)\ge C_0(1+K(u))$ where $2C_0=\min\{1,K_0\}$. Thus, letting

$$E_m(t)=\frac{1}{2}\Big(\|u_m'(t)\|^2+\bar{M}(\|\nabla u_m(t)\|^2)+C_0\int_{\Gamma_1}(1+K(u_m(t)))|u_m'(t)|^2 d\Gamma\Big),$$

from (3.7) we have $E_m(t)\le c_5(1+\int_0^t E_m(s)ds)$. Thus, by Gronwall's lemma, we conclude that

$$E_m(t)\le c_6,\quad \forall t\in[0,T]. \tag{3.7}$$

This inequality and (3.7) imply that for all $t \in (0,T)$

$$\int_0^t \|\nabla u_m'(s)\|^2 ds \le c_7, \quad \int_{\Gamma_1} |u_m'(t)|^2 d\Gamma \le c_8. \tag{3.8}$$

By imbedding theorem, from (3.9) we have

$$\int_0^t \|u_m'(s)\|^2 ds \le c_9. \tag{3.9}$$

Furthermore, from (3.4) and (3.9) we obtain

$$\int_0^t \|b^m(u_m'(s))\|_{\Gamma_1}^2 ds \le c_{10}. \tag{3.10}$$

Since $\bar{M}(\|\nabla u_m(t)\|^2) \ge m_0\|\nabla u_m(t)\|^2$, by (3.8)

$$\|\nabla u_m(t)\|^2 \le c_{11}. \tag{3.11}$$

Next, multiplying (3.1) by $g_{jm}''(t)$ and summing from $j=1$ to $j=m$, we have

$$\begin{aligned}
&\|u_m''(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m'(t)\|^2 + \ M(\|\nabla u_m(t)\|^2)\frac{d}{dt}(\nabla u_m(t), \nabla u_m'(t)) \\
&- M(\|\nabla u_m(t)\|^2)\|\nabla u_m'(t)\|^2 + (b^m(u_m'(t)), u_m''(t))_{\Gamma_1} \\
&+ \int_{\Gamma_1} K(u_m(t))(u_m''(t))^2 d\Gamma r + \frac{1}{\rho+2}\frac{d}{dt}\|u_m'(t)\|_{\rho+2,\Gamma_1}^{\rho+2} \\
&= (f(t), u_m''(t)).
\end{aligned} \tag{3.12}$$

Integrating this inequality over $(0,t)$ and using (2.1) and Young's inequality, we obtain

$$\begin{aligned}
&\int_0^t \|u_m''(s)\|^2 ds + \frac{1}{2}\|\nabla u_m'(t)\|^2 + K_0\int_0^t \|u_m''(s)\|_{\Gamma_1}^2 ds + \frac{1}{\rho+2}\|u_m'(t)\|_{\rho+2,\Gamma_1}^{\rho+2} \\
&\le \int_0^t M(\|\nabla u_m(s)\|^2)\|\nabla u_m'(s)\|^2 ds - M(\|\nabla u_m(t)\|^2)(\nabla u_m(t), \nabla u_m'(t)) \\
&\quad + 2M'(\|\nabla u_m(t)\|^2)(\nabla u_m(t), \nabla u_m'(t))^2 + \epsilon\int_0^t \|u_m''(s)\|_{\Gamma_1}^2 ds \\
&\quad + C(\epsilon)\int_0^t \|b^m(u_m'(s))\|_{\Gamma_1}^2 ds + \epsilon\int_0^t \|u_m''(s)\|^2 ds + C(\epsilon)\int_0^t \|f(s)\|^2 ds,
\end{aligned} \tag{3.13}$$

where we have used $u_m(0) = u_m'(0) = 0$. Since ϵ is arbitrary and $M(s)$ is a C^1 function, from (3.8),(3.9),(3.11) and (3.12), we conclude that

$$\int_0^t \|u_m''(s)\|^2 ds + \|\nabla u_m'(t)\|^2 + \int_0^t \|u_m''(s)\|_{\Gamma_1}^2 ds + \|u_m'(t)\|_{\rho+2,\Gamma_1}^{\rho+2} \le c_{12}. \tag{3.14}$$

From (3.8)-(3.12), and (3.15), taking into consideration that $u\big|_{\Gamma_0} = 0$, we obtain

$$\begin{gathered}
(u_m) \text{ is bounded in } L^\infty(0,T; H_1(\Omega)), \\
(u_m') \text{ is bounded in } L^\infty(0,T; H_1(\Omega)) \cap L^\infty(0,T; L^{\rho+2}(\Gamma_1)), \\
(u_m'') \text{ is bounded in } L^2(0,T; L^2(\Omega) \cap L^2(\Gamma_1)), \\
(b^m(u_m')) \text{ is bounded in } L^2(0,T; L^2(\Gamma_1)).
\end{gathered} \tag{3.15}$$

Step 2 : Passage to the limit. Multiplying (3.1) by $\psi \in C^1(0,T)$ with $\psi(T)=0$ and integrating over $(0,T)$, we obtain

$$\begin{aligned}&\int_0^T \Big\{(u_m''(t),w_j)+(\nabla u_m'(t),\nabla w_j)+M(\|\nabla u_m(t)\|^2)(\nabla u_m(t),\nabla w_j)\\&+(b^m(u_m'(t)),w_j)_{\Gamma_1}+(|u_m'(t)|^\rho u_m'(t)-K'(u_m(t))(u_m'(t))^2,w_j)_{\Gamma_1}\Big\}\psi(t)dt\\&-\int_0^T(K(u_m(t))u_m'(t),w_j)_{\Gamma_1}\psi'(t)dt\\&=\int_0^T(f(t),w_j)\psi(t)dt.\end{aligned}\tag{3.16}$$

