Electronic Journal of Differential Equations, Vol. 2003(2003), No. 78, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

ESTIMATES FOR CRITICAL GROUPS OF SOLUTIONS TO QUASILINEAR ELLIPTIC SYSTEMS

JOSÉ CARMONA, SILVIA CINGOLANI, & GIUSEPPINA VANNELLA

Abstract. In this work we study a class of functionals, defined on Banach spaces, associated with quasilinear elliptic systems. Firstly, we prove some regularity results about the critical points of such functionals and then we estimate the critical groups in each critical point via its Morse index.

1. Introduction and statement of the results

Let X be a Banach space and $f : X \to \mathbb{R}$ a C^2 functional. For any $a \in \mathbb{R}$, we denote by f^a the sublevel $\{v \in X : f(v) \leq a\}$. Let u be a critical point of f, at level $c = f(u)$. We call

$$C_q(f,u) = H^q(f^c, f^c \setminus \{u\})$$

the q-th critical group of f at u, $q = 0, 1, 2, ..$, where $H^q(A,B)$ stands for the q-th Alexander-Spanier cohomology group of the pair (A,B) with coefficients in $\mathbb{K}$ [2].

If X is a Hilbert space and f is a C^2 Euler functional on X, which satisfies some *apriori* compactness conditions on the sublevels, the changes of topology of its sublevels can be checked by computing the *critical groups* of the functional in the critical points. The estimates of the critical groups in a critical point become a quite clear fact if the critical point is *non-degenerate*, namely the second derivative of the functional in the critical point is an isomorphism. A classical result, based on Morse Lemma, allows us to relate the critical groups in the non-degenerate critical point u to its Morse index, namely the supremum of the dimensions in which the second derivative $f''(u)$ of f in u is negative definite. For reader's convenience, we recall the following theorem (see also [1]).

Theorem 1.1. *Suppose H a Hilbert space and $I \in C^2(H,\mathbb{R})$. Let u be a non-degenerate critical point of I with Morse index m. Then*

$$C_q(I,u) \cong \mathbb{K} \quad \text{if } q = m, \text{ and} \quad C_q(I,u) = \{0\} \text{ if } q \neq m.$$

Nevertheless, if $m = +\infty$, we always have $C_q(I,u) = \{0\}$.

Gromoll and Meyer extended the previous theorem for the case of an isolated critical point u, possibly degenerate, when the second derivative of the functional

2000 *Mathematics Subject Classification.* 58E60, 35J60, 35B65.
Key words and phrases. Quasilinear elliptic systems, critical groups, Morse index.

Submitted January 15, 2003. Published July 28, 2003.

in the critical point u is a Fredholm operator with zero index. These extensions are based on generalizations of the Morse splitting lemma (see [1]).

In a Banach space framework, the estimate of the critical groups is a very interesting, but not classical fact. A lot of problems arise when trying to develop a local Morse theory, as Morse splitting type lemma does not hold and it is not *a priori* clear what can be a reasonable definition of non-degenerate critical point in a Banach space, which is not isomorphic to its dual space. Moreover, some crucial ingredients in Morse theory for Hilbert spaces, like the Fredholm properties of the second derivative of the functional in the critical points, lack in a Banach space setting.

In recent papers by Cingolani and Vannella [3, 4], the authors give a connection between critical groups of a solution to a quasilinear equation, involving p-Laplacian, and its Morse index. In [4], a new definition of non-degenerate critical point (see also [1, 15, 16] for different definitions) is introduced involving only the injectivity of the second derivative of the Euler functional in the critical point. Such weak nondegeneracy is enough to regain a suitable splitting of the Banach space and to evaluate the critical groups of the critical points via Morse index. See also [5] for Marino-Prodi perturbation type results [11], using the new definition of nondegeneracy given in [4].

In this paper we aim to extend the results in [4] for a class of quasilinear elliptic systems. We deal with functionals $I : W_0^{1,p}(\Omega,\mathbb{R}^m) \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega \Big(\frac{1}{p}|\nabla u|^p + \frac{1}{2}|\nabla u|^2 + g(u)\Big)dx, \quad u \in W_0^{1,p}(\Omega,\mathbb{R}^m),$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n \geq 2$, $m \geq 2$, $p \geq 2$ and, as usual

$$|\nabla u|^p = \Big(\sum_{i=1}^m \sum_{\alpha=1}^n \big(\frac{\partial u^i}{\partial x_\alpha}\big)^2\Big)^{p/2}.$$

We assume that $g \in C^2(\mathbb{R}^m,\mathbb{R})$ and that

(G1) For any $\xi \in \mathbb{R}^m$, $|g''(\xi)| \leq c_1|\xi|^q + c_2$ with c_1, c_2 positive constants and $0 \leq q < \frac{np}{(n-p)} - 2$ if $n > p$, while q is any positive number, if $n = p$.

If $n < p$, no restrictive assumption is required.

The functional I is C^2 on $W_0^{1,p}(\Omega,\mathbb{R}^m)$ and critical points of I correspond to weak solutions of the quasilinear elliptic system

$$\begin{gathered} -\Delta u - \Delta_p u + g'(u) = 0 \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where, as usual $\Delta_p u = (\operatorname{div}(|\nabla u|^{p-2}\nabla u^i))_{i=1}^m$ for any $p \geq 2$, and $\Delta u = \Delta_2 u$.

We underline that the variational setting for problem (1.1) is the Banach space $W_0^{1,p}(\Omega,\mathbb{R}^m)$, which cannot be equipped by an equivalent Hilbert norm if $p \neq 2$. In this case we also note that:

- Any critical point of I is degenerate in the classical sense used in Hilbert space, in fact $I''(u) : W_0^{1,p}(\Omega,\mathbb{R}^m) \to W^{-1,p'}(\Omega,\mathbb{R}^m)$ $(1/p' + 1/p = 1)$ is not invertible, since $W_0^{1,p}(\Omega,\mathbb{R}^m)$ is not isomorphic to its dual;
- $I''(u)$ cannot be a Fredholm operator. Hence extensions of Morse's lemma such as that of Gromoll and Meyer type cannot be applied.

