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AN ABSTRACT APPROACH TO SOME SPECTRAL PROBLEMS OF DIRECT SUM DIFFERENTIAL OPERATORS

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Abstract. In this paper, we study the common spectral properties of abstract self-adjoint direct sum operators, considered in a direct sum Hilbert space. Applications of such operators arise in the modelling of processes of multi-particle quantum mechanics, quantum field theory and, specifically, in multi-interval boundary problems of differential equations. We show that a direct sum operator does not depend in a straightforward manner on the separate operators involved. That is, on having a set of self-adjoint operators giving a direct sum operator, we show how the spectral representation for this operator depends on the spectral representations for the individual operators (the coordinate operators) involved in forming this sum operator. In particular it is shown that this problem is not immediately solved by taking a direct sum of the spectral properties of the coordinate operators. Primarily, these results are to be applied to operators generated by a multi-interval quasi-differential system studied, in the earlier works of Ashurov, Everitt, Gesztezy, Kirsch, Markus and Zettl. The abstract approach in this paper indicates the need for further development of spectral theory for direct sum differential operators.

1. Preliminaries

Below, we follow the idea for multi-interval quasi-differential operators, see [1, 2], to construct and extend their results to the general case of abstract operators in Hilbert space. All the information on spectral theory of abstract linear operators required in this paper may be found, for instance, in the texts [4, 5].

Let Ω be finite or countable set of indices; designate $\omega = \mathtt{card}\,(\Omega)$. Consider a family of separable Hilbert spaces $\{\mathcal{H}_i\}_{i\in\Omega}$ and a family of self-adjoint operators $\{T_i\}_{i\in\Omega}$, such that

$$T_i : D(T_i) \subseteq \mathcal{H}_i \to \mathcal{H}_i.$$

We introduce the sum Hilbert space $\mathcal{H} = \oplus_{i\in\Omega}\mathcal{H}_i$, consisting of vectors $\overline{x} = \oplus_{i\in\Omega} x_i$, such that $x_i \in \mathcal{H}_i$ and

$$\|\overline{x}\|^2_{\mathcal{H}} = \sum_{i\in\Omega} \|x_i\|^2_i < \infty, \tag{1.1}$$

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where $\|\cdot\|_i^2$ are norms in $\mathcal{H}_i$. In this direct sum space $\mathcal{H}$ consider the operator

$$T : D(T) \subseteq \mathcal{H} \to \mathcal{H},$$

defined on the domain

$$D(T) = \{\overline{x} \in \mathcal{H} : \sum_{i\in\Omega} \|T_i x_i\|_i^2 < \infty\} \tag{1.2}$$

by $T\overline{x} = \oplus_{i\in\Omega} T_i x_i$. Clearly, if the operator T_i is self-adjoint, for all $i \in \Omega$, then and only then is T self-adjoint.

For each T_i there is a unique resolution of the identity E_λ^i and a unitary operator U_i, giving an isometric isomorphic mapping of the space $\mathcal{H}_i$ on to the space $L^2(M_i, \mu_i)$, such that the operator T_i in $\mathcal{H}_i$ is represented as the multiplication operator in $L^2(M_i, \mu_i)$. Below, we present a more detailed structure of the mapping U_i.

Fix $i \in \Omega$. We call $\phi \in \mathcal{H}_i$ a *cyclic vector*, if for each $z \in \mathcal{H}_i$ there exists a Borel measureable function f, such that $z = f(T_i)\phi$. Generally, such a cyclic vector does not exist in $\mathcal{H}_i$, but there is a collection $\{\phi^k\}$ of vectors in $\mathcal{H}_i$, such that $\mathcal{H}_i = \oplus^k \mathcal{H}_i(\phi^k)$, where $\mathcal{H}_i(\phi^k)$ are T_i-invariant subspaces in $\mathcal{H}_i$, generated by cyclic vectors ϕ^k, that is

$$\mathcal{H}_i(\phi^k) = \overline{\{f(T_i)\phi^k\}},$$

varying Borel function f, such that $\phi \in D(f(T_i))$. Then there exist unitary operators

$$U^k : \mathcal{H}_i(\phi^k) \to L^2(\mathbb{R}, \mu^k),$$

where $\mu^k(\Delta) = \|E_\Delta^i \phi^k\|_i^2$, for any Borel set Δ. In $L^2(\mathbb{R}, \mu^k)$, the operator T_i has the form of multiplication by λ, i.e.

$$\left(U^k T_i|_{\mathcal{H}_i(\phi^k)} {U^k}^{-1} z\right)(\lambda) = \lambda z(\lambda).$$

Then the operator

$$U_i = \oplus^k U^k : \oplus^k \mathcal{H}_i(\phi^k) \to \oplus^k L^2(\mathbb{R}, \mu^k)$$

gives the spectral representation of the space $\mathcal{H}_i$ in the space $L^2(M_i, \mu_i)$, where M_i is a union of nonintersecting copies of the real line (*sliced union*) and $\mu_i = \sum_k \mu^k$. That is

$$(U_i T_i U_i^{-1} z)(\lambda) = f(\lambda) z(\lambda),$$

where $z \in U[D(T_i)]$ and f is a Borel function defined almost everywhere according to the measure μ_i.