From (3.16), we have subsequences (in the sequel we denote subsequences by the same symbols as original sequences) such that

$$u_m\to u \quad \text{weakly star in } L^\infty(0,T;H_1(\Omega)),\tag{3.17}$$
$$u_m'\to u' \quad \text{weakly star in } L^\infty(0,T;H_1(\Omega))\cap L^\infty(0,T;L^{\rho+2}(\Gamma_1)),\tag{3.18}$$
$$u_m''\to u'' \quad \text{weakly in } L^2(0,T;L^2(\Omega)\cap L^2(\Gamma_1)),\tag{3.19}$$
$$b^m(u_m')\to \Xi \quad \text{weakly in } L^2(0,T;L^2(\Gamma_1)).\tag{3.20}$$

From (3.18)–(3.21), considering that the imbedding $H_1(\Omega)\hookrightarrow L^2(\Gamma)$ is continuous and compact and using Aubin compactness theorem [2], we have

$$|u_m'|^\rho u_m', K(u_m)u_m', K'(u_m)(u_m')^2\in L^q(\Sigma_1),\quad q=\frac{\rho+2}{\rho+1}>1,\tag{3.21}$$
$$u_m\to u \text{ a.e. on } \Sigma_1 \quad\text{and}\quad u_m'\to u' \text{ a.e. on } \Sigma_1\,.\tag{3.22}$$

Therefore,

$$\begin{gathered}|u_m'|^\rho u_m'\to|u'|^\rho u',\quad K(u_m)u_m'\to K(u)u',\\K'(u_m)(u_m')^2\to K'(u)(u')^2\quad \text{a.e. on }\Sigma_1.\end{gathered}\tag{3.23}$$

Step 3 : (u,Ξ) is a solution of (1.1). Letting m tend to infinity in (3.17) and using (3.18)-(3.24), we have

$$\begin{aligned}&\int_0^T \big\{(u''(t),w_j)+(\nabla u'(t),\nabla w_j)+M(\|\nabla u(t)\|^2)(\nabla u(t),\nabla w_j)\\&+(\Xi(t),w_j)_{\Gamma_1}+(|u'(t)|^\rho u'(t)-K'(u(t))(u'(t))^2,w_j)_{\Gamma_1}\big\}\psi(t)dt\\&-\int_0^T(K(u(t))u'(t),w_j)_{\Gamma_1}\psi'(t)dt\\&=\int_0^T(f(t),w_j)\psi(t)dt.\end{aligned}\tag{3.24}$$

Since $\{w_j\}$ is dense in $H_1(\Omega)\cap L^{\rho+2}(\Gamma)$, we conclude that (2.4) hold. It remains to show that (2.5), i.e., $\Xi(x,t)\in\varphi(u'(x,t))$ a.e. $(x,t)\in\Sigma_1$. By the Aubin-Lions compactness Lemma[2], we get from (3.19)-(3.20) that

$$u_m'\to u' \quad \text{strongly in } L^2(0,T;L^2(\Gamma_1)).$$

This implies $u_m'(x,t)\to u'(x,t)$ a.e. in Σ_1. Thus, for given $\eta>0$, using the theorems of Lusin and Egoroff, we can choose a subset $\omega\subset\Sigma_1$ such that meas$(\omega)<$

$\eta, u' \in L^\infty(\Sigma_1 \setminus \omega)$ and $u'_m \to u'$ uniformly on $\Sigma_1 \setminus \omega$. Thus, for each $\epsilon > 0$, there is an $N > \frac{2}{\epsilon}$ such that

$$|u'_m(x,t) - u'(x,t)| < \frac{\epsilon}{2}, \quad \forall (x,t) \in \Sigma_1 \setminus \omega.$$

Then, if $|u'_m(x,t) - s| < 1/m$, we have $|u'(x,t) - s| < \epsilon$ for all $m > N$ and $(x,t) \in \Sigma_1 \setminus \omega$. Therefore,

$$\underline{b}_\epsilon(u'(x,t)) \le b^m(u'_m(x,t)) \le \overline{b}_\epsilon(u'(x,t)), \quad \forall m > N, (x,t) \in \Sigma_1 \setminus \omega.$$

Let $\phi \in L^\infty(\Sigma_1), \phi \ge 0$. Then

$$\begin{aligned} \int_{\Sigma_1 \setminus \omega} \underline{b}_\epsilon(u'(x,t))\phi(x,t) d\Gamma dt &\le \int_{\Sigma_1 \setminus \omega} b^m(u'_m(x,t))\phi(x,t) d\Gamma dt \\ &\le \int_{\Sigma_1 \setminus \omega} \overline{b}_\epsilon(u'(x,t))\phi(x,t) d\Gamma dt. \end{aligned} \tag{3.25}$$

Letting m approach ∞ in (3.26) and using (3.21), we obtain

$$\begin{aligned} \int_{\Sigma_1 \setminus \omega} \underline{b}_\epsilon(u'(x,t))\phi(x,t) d\Gamma dt &\le \int_{\Sigma_1 \setminus \omega} \Xi(x,t)\phi(x,t) d\Gamma dt \\ &\le \int_{\Sigma_1 \setminus \omega} \overline{b}_\epsilon(u'(x,t))\phi(x,t) d\Gamma dt. \end{aligned} \tag{3.26}$$

Letting $\epsilon \to 0^+$ in (3.27), we infer that

$$\Xi(x,t) \in \varphi(u'(x,t)) \quad \text{a. e. in } \Sigma_1 \setminus \omega,$$

and letting $\eta \to 0^+$ we get

$$\Xi(x,t) \in \varphi(u'(x,t)) \quad \text{a.e. in } \Sigma_1.$$

This completes the proof.

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