In spite of the above difficulties, we are able to relate critical groups to differential properties of the critical points of I, like in the Hilbert space case. Before stating the main results, let us denote by $m(I,u)$ the Morse index of I at u and by $m^*(I,u)$ the sum of $m(I,u)$ and the dimension of the kernel of $I''(u)$ in $W_0^{1,p}(\Omega,\mathbb{R}^m)$.

Theorem 1.2. *Let u be a critical point of I such that $I''(u)$ is injective. Then $m(I,u)$ is finite and*

$$\begin{gathered} C_q(I,u) \cong \mathbb{K}, \quad \text{if } q = m(I,u), \\ C_q(I,u) = \{0\}, \quad \text{if } q \neq m(I,u). \end{gathered}$$

The main idea in the proof of Theorem 1.2 is the introduction of a Hilbert space H which is obtained as closure of the space $C_0^\infty(\Omega,\mathbb{R}^m)$ with respect to a scalar product depending on the critical point $u \in W_0^{1,p}(\Omega,\mathbb{R}^m)$. We remark that the Banach space $W_0^{1,p}(\Omega,\mathbb{R}^m)$ is continuously embedded in such an auxiliary Hilbert space. To this aim it is crucial to prove some regularity results concerning the weak solutions to problem (1.1) (see Section 2). Furthermore it is possible to extend the second derivative of the Euler functional I at the critical point u to a Fredholm operator with zero index from H to its dual space and so to obtain a suitable splitting of the variational space $W_0^{1,p}(\Omega,\mathbb{R}^m)$.

Finally, by using some ideas recently developed in [4], we are able to obtain a finite dimensional reduction and to prove that the critical groups of I in u are isomorphic to the critical groups of a suitable function defined on a finite dimensional subspace of $W_0^{1,p}(\Omega,\mathbb{R}^m)$.

It is important to note that when Theorem 1.2 is applied to problems like (1.1), the Morse series is not yet formal, but it is a real equality between polynomials.

In the case of isolated, possibly degenerate, critical points we also prove the following statement.

Theorem 1.3. *Let u be an isolated critical point of I. Then $m(I,u)$ and $m^*(I,u)$ are finite and we have*

$$C_q(I,u) = \{0\}, \quad \text{if } q < m(I,u) \quad \text{or } q > m^*(I,u).$$

Notation. We denote by $(\cdot|\cdot)$ the usual scalar product in $\mathbb{R}^n$.

$L^p(\Omega)$ denotes the usual Lebesgue space with norm $\left(\int_\Omega |v|^p\right)^{1/p}$. In the vector valued space $L^p(\Omega,\mathbb{R}^m) = L^p(\Omega)\times\cdots\times L^p(\Omega) = (L^p(\Omega))^m$ we consider the norm $\|u\|_p^p = \int_\Omega |u|^p$.

$W_0^{1,p}(\Omega)$ denotes the usual Sobolev space with norm $\left(\int_\Omega |\nabla v|^p\right)^{1/p}$. In the vector valued space $W_0^{1,p}(\Omega,\mathbb{R}^m) = W_0^{1,p}(\Omega)\times\cdots\times W_0^{1,p}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^m$ we consider the norm $\|u\| = \|\nabla u\|_p$. For any $p < n$ the critical Sobolev exponent will be denoted by $p^* = \frac{pn}{n-p}$.

$W^{-1,p'}(\Omega,\mathbb{R}^m)$ denotes the dual space of $W^{1,p}(\Omega,\mathbb{R}^m)$, and $1/p' + 1/p = 1$. Moreover we denote by $\langle\cdot,\cdot\rangle : W^{-1,p'}(\Omega,\mathbb{R}^m)\times W_0^{1,p}(\Omega,\mathbb{R}^m) \to \mathbb{R}$ the duality pairing.

$B_r(u) = \{v \in W_0^{1,p}(\Omega,\mathbb{R}^m) : \|v-u\| < r\}$, where $u \in W_0^{1,p}(\Omega,\mathbb{R}^m)$ and $r > 0$.

$|A|$ and $\operatorname{meas} A$ denotes the Lebesgue measure of $A \subset \mathbb{R}^n$.

2. Regularity

In this section, using the interior regularity result contained in [13] and an argument similar to those contained in [6, 14] for the scalar case, we prove that every solution $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ of (1.1) also belongs to $C^1(\overline{\Omega}, \mathbb{R}^m)$.

We recall here the Stampacchia lemma [12, Lemma 4.1], which will be the main tool in proving that $u \in L^\infty(\Omega, \mathbb{R}^m)$.

Lemma 2.1. *Let $\varphi : [0, +\infty[\to \mathbb{R}$ be a non negative, non increasing real valued function. If there exists positive constants C, α, β such that*

$$\varphi(h) \leq \frac{C}{(h-k)^\alpha} \varphi(k)^\beta, \; h > k \geq 0,$$

then

(i) *if $\beta > 1$ then $\varphi(d) = 0$ with $d^\alpha = C\varphi(0)^{\beta-1} 2^{\alpha\beta/(\beta-1)}$*
(ii) *if $\beta = 1$ then $\varphi(h) \leq \varphi(0) \exp\left(1 - \frac{h}{(eC)^{(1/\alpha)}}\right)$*
(iii) *if $\beta < 1$ then $\varphi(h) \leq \frac{2^{\frac{\alpha}{(1-\beta)^2}} [C^{\frac{1}{1-\beta}} + 2^{\frac{\alpha}{1-\beta}} \varphi(1)]}{h^{\frac{\alpha}{1-\beta}}}$, $h > 1$.*

Next we prove the main results for this section.