2. Spectral properties of the operator T

In this section it is shown how spectral representation of the direct sum operator T may depend on spectral representations of the given operators T_i. For this purpose, we first prove some auxiliary statements.

Definition 2.1. For $i \in \Omega$, we introduce a *sliced union* of sets M_i (see preliminaries) as a set M, containing all M_i on different copies of $\cup_{i\in\Omega} M_i$. In this set M, the sets M_i do not intersect, but they may *superpose*, i.e. two sets M_i and M_j superpose, if their projections in the set $\cup_{i\in\Omega} M_i$ intersect.

Separate arguments show that the following auxiliary proposition is valid

Proposition 2.2. *Let us have a set of measures μ_i, $i \in \Omega$, defined on nonintersecting supports. If*
$$\sum_{i\in\Omega}\int_{-\infty}^{\infty} f(\lambda)\,d\mu_i(\lambda) < \infty,$$
for any borel function $f(\lambda)$, then the following equality is true:
$$\sum_{i\in\Omega}\int_{-\infty}^{\infty} f(\lambda)\,d\mu_i(\lambda) = \int_{-\infty}^{\infty} f(\lambda)\,d\sum_{i\in\Omega}\mu_i(\lambda). \tag{2.1}$$

Using this proposition, we prove the following lemmas:

Lemma 2.3. *The resolution of the identity E_λ of the system operator T is given by the direct sum of resolutions of the identity E^i_λ of the operators T_i; that is*
$$E_\lambda = \oplus_{i\in\Omega}E^i_\lambda \tag{2.2}$$

Proof. Consider $\overline{x} \in D(T)$; this holds if and only if
$$\|T\overline{x}\|^2_{\mathcal{H}} = \sum_{i\in\Omega}\|T_i x_i\|^2_i = \sum_{i\in\Omega}\int_{-\infty}^{\infty}\lambda^2\,d\|E^i_\lambda x_i\|^2_i < \infty.$$
Recall that we consider the sets $M_i = \operatorname{supp}\{\|E^i_\lambda x_i\|^2_i\}$ divided in the sense of slicing, so that they do not intersect. Using Proposition 2.2 implies that the following equality is true:
$$\sum_{i\in\Omega}\int_{-\infty}^{\infty}\lambda^2\,d\|E^i_\lambda x_i\|^2_i = \int_{-\infty}^{\infty}\lambda^2\,d\sum_{i\in\Omega}\|E^i_\lambda x_i\|^2_i. \tag{2.3}$$
In turn this implies that $\overline{x} \in D(T)$, if and only if
$$\int_{-\infty}^{\infty}\lambda^2\,d\sum_{i\in\Omega}\|E^i_\lambda x_i\|^2_i < \infty;$$
and
$$\|T\overline{x}\|^2_{\mathcal{H}} = \int_{-\infty}^{\infty}\lambda^2\,d\sum_{i\in\Omega}\|E^i_\lambda x_i\|^2_i.$$
Using the uniqueness property of a resolution of the identity, these two statements show that the operator $\oplus_{i\in\Omega}E^i_\lambda$ is the resolution of the identity of the system operator T, that is, according to our notations $E_\lambda = \oplus_{i\in\Omega}E^i_\lambda$. This completes the proof of the lemma. □

Lemma 2.4. *For any Borel function f, and any vector $\overline{x} \in D(f(T))$, the following equality is satisfied: $f(T)\overline{x} = [\oplus_{i\in\Omega}f(T_i)]\overline{x}$.*

Proof. Let $\overline{x} \in D(f(T))$. Then from Proposition 2.2 and Lemma 2.3, we obtain, for any $\overline{y} \in \mathcal{H}$:
$$\begin{aligned}(f(T)\overline{x},\overline{y})_{\mathcal{H}} &= \int_{-\infty}^{\infty} f(\lambda)\,d(E_\lambda\overline{x},\overline{y})_{\mathcal{H}} = \int_{-\infty}^{\infty} f(\lambda)\sum_{i\in\Omega}(E^i_\lambda x_i, y_i)_i = \\ &= \sum_{i\in\Omega}\int_{-\infty}^{\infty} f(\lambda)\,d(E^i_\lambda x_i, y_i)_i = \sum_{i\in\Omega}(f(T_i)x_i, y_i)_i = ([\oplus_{i\in\Omega}f(T)]\overline{x},\overline{y})_{\mathcal{H}}.\end{aligned}$$
Since $\overline{y}$ is arbitrary, we have $f(T)\overline{x} = [\oplus_{i\in\Omega}f(T_i)]\overline{x}$. This completes the proof of the lemma. □