Lemma 2.2. *If $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ is a solution to (1.1) then $L^\infty(\Omega)$. Moreover there exists $M > 0$ such that $\|u\|_{L^\infty} \leq M$.*

Proof. During this proof c, c_i will denote positive constants independent on u that may change between consecutive steps. We focus on the case $p \leq n$. In the other case the proof follows directly by the Sobolev embedding.

Firstly, we note that condition (G1) implies in particular that

(G2) $|g'(\xi)| \leq c_1|\xi|^s + c_2$ with c_1, c_2 positive constants and $1 \leq s < p^* - 1$ if $n > p$, while $s \geq 1$, if $n = p$.

We consider now for every $k \in \mathbb{R}^+$ the function $G_k : \mathbb{R} \to \mathbb{R}$ given by

$$G_k(s) = \begin{cases} s + k & s \leq -k, \\ 0 & -k < s \leq k, \\ s - k & s > k\,. \end{cases}$$

Thus, for every $j = 1, \dots, m$ we can take $\phi_j = (\delta_{ij} G_k(u^j))_{i=1,\dots,m}$ as test function in the weak equation satisfied by u and using (G_2) we have

$$\|\nabla G_k(u^j)\|_p^p \leq \int_\Omega (1 + |\nabla u|^{p-2}) \nabla u \nabla(\delta_{ij} G_k(u^j)) \leq \int_{\Omega_k} (c_1|u|^s + c_2)|G_k(u^j)|, \quad (2.1)$$

where $\Omega_k \equiv \{x \in \Omega : |u^j(x)| > k\}$.

We first consider the case $p < n$. Let $r > s\frac{p^*}{p^*-1}$ be such that $u \in L^r(\Omega)$. Writing $|v|^s = c_1|u|^s + c_2$ and using the Sobolev and Hölder inequalities, by (2.1) we yield,

$$\|G_k(u^j)\|_{p^*}^p \leq c\|v\|_r^s \|G_k(u^j)\|_{p^*} (\operatorname{meas} \Omega_k)^{(1 - s/r - 1/p^*)}. \quad (2.2)$$

Taking into account that, for every $h > k$, $|G_k(u^j)| \geq h - k$ in Ω_h, (2.2) implies

$$(h-k)^{p-1} (\operatorname{meas} \Omega_h)^{(p-1)/p^*} \leq c\|v\|_r^s (\operatorname{meas} \Omega_k)^{(1 - s/r - 1/p^*)},$$

or equivalently

$$\operatorname{meas}\Omega_h \le \frac{c\|v\|_r^{\frac{sp^*}{p-1}}(\operatorname{meas}\Omega_k)^{\frac{(p^*-1-sp^*/r)}{p-1}}}{(h-k)^{p^*}}. \tag{2.3}$$

Now we apply Stampacchia's Lemma with $\varphi(h)=\operatorname{meas}\Omega_h$, $C=c\|v\|_r^{\frac{sp^*}{p-1}}$, $\alpha=p^*$ and $\beta=\frac{(p^*-1-sp^*/r)}{p-1}$, to prove that:

- (i) if $u\in L^r(\Omega)$ with $r>\frac{sp^*}{p^*-p}=\frac{sn}{p}$, then $u^j\in L^\infty(\Omega)$ and $\|u^j\|_\infty\le c\|v\|_r^{s/(p-1)}$,
- (ii) if $u\in L^r(\Omega)$ with $r=\frac{sn}{p}$, then $u^j\in L^t(\Omega)$ for $t\in[1,\infty)$ and $\|u^j\|_t^t\le c+c'\|v\|_r^{ts/(p-1)}$,
- (iii) if $u\in L^r(\Omega)$ with $r<\frac{sn}{p}$, then $u^j\in L^t(\Omega)$ for $t=\frac{p^*r(p-1)}{(p-p^*)r+p^*s}-\delta$ and $\delta>0$ arbitrarily small. Moreover, $\|u^j\|_t^t\le c+c'\|v\|_r^{(t+\delta)s/(p-1)}$.

Item (i) follows from the fact that $\operatorname{meas}\Omega_d=0$. Hence

$$u^j\le d=(C\varphi(0)^{\beta-1}2^{\alpha\beta/(\beta-1)})^{1/\alpha}=c_1\|v\|_r^{\frac{s}{p-1}}.$$

To prove items (ii) and (iii) we take

$$T_h(s)=\begin{cases}-h & s\le -h,\\ s & -h<s\le h,\\ h & s>h.\end{cases}$$

It is clear that $T_h(u^j)\to u^j$ a.e. in Ω. Now we shall apply the Vitali Theorem (cf. [8]) to prove that $u^j\in L^t(\Omega)$ and the convergence is strong in $L^t(\Omega)$.

Firstly by Stampacchia Lemma we deduce that

$$h^t(\operatorname{meas}\Omega_h)\to 0 \tag{2.4}$$

for any $t>1$ in the case ii) and for $t<\frac{\alpha}{1-\beta}=\frac{p^*r(p-1)}{(p-p^*)r+p^*s}$ in the case iii). We note that, in both cases, it is possible to choose $t>1$.

To apply Vitali Theorem, we need to prove that for any $E\subset\Omega$,

$$\lim_{|E|\to 0}\int_E |T_n(u^j)|^t=0,$$

uniformly in n. A sufficient condition is to show that

$$\lim_{k\to+\infty}\int_{B_{n,k}\equiv\{|T_n|\ge k\}}|T_n(u^j)|^t=0, \tag{2.5}$$

uniformly in n. First we observe that $B_{n,k}=\emptyset$ if $k>n$ and $B_{n,k}=B_{k,k}=\Omega_k$ for every $k\le n$. Moreover, if $n\ge k$, we can write

$$\begin{aligned}\int_{B_{n,k}}|T_n(u^j)|^t &= \int_{\Omega_k}|T_n(u^j)|^t\\ &= \sum_{s=k}^{n-1}\int_{\Omega_s\setminus\Omega_{s+1}}|T_n(u^j)|^t+\int_{\Omega_n}|T_n(u^j)|^t\\ &\le \sum_{s=k}^{n-1}(s+1)^t\left(|\Omega_s|-|\Omega_{s+1}|\right)+n^t|\Omega_n|.\end{aligned}$$

Since (2.4) holds, to prove (2.5) uniformly in n, it is sufficient to show the convergence of

$$\sum (n+1)^t \left(|\Omega_n| - |\Omega_{n+1}|\right).$$

To this aim, it is sufficient to prove the convergence of the two series

$$\sum \left((n+1)^t|\Omega_n| - n^t|\Omega_n|\right) \quad \text{and} \quad \sum \left(n^t|\Omega_n| - (n+1)^t|\Omega_{n+1}|\right).$$

The first series converges as, by Stampacchia lemma, we can observe that

$$\left((n+1)^t - n^t\right)|\Omega_n| \le t(n+1)^{t-1}|\Omega_n| \le C_t\big(\frac{n+1}{n}\big)^{t-1}\frac{1}{n^{1+\gamma}}$$

for some constant $C_t \in \mathbb{R}$ and $0 < \gamma = \frac{\alpha}{1-\beta} - t$. Also the second series is convergent, due to (2.4). As a consequence we have that

- If $r = \frac{sn}{p}$ then $u^j \in L^t(\Omega)$ for $t \in [1, \infty)$
- If $r < \frac{sn}{p}$ then $u^j \in L^t(\Omega)$ for $t = \frac{p^* r(p-1)}{(p-p^*)r + p^* s} - \delta$ and $\delta > 0$ arbitrarily small.

Finally, the estimates on the norms follows from the convexity of the real function s^t with $t > 1$ and the estimates of $\varphi(h)$ in Lemma 2.1. More precisely,

$$\|u^j\|_t^t \le (\|u^j - T_h(u^j)\|_t + \|T_h(u^j)\|_t)^t \le c_1 + c_2\|T_h(u^j)\|_t^t \le c_1 + c_3 h^t,$$

for each fixed h big enough. Hence by Lemma 2.1 we have, in the case (ii), that

$$h^t \le (\log e\varphi(0)/\varphi(h))^t (ec)^{t/p^*} \|v\|_r^{st/(p-1)}$$

and in the case (iii)

$$h^{t+\delta} \le \frac{c_4\|v\|_r^{\frac{sp^* r}{r(p-p^*)+sp^*}} + c_5\varphi(1)}{\varphi(h)},$$

which concludes the estimates in items (ii) and (iii) above. We observe that this argument can be done for every $j = 1, \dots, m$, and hence those estimates above, remain valid if we replace u^j by u.

Since $u \in L^{p^*}(\Omega, \mathbb{R}^m)$ and $p^* > s\frac{p^*}{p^*-1}$, we can argue as before for $r_0 = p^*$. Thus, if $p^* > \frac{sn}{p}$ we conclude by item i). In the case $p^* = \frac{sn}{p}$ we use item ii) in order to take $r_1 > \frac{sn}{p}$ and conclude again by item i). Finally, in the case $p^* < \frac{sn}{p}$ we can take

$$r_1 = \frac{p^* r_0(p-1)}{(p-p^*)r_0 + p^* s} - \delta_1 > r_0.$$

As before, if $r_1 \ge \frac{sn}{p}$ we easily conclude. In other case we take

$$r_2 = \frac{p^* r_1(p-1)}{(p-p^*)r_1 + p^* s} - \delta_2.$$

By an iterative argument we conclude after a finite number of steps. Indeed, in other case, we have that r_n is bounded, where r_n is defined recurrently by

$$r_0 = p^*$$

$$r_{n+1} = \frac{p^* r_n(p-1)}{(p-p^*)r_n + p^* s} - \delta_{n+1},$$

where $\lim \delta_n = 0$. Moreover, r_n is non decreasing and so it converges to $r \in (p^*, \frac{sn}{p}]$ that satisfies

$$r = \frac{p^* r(p-1)}{(p-p^*)r + p^* s},$$

that is, $p^*(p-1) = (p-p^*)r + p^*s$, which implies that $r = \frac{p^*(p-1-s)}{p-p^*} < p^*$ and this is a contradiction.

In the case $p = n$ we can choose $q > p$ and $r > \frac{qs}{q-p}$ and argue as before with p^* replaced by q. In this case we finish by item i) in the Stampacchia Lemma. $\square$

Theorem 2.3. *If u is a solution to* (1.1)*, then $u \in C^1(\overline{\Omega}, \mathbb{R}^m)$.*

Proof. Let $u_0 \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ be a solution of (1.1). We consider the problem

$$\begin{gathered} -\Delta u - \Delta_p u + g'(u_0) = 0, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{2.6}$$

and we will prove that every $L^\infty(\Omega, \mathbb{R}^m)$-solution of (2.6) (by Lemma 2.2, in particular u_0) is $C^1(\overline{\Omega}, \mathbb{R}^m)$-regular. Indeed, the interior regularity follows immediately from the estimates in [13]. Let us show how to obtain the regularity up to the boundary. Since $\partial\Omega$ is smooth, we can assume without lost of generality that, near each fixed $x^0 \in \partial\Omega$,

$$x \in \Omega \iff x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ and } x_n > a(x')$$

$a : \mathbb{R}^{n-1} \to \mathbb{R}$ being a C^2-function. Thus, the change of variables $y(x_1, \dots, x_n)$ given by

$$\begin{aligned} &y_1 = x_1 - x_1^0, \\ &y_2 = x_2 - x_2^0, \\ &\dots . \\ &y_{n-1} = x_{n-1} - x_{n-1}^0, \\ &y_n = x_n - a(x_1, \dots, x_{n-1}), \end{aligned}$$

is an invertible map between $\Omega \cap U$ and $D = \{y = (y_1, \dots, y_n) : |y| < \delta, \, y_n > 0\}$, where $\delta > 0$ and U neighborhood of x_0 are suitably chosen.