For $\varphi_i \in \mathcal{H}_i$, $i \in \Omega$, define

$$\overline{\varphi_i} = \{0, \dots, 0, \varphi_i, 0, \dots, 0\} \in \mathcal{H},$$

where φ_i is on i-th place. Consider a projection $P : M \to \cup_{i\in\Omega} M_i$ (see Definition 2.1), such that $P(\sigma(T_i)) = \sigma(T_i)$.

Definition 2.5. Divide Ω into non-intersecting sets

$$\begin{aligned} A_k = \{s \in \Omega : \forall s, l \in A_k, s \neq l, P(\sigma(T_s)) \cap P(\sigma(T_l)) = B_{sl}, \\ \text{where } \|E^t_{B_{sl}} \varphi_t\|^2_t = 0, \text{ for any cyclic } \varphi_t \in \mathcal{H}_t,\ t = s, l\}. \end{aligned} \tag{2.4}$$

From all possible divisions of this type, we choose and fix the one which contains the minimal number of the sets A_k. With this notation, we call the number $\Lambda = \min\{k\}$ a *spectral index* of the direct sum operator T.

Theorem 2.6. *Let each T_i has a cyclic vector ϕ_i in $\mathcal{H}_i$. Then the operator T has minimum Λ of cyclic vectors $\{\overline{\xi}_k\}_{k=1}^{\Lambda}$, having the form*

$$\overline{\xi}_k = \sum_{i\in A_k} \overline{\phi_i}. \tag{2.5}$$

Proof. First consider the case of two operators. Let $s, l \in \Omega$; then, in order to obtain one cyclic vector in $\mathcal{H}_s \oplus \mathcal{H}_l$ having the form $\phi_s \oplus \phi_l$, for any $\overline{x} = x_s \oplus x_l \in \mathcal{H}_s \oplus \mathcal{H}_l$ it is necessary to find a Borel function f, such that

$$\overline{x} = f(T_s \oplus T_l)[\phi_s \oplus \phi_l].$$

From Lemma 2.4 it follows that

$$\overline{x} = [f(T_s) \oplus f(T_l)][\phi_s \oplus \phi_l].$$

On the other hand we must determine each space $\mathcal{H}_p$ $(p = s, l)$ by closing the set $\{f_p(T_p)\phi_p\}$, where f_p varies over all Borel functions such that $\phi_p \in D(f_p(T_p))$. If $s, l \in A_k$, then supposing that $f = f_p$ on $P(\sigma(T_p))$, we obtain the required function f, since functions in the isomorphic space L^2 are considered equal on any set of measure zero. Hence, it is clear, that for all $i \in A_k$, we may construct a single cyclic vector of the form

$$\overline{\xi}_k = \oplus_{i\in A_k} \phi_i = \sum_{i\in A_k} \overline{\phi_i},$$

using the process described above, considering pairs of operators.

We recall that we have a minimal number of the sets A_k. Consider the Hilbert space

$$[\oplus_{i\in A_k} \mathcal{H}_i] \oplus [\oplus_{j\in A_q} \mathcal{H}_j] \text{ with } k \neq q. \tag{2.6}$$

It follows then that the set

$$[\cup_{i\in A_k} P(\sigma(T_i))] \cap [\cup_{j\in A_q} P(\sigma(T_j))] = B_{kq}$$

has a non-zero spectral measure. From the above results it follows that, by joining cyclic vectors $\overline{\xi}_k = \oplus_{i\in A_k} \phi_i$ and $\overline{\xi}_q = \oplus_{j\in A_q} \phi_j$ into one

$$\overline{\xi}_k + \overline{\xi}_q = \sum_{i\in A_k} \overline{\phi_i} + \sum_{j\in A_q} \overline{\phi_j},$$

we have to obtain the Hilbert space (2.6), by closing the set

$$\{f_k(\oplus_{i\in A_k} T_i)\overline{\xi}_k\} \oplus \{f_q(\oplus_{j\in A_q} T_j)\overline{\xi}_q\},$$

with varying Borel functions f_k and f_q, which coincide on B_{kq}. This is not possible, since the set of such functions is not dense in the isomorphic space L^2 (isomorphism is understood under spectral representation of the space (2.6)). Hence, we have Λ cyclic vectors

$$\bar{\xi}_k = \sum_{i\in A_k} \overline{\phi_i} \in \mathcal{H},\ k = \overline{1,\Lambda}.$$

This completes the proof of the theorem. □

Corollary 2.7. *Let each T_i has a single cyclic vector. Then*
1. $\Lambda = 1$ if and only if the operators $T_i, i \in \Omega$, have almost everywhere (relatively to the spectral measure) pairwise non-superposing spectra.
2. a) $\omega < \aleph_0$. $\Lambda = \omega$, if and only if all the operators T_i have pairwise superposing spectra. b) $\omega = \aleph_0$. $\Lambda = \infty$, if and only if all the operators T_i have pairwise superposing spectra, except maybe a finite number of these operators.