Let us note that, given $\phi \in C_0^1(D, \mathbb{R}^m)$, then $\phi(y^{-1}) \in C_0^1(\Omega \cap U, \mathbb{R}^m)$. This leads, for any solution $u \in W_0^{1,p}(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m)$ of (2.6), to the integral equality

$$\begin{aligned} &\int_{\Omega\cap U} \Big(1 + \Big|\sum_{i=1}^m \sum_{j=1}^n \frac{\partial u^i}{\partial x_j}\frac{\partial u^i}{\partial x_j}\Big|^{\frac{p-2}{2}}\Big) \sum_{i=1}^m \sum_{j=1}^n \frac{\partial u^i}{\partial x_j}\frac{\partial \phi^i(y^{-1})}{\partial x_j}\,dx \\ &\quad + \int_{\Omega\cap U} \sum_{i=1}^m g_i'(u_0)\phi^i(y^{-1})\,dx = 0. \end{aligned} \tag{2.7}$$

We perform the change of variables in (2.7) and taking into account that $|\frac{dx}{dy}| = 1$, we obtain

$$\begin{aligned} &\int_D \Big(1 + \Big|\sum_{i=1}^m \sum_{j,l,k=1}^n \frac{\partial y_l}{\partial x_j}\frac{\partial y_k}{\partial x_j}\frac{\partial u^i}{\partial y_l}\frac{\partial u^i}{\partial y_k}\Big|^{\frac{p-2}{2}}\Big) \sum_{i=1}^m \sum_{j,l,k=1}^n \frac{\partial y_l}{\partial x_j}\frac{\partial y_k}{\partial x_j}\frac{\partial u^i}{\partial y_l}\frac{\partial \phi^i}{\partial y_k}\,dy \\ &\quad + \int_D \sum_{i=1}^m g_i'(u_0)\phi^i\,dy = 0, \end{aligned}$$

for any $\phi \in C_0^1(D, \mathbb{R}^m)$, where we have denoted by $u(y)$ and $g'(u_0(y))$, the functions

$$\begin{gathered} u(y_1 + x_1^0, \dots, y_{n-1} + x_{n-1}^0, y_n + a(y_1 + x_1^0, \dots, y_{n-1} + x_{n-1}^0)), \\ g'(u_0(y_1 + x_1^0, \dots, y_{n-1} + x_{n-1}^0, y_n + a(y_1 + x_1^0, \dots, y_{n-1} + x_{n-1}^0))). \end{gathered}$$

Now we extend u to the whole ball $B = \{y : |y| < \delta\}$, in order to have the equation above satisfied in B. More precisely, for $y_n > 0$ we define

$$\begin{gathered} u(y', -y_n) = -u(y', y_n), \\ g_i'(u_0(y', -y_n)) = -g_i'(u_0(y', y_n)). \end{gathered}$$

Moreover, denoting by e_{lkj} the product $\frac{\partial y_l}{\partial x_j}\frac{\partial y_k}{\partial x_j}$, we also extend e_{lkj} as

$$e_{lkj}(y', -y_n) = \begin{cases} e_{lkj}(y', y_n), & l, k < n \ l = k = n, \\ -e_{lkj}(y', y_n), & \text{otherwise.} \end{cases}$$

With this extensions it is possible to verify that:

$$\begin{aligned} \int_B \Big(1 + \Big|\sum_{i=1}^m \sum_{j,l,k=1}^n e_{lkj} \frac{\partial u^i}{\partial y_l}\frac{\partial u^i}{\partial y_k}\Big|^{\frac{p-2}{2}}\Big) \sum_{i=1}^m \sum_{j,l,k=1}^n e_{lkj} \frac{\partial u^i}{\partial y_l}\frac{\partial \phi^i}{\partial y_k}\, dy & \\ + \int_B \sum_{i=1}^m g_i'(u_0)\phi^i \, dy = 0, & \quad (2.8) \end{aligned}$$

for any $\phi \in C_0^1(B, \mathbb{R}^m)$. Setting $b(\eta, \eta') = \sum_{i,j=1}^m \sum_{l,k=1}^n \gamma_{lk}\delta_{ij}\eta_l^i {\eta'}_k^j$, where $\gamma_{lk} = \sum_{j=1}^n e_{lkj}$, and $A(t) = 1 + |t|^{\frac{p-2}{2}}$, (2.8) becomes

$$\int_B A\left(b(\nabla u, \nabla u)\right) b(\nabla u, \nabla\phi)\; dy + \int_B \sum_{i=1}^m g_i'(u_0)\phi^i \, dy = 0, \quad \forall \phi \in C_0^1(B, \mathbb{R}^m)$$

We now prove that (γ_{lk}) is positive definite. Indeed, it is sufficient to observe that in D

$$e_{lkj} = \begin{cases} \delta_{lj}\delta_{kj}, & l, k < n, \\ -\delta_{lj}\frac{\partial a}{\partial x_j}, & l < n, k = n, \\ -\delta_{kj}\frac{\partial a}{\partial x_j}, & k < n, l = n, \\ (\frac{\partial a}{\partial x_j})^2, & l = k = n, \end{cases}$$

in the case $j < n$, while $e_{lkn} = \delta_{ln}\delta_{kn}$. This implies that

$$\gamma_{lk} = \begin{cases} 1, & l = k < n, \\ -\frac{\partial a}{\partial x_l}, & l < n, k = n, \\ -\frac{\partial a}{\partial x_k}, & l = n, k < n, \\ 1 + |\nabla a|^2, & l = k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently there exist two positive constants λ, λ' such that

$$\begin{gathered} b(\eta, \eta) \geq \lambda|\eta|^2, \ \forall \eta \in \mathbb{R}^{nm}, \\ \sum_{i,j=1}^m \delta_{ij} + \sum_{l,k=1}^n \|\gamma_{lk}\|_{C^1(\overline{B})} \leq \lambda'. \end{gathered}$$

For every $t > 0$,

$$\lambda(1+t)^{p-2} \le A(t^2) \le \lambda'(1+t)^{p-2},$$
$$(\lambda - \frac{1}{2})A(t) \le t\frac{\partial A}{\partial t}(t) \le \lambda' A(t),$$
$$t^2 \left| \frac{\partial^2 A}{\partial t^2}(t) \right| \le \lambda' A(t),$$
$$\sum_{i=1}^{m} |g_i'(u_0(y))| \le \lambda', \ \forall y \in B.$$

In particular we can apply the interior estimate in [13] to get that $u \in C^1(B, \mathbb{R}^m)$ and so the proof of the theorem. $\square$

3. Critical groups computations

The functional I is C^2 on $W_0^{1,p}(\Omega, \mathbb{R}^m)$ and critical points of I on $W_0^{1,p}(\Omega, \mathbb{R}^m)$ are weak solutions to (1.1). In fact, since

$$I(u) = \frac{1}{p}\int_\Omega \Big(\sum_{i=1}^{m}\sum_{\alpha=1}^{n} (\frac{\partial u^i}{\partial x_\alpha})^2\Big)^{\frac{p}{2}} dx + \frac{1}{2}\int_\Omega \sum_{i=1}^{m}\sum_{\alpha=1}^{n} (\frac{\partial u^i}{\partial x_\alpha})^2 dx + \int_\Omega g(u)\, dx,$$

we easily have for any $u, v \in W_0^{1,p}(\Omega, \mathbb{R}^m)$

$$\begin{aligned}\langle I'(u), v\rangle &= \int_\Omega \Big(\sum_{i=1}^{m}\sum_{\alpha=1}^{n} (\frac{\partial u^i}{\partial x_\alpha})^2\Big)^{\frac{p-2}{2}} \sum_{i=1}^{m}\sum_{\alpha=1}^{n} \frac{\partial u^i}{\partial x_\alpha}\frac{\partial v^i}{\partial x_\alpha}\, dx \\ &\quad + \int_\Omega \sum_{i=1}^{m}\sum_{\alpha=1}^{n} \frac{\partial u^i}{\partial x_\alpha}\frac{\partial v^i}{\partial x_\alpha}\, dx + \int_\Omega \sum_{i=1}^{m} g_i'(u) v^i dx, .\end{aligned}$$

This will be expressed briefly as

$$\langle I'(u), v\rangle = \int_\Omega \Big(1 + |\nabla u|^{p-2}\Big)(\nabla u|\nabla v)\, dx + \int_\Omega (g'(u)|v)\ dx.$$

It is easy to prove that the second order differential of I in u is given by

$$\begin{aligned}\langle I''(u)\, v, w\rangle &= \int_\Omega (1 + |\nabla u|^{p-2})(\nabla v|\nabla w)\, dx \\ &\quad + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u|\nabla v)(\nabla u|\nabla w)\, dx + \int_\Omega (g''(u)v|w)\, dx\end{aligned}$$

for any $v, w \in W_0^{1,p}(\Omega, \mathbb{R}^m)$.

Now let us fix $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ a critical point of I. As mentioned in the introduction, the operator $I''(u)$ is not a Fredholm operator, thus any generalized splitting Morse lemma fails. To compute the critical groups in the critical point u, we introduce an auxiliary Hilbert space, depending on the critical point u, in which $W_0^{1,p}(\Omega, \mathbb{R}^m)$ can be embedded, so that a natural splitting can be obtained. By Theorem 2.3, we have that $u \in C^1(\overline{\Omega}, \mathbb{R}^m)$. Therefore the vector function $r^i(x) = |\nabla u(x)|^{(p-4)/2}\nabla u^i(x)$ belongs to $C^0(\overline{\Omega}, \mathbb{R}^m)$, for each $i = 1, \dots, m$. We write $r = (r^1, \dots, r^m)$. Let H_r be the closure of $C_0^\infty(\Omega, \mathbb{R}^m)$ under the scalar product

$$(v, w)_r = \int_\Omega (1 + |r|^2)(\nabla v|\nabla w)\ dx + (p-2)(r|\nabla v)(r|\nabla w)\, dx$$

and let $\langle \cdot, \cdot \rangle_r : H_r^* \times H_r \to \mathbb{R}$ denote the duality pairing in H_r.

We emphasize that the space H_r is $W_0^{1,2}(\Omega, \mathbb{R}^m)$ equipped by an equivalent Hilbert structure, which depends on the critical point u, being suggested by $I''(u)$ itself. In such a way $W_0^{1,p}(\Omega, \mathbb{R}^m) \subset H_r$ continuously. Furthermore $I''(u)$ can be extended to a Fredholm operator $L_r : H_r \to H_r^*$ defined by setting

$$\langle L_r v, w \rangle_r = (v, w)_r \ + \langle Kv, w \rangle_r$$

where $\langle Kv, w \rangle_r = \int_\Omega (g''(u)v|w)\, dx$ for any $v, w \in H_r$. As L_r is a compact perturbation of the Riesz isomorphism from H_r to H_r^*, then L_r is a Fredholm operator with index zero. We can consider the natural splitting

$$H_r = H^- \oplus H^0 \oplus H^+$$

where H^-, H^0, H^+ are, respectively, the negative, null, and positive spaces, according to the spectral decomposition of L_r in $L^2(\Omega, \mathbb{R}^m)$.