Proof. The proof directly follows from the results given in the proof of Theorem 2.6. □

Remark 2.8. Note, that these two cases are contiguous. Case 2 is the most natural for applications, in particular for the direct sum of differential operators.

Now suppose that each operator T_i has m_i cyclic vectors. Then, there exists a decomposition

$$T = \oplus_{i\in\Omega} T_i = \oplus_{i\in\Omega}\oplus_{k=i}^{m_i} T_i^k = \oplus_s T_s, \tag{2.7}$$

where each T_s has a single cyclic vector. For the operator T, decomposed as above, we apply Theorem 2.6. This implies that we can find spectral index Λ for the operator T, decomposed as in (2.7). It is clear, in this case, that there exists an estimate for the spectral index given by

$$\Lambda \geqslant \max\{m_i\}. \tag{2.8}$$

As it has been stated above, for each operator T_i there exists a unitary operator U_i, such that $U_i : \mathcal{H}_i \to L^2(M_i, \mu_i)$. Hence

$$\oplus_{i\in\Omega} U_i : \oplus_{i\in\Omega}\mathcal{H}_i \to \oplus_{i\in\Omega} L^2(M_i, \mu_i).$$

In the general case (i.e. when there are T_i with more then one cyclic vector),

$$\oplus_{i\in\Omega} U_i : \oplus_{i\in\Omega}\oplus_{k=1}^{m_i}\mathcal{H}_i^k \to \oplus_{i\in\Omega}\oplus_{k=1}^{m_i} L^2(\mathbb{R}, \mu_i^k).$$

From Theorem 2.6 follows that there exists a unitary operator

$$V : \oplus_{i\in\Omega}\oplus_{k=1}^{m_i} L^2(\mathbb{R}, \mu_i^k) = \oplus_s L^2(\mathbb{R}, \mu_s) \to \oplus_{q=1}^{\Lambda} L^2\left(\mathbb{R}, \sum_{j\in A_q} \mu_j\right). \tag{2.9}$$

This implies that for any direct sum operator T, there exists a unitary operator $V \oplus_{i\in\Omega} U_i$, which represents the space $\mathcal{H}$ in the space L_2:

$$V \oplus_{i\in\Omega} U_i : \mathcal{H} \to L^2(N, \mu), \tag{2.10}$$

where N is a sliced union of Λ copies of $\mathbb{R}$ and

$$\mu = \sum_{q=1}^{\Lambda} \sum_{j\in A_q} \mu_j,$$

according to the symbols in (2.9). Furthermore we know, that for each T_s (see (2.7)), there exists a real-valued almost everywhere finite function f_s on $\mathbb{R}$, such that

1) $\psi_s \in D(T_s)$ if and only if $f(\cdot)(U_s\psi)(\cdot) \in L^2(\mathbb{R}, \mu_s)$;

2) if $\phi_s \in U_s[D(T_s)]$, then $(U_s T_s U_s^{-1}\phi_s)(m) = f_s(m)\phi_s(m)$.

Defining $f = f_s\chi_s$, where $\chi_s = 1$ on s-th copy of $\mathbb{R}$, and zero elsewhere, according to the above notations, we obtain

Theorem 2.9. *If the unitary operators U_i give spectral representations of Hilbert spaces $\mathcal{H}_i$ onto the spaces $L^2(M_i, \mu_i)$, then the unitary operator*

$$W = V \oplus_{i\in\Omega} U_i$$

gives a spectral representation of the space $\mathcal{H}$ onto the space $L^2(N, \mu)$. According to this representation, there exists a real-valued almost everywhere finite function f on N, such that

1) $\overline{\psi} \in D(T)$ if and only if $f(\cdot)(W\overline{\psi})(\cdot) \in L^2(N, \mu)$;

2) if $\overline{\phi} \in W[D(T)]$, then $(WTW^{-1}\overline{\phi})(m) = f(m)\overline{\phi}(m)$.

These abstract results appear to be the foundation for our further works where we shall build an ordered representation for a direct sum operator and consider matters connected with eigenvalue expansions for self-adjoint direct sum differential operators.

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