Furthermore, denoting by $\|\cdot\|_r$ the norm induced by $(\cdot, \cdot)_r$, it is obvious that there exists $c > 0$ such that

$$\langle L_r v, v \rangle_r + c \int_\Omega |v|^2\, dx \geq \|v\|_r^2 \quad \forall v \in H_r.$$

Hence it follows that there exists $\mu > 0$ such that

$$\langle L_r v, v \rangle_r \geq \mu \|v\|_r^2 \quad \forall v \in H^+. \tag{3.1}$$

We claim now that $H^- \oplus H^0 \subset W_0^{1,p}(\Omega, \mathbb{R}^m)$. Indeed, this is a direct consequence of the fact that every $v \in H_r$ which is a weak solution of the equation $L_r v + \eta v = 0$, belongs to $W_0^{1,p}(\Omega, \mathbb{R}^m)$. For such a solution v,we have

$$\begin{aligned} &\int_\Omega (1 + |r|^2)(\nabla v|\nabla w)\, dx + (p-2)(r|\nabla v)(r|\nabla w)\, dx \\ &\qquad + \int_\Omega (g''(u)v|w)\, dx + \int_\Omega (\eta v|w)\, dx = 0, \end{aligned}$$

or equivalently

$$\begin{aligned} &\int_\Omega (1+|r|^2) \sum_{i=1}^m \sum_{\alpha=1}^n \frac{\partial v^i}{\partial x_\alpha} \frac{\partial w^i}{\partial x_\alpha} dx \\ &+(p-2) \int_\Omega \Big(\sum_{i=1}^m \sum_{\alpha=1}^n r_\alpha^i \frac{\partial v^i}{\partial x_\alpha} \Big) \Big(\sum_{j=1}^m \sum_{\beta=1}^n r_\beta^j \frac{\partial w^j}{\partial x_\beta} \Big)\, dx \\ &+ \int_\Omega \sum_{i=1}^m \sum_{j=1}^m D_{ij} g(u) v^j w^i\, dx + \int_\Omega \sum_{i=1}^m \eta v^i w^i\, dx = 0. \end{aligned}$$

After re-ordering the sums, we have that for all $w \in W_0^{1,2}(\Omega,\mathbb{R}^m)$,

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^m \sum_{\alpha,\beta=1}^n \Big(\big(\delta_{ij}\delta_{\alpha\beta}(1+|r|^2)+(p-2)r_\alpha^i r_\beta^j\big)D_\alpha v^i D_\beta w^j\Big)&\\ +\int_\Omega \sum_{i=1}^m \Big(\sum_{j=1}^m (D_{ij}g(u)+\eta\delta_{ij})v^j\Big)w^i = 0,&\end{aligned} \tag{3.2}$$

Let us set $A_{\alpha\beta}^{ij}(x) = \big(\delta_{ij}\delta_{\alpha\beta}(1+|r|^2)+(p-2)r_\alpha^i r_\beta^j\big)$ and $l_i(x) = \sum_{j=1}^m (D_{ij}g(u)+\eta\delta_{ij})v^j$, then (3.2) is equivalent to

$$\int_\Omega \sum_{i,j=1}^m \sum_{\alpha,\beta=1}^n A_{\alpha\beta}^{ij}(x) D_\alpha v^i D_\beta w^j + \int_\Omega \sum_{i=1}^m l_i(x) w^i = 0, \quad \forall w \in W_0^{1,2}(\Omega,\mathbb{R}^m).$$

By Theorem 2.3 we have $A_{\alpha\beta}^{ij} \in C^{0,\mu}(\overline{\Omega},\mathbb{R}^m)$ and clearly satisfies a strict ellipticity condition, in fact

$$\sum_{i,j}\sum_{\alpha,\beta} A_{\alpha\beta}^{ij}(x)\xi_\alpha^i\xi_\beta^j \ge |\xi|^2, \quad \forall\, x \in \Omega.$$

Moreover, for v fixed in the definition of $l_i(x)$ we have that $l_i \in L^q(\Omega)$ if $v \in L^q(\Omega,\mathbb{R}^m)$ for some $q > 1$. So we can use the result in [7, pp. 73-74] with $q = 2$ to conclude that $v \in W^{1,2^*}(\Omega,\mathbb{R}^m)$. Then we can choose $q = 2^*$ and apply again the result in an iterative scheme, thus after a finite number of steps we have $v \in W^{1,q}(\Omega,\mathbb{R}^m)$ for some $q > n$, which implies in particular that v is locally Hölder continuous.

At this point we can use [7, Theorem 3.5] to get that $v \in C^{1,\mu}(\Omega,\mathbb{R}^m)$ and as a consequence $v \in W_0^{1,p}(\Omega,\mathbb{R}^m)$. Consequently, denoted by $W = H^+ \cap W_0^{1,p}(\Omega,\mathbb{R}^m)$ and $V = H^- \oplus H^0$, we get the splitting

$$W_0^{1,p}(\Omega,\mathbb{R}^m) = V \oplus W$$

and, by (3.1) we infer

$$\langle I''(u)v,v\rangle \ge \mu\|v\|_r^2 \quad \forall v \in W.$$

In particular $m^*(I,u) = \dim V$ is finite.

Following the arguments in Lemma 4.4 in [4], it is possible to prove a sort of local weak convexity along the direction of W. More precisely, for any $M > 0$ there exist $r_0 > 0$ and $C > 0$ such that for any $z \in W_0^{1,p}(\Omega,\mathbb{R}^m) \cap L^\infty(\Omega,\mathbb{R}^m)$ with $\|z\|_\infty \le M$, $\|z-u\| < r_0$, we have

$$\langle I''(z)w,w\rangle \ge C\|w\|_r^2 \quad \forall w \in W. \tag{3.3}$$

An essential tool in these arguments is an abstract result, due to Ioffe [9], concerning sequentially lower semicontinuity of integral functionals with respect to mixed strong-weak convergence, both in the scalar and vectorial case. The inequality (3.3) is sufficient to obtain a finite-dimensional reduction.

Lemma 3.1. *There exist $r > 0$ and $\rho \in]0,r[$ such that for any $v \in V \cap \overline{B}_\rho(0)$ there exists one and only one $\overline{w} \in W \cap \overline{B}_r(0)$ such that for any $z \in W \cap \overline{B}_r(0)$ we have*

$$I(v+\overline{w}+u) \le I(v+z+u).$$

Moreover $\overline{w}$ is the only element in $W \cap \overline{B}_r(0)$ such that $\langle I'(u+v+\overline{w}), z\rangle = 0$ for all $z \in W$.

Proof. Firstly, we observe that 0 is a local minimum for I along the direction of W. This can be proved arguing as in [4, Lemma 4.5]. As in Lemma 2.2, it is possible to prove an uniform L^∞-bound for the critical points of I along W, which are sufficiently close to u. The claim of Lemma 3.1 can be finally deduced by similar arguments to those used in [4, Lemma 4.6]. $\square$

Proof of Theorem 1.2 and Theorem 1.3 completed. We can introduce the map

$$\psi : V \cap \overline{B}_\rho(0) \to W \cap B_r(0)$$

where $\psi(v) = \overline{w}$ is the unique minimum point of the function $w \in W \cap \overline{B}_r(0) \mapsto I(u + v + w)$, and the function

$$\phi : V \cap \overline{B}_\rho(0) \to \mathbb{R}$$

defined by $\phi(v) = I(u + v + \psi(v))$. It is not difficult to show that the maps ψ and ϕ are continuous. Using a suitable pseudogradient flow, like in section 5 of [4] it can be proved that

$$C_j(I, u) = C_j(\phi, 0). \tag{3.4}$$

In particular, if $I''(u)$ is injective, it can be deduced that 0 is a local maximum of ϕ in $V \cap \overline{B}_\rho(0)$, so that by (3.4) Theorem 1.2 comes.

More generally, not requiring the injectivity of $I''(u)$, it is clear that $C_j(\phi, 0) = \{0\}$ when $j \geq m^*(I, u) + 1 = \dim V + 1$. Finally [10, Theorem 2.6] assures that $C_j(\phi, 0) = \{0\}$ if $j \leq m(I, u) - 1$ and thus Theorem 1.3 follows.

Acknowledgement. This work has been developed while the first author was visiting the Dipartimento Interuniversitario di Matematica, Università degli Studi di Bari-Politecnico di Bari. He gratefully acknowledges the whole department for the warm hospitality and the friendly atmosphere. This work was supported by Azione Integrata Italia-Spagna HI2000-0108.

References

[1] K.C. Chang, *Morse Theory on Banach space and its applications to partial differential equations*, Chin. Ann. of Math., **4B** (1983), 381–399.

[2] K.C. Chang, *Morse theory in nonlinear analysis*, in Nonlinear Functional Analysis and Applications to Differential Equations, A. Ambrosetti, K. C. Chang, I. Ekeland Eds., Word Scientific Singapore, 1998.

[3] S. Cingolani, G. Vannella, *Some results on critical groups estimates for a class of functionals defined on Sobolev Banach spaces*, Rend. Mat. Acc. Naz. Lincei, **12**, 199–201 (2001).

[4] S. Cingolani, G. Vannella, *Critical groups computations on a class of Sobolev Banach spaces via Morse index*, Ann. Inst. H. Poincaré: analyse non linéaire, **2**, 271–292 (2003).

[5] S. Cingolani, G. Vannella, *Marino-Prodi perturbation type results and Morse indices of minimax critical points for a class of functionals defined in Banach spaces*, preprint.

[6] H. Egnell, *Existence and Nonexistence Results for m-Laplace Equations Involving Critical Sobolev Exponents*, Arch. Rational Mech. Anal., **104** (1988), 57-77.

[7] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Lectures in Mathematics. Birkhauser, 1993.

[8] E. Hewitt, K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York Heidelberg Berlin, 1975.

[9] A. D. Ioffe, *On lower semicontinuity of integral functionals I and II*, S.I.A.M. J. Control Optim., **15** (1977), 521–538, and 991–1000.

[10] S. Lancelotti, *Morse index estimates for continuous functionals associated with quasilinear elliptic equations*, Advances Diff. Eqs., **7** (2002), 99–128.

[11] A. Marino, G. Prodi, *Metodi perturbativi nella teoria di Morse*, Boll. U.M.I., (4) **11** Suppl. fase 3 (1975), 1–32.

[12] G. Stampacchia, *Équations Elliptiques du second ordre á Coefficients Discontinus*, Les Presses de L'Université de Montréal., 1965.
[13] P. Tolksdorf, *Everywhere-Regularity for some quasilinear systems with a lack of ellipticity*, Annali di Matematica Pura ed Applicata, **134** (1983), 241-266.
[14] P. Tolksdorf, *On the Dirichlet problem for Quasilinear Equations in Domains with Conical Boundary Points*, Communications in Partial Differential Equations, **8** (1983), 773-811.
[15] A. J. Tromba, *A general approach to Morse theory*, J. Diff. Geometry, **12** (1977), 47–85.
[16] K. Uhlenbeck, *Morse theory on Banach manifolds*, J. Funct. Anal., **10** (1972), 430–445.

José Carmona
Department Álgebra y Análisis Matemático Universidad de Almería, Almería, Spain
E-mail address: jcarmona@ual.es

Silvia Cingolani
Dipartimento di Matematica, Politecnico di Bari, Bari, Italy
E-mail address: cingolan@dm.uniba.it

Giuseppina Vannella
Dipartimento di Matematica, Politecnico di Bari, Bari, Italy
E-mail address: vannella@dm.uniba.it