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A COMPARISON PRINCIPLE FOR AN AMERICAN OPTION ON SEVERAL ASSETS: INDEX AND SPREAD OPTIONS

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Abstract. Using the method of symmetrization, we compare the price of the American option on an index or spread to that of the solution of a parabolic variational inequality in one spatial variable. This comparison principle is established for a broad class of diffusion operators with time and state dependent coefficients. The purpose is to take a first step towards deriving symmmetrized problems whose solutions bound solutions of multidimensional American option problems with variable coefficients when the computation of the latter lies beyond the scope of the most powerful numerical methods.

1. Introduction

An American option gives its holder the right to buy a stock or basket of stocks at a given price called the strike prior to but not later than a given time T, from the time of inception of the contract. What distinguishes an American option from a European option is the possibility of early exercise. In this paper we focus on American options on two or more assets. A standard example is an index option that is based on the geometric or arithmetic means of several assets. The S&P 100 index option, traded on the Chicago Board of Options Exchange is an American option on a value weighted index of 100 stocks. Two other examples are American options on the maximum or on the spread of two stocks.

It is well known since 1973 [37] that an American call option on a single stock which pays no dividends and which follows a geometric Brownian motion will not be exercised prior to expiration and therefore is, for valuation purposes, equivalent to a European option. This is not the case for American put options. Moreover, in most cases of practical interest, the stocks underlying the call options pay dividends and early exercise is then often not optimal.

No closed form solutions are known for American options, even in the case of one asset, except for the so-called perpetual option, which is of limited practical interest. In the case of a single underlying asset, a long tradition exists in the finance literature of seeking analytical solutions which yield good approximations to the value of the option and to the value $S_f(t)$, $0 \leq t \leq T$, at which it is optimal to exercise the option. Some good references in this direction are Barone-Adesi and

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Whaley [10] and the recent paper by Ju and Zhong [30] which contains an extensive bibliography of previous work.

The literature on options with several underlying assets is less extensive. A pioneering paper is that of Broadie and Detemple [15] who use probabilistic techniques to describe the shape of the free boundary for some of the most important contracts. Villeneuve [47] further enriched and strengthened the mathematical underpinnings of Broadie and de Temple's work. It should be emphasized that the qualitative results obtained in these papers are set in the standard Black-Scholes framework, in which all parameters, such as the volatility, are constant, and to our knowledge, little is known about the free boundary or about the options value, when we leave this setting. Thus it is desirable to identify *features of the underlying volatility, dividend and interest processes* from which bounds can be derived, that depend on only *partial information* concerning these processes. Note that even in one dimension, the effect of a state dependent or of a stochastically driven volatility, has significant influence on American option's value, as discussed in the recent papers by Broadie, Detemple, Ghysels and Torres [16, 17]) and the assumption that American index option follows a geometric Brownian motion is weakly founded. Moreover in the case of American options on several underlying assets even the most recent numerical methods based on Monte-Carlo algorithms, require extensive computational power, especially in the case of a continuous exercise envisaged here [1] of multiple underlying assets and state and time dependent parameters. Thus it is desirable to develop analytic methods which yield useful and stable bounds, that can be used as benchmarks by the investor. These bounds will not be optimal in general, ie. if all the parameters, such as the volatility, are known with precision, one can in principle obtain sharper bounds or better comparison equations. By "stable" we mean that the bounds change little if the partial information we have about the parameters is altered a little.

In this paper, we address the following question. Given an American option on several assets we seek to obtain upper bounds for its value which rely only on partial information about the volatility, interest rate and dividend processes governing the stocks evolution under the risk-neutral measure. Such partial information is the best that one can expect in most cases. Indeed the process of calibration or "backing out" a reasonable estimate of the stocks volatility from market data is one of the most active areas of research in modern finance, see for instance Bakshi, Cao, and Zhiwu [8]. Of the many approaches that have been taken, perhaps the ones most similar, in the financial context, to the one we will take here, is that of Avellaneda, Levy and Paras [5], and that of Lyons [36], which was applied in the American option setting by Buff [18]. These are similar not in the techniques used, but rather in the types of assumptions that are made about the volatility process. For instance in Buff's development of Avellaneda et al.'s uncertain volatility approach, its assumed that the volatility lies in a "band" $[\sigma_{min}, \sigma_{max}]$.

Many pricing methods for index options assume that the latter follows a geometric Brownian motion. Recent work has found this assumption to be weakly founded in some cases. See Broadie, Detemple, Ghysels and Torres [16]. In considering the

[1] In practice American options permit at best daily exercise. Using even the best Monte Carlo methods, which involves one hundred regressions at each exercise date, an option on 3 underlying assets, with 100 days to expiration and 5000 paths per stock generated, yields a conservative estimate of $100 \times 5000 \times 3 = 150000$ bits to be stored

use of more complicated models, an important consideration is their *tractability*. Monte Carlo methods for options on multiple assets that take into account the daily exercise feature and the multifactor structure are expensive, especially when calibrating market data to a rich structure of input parameters and allowing these to have a non trivial functional form. The method proposed here explores a direction which trades off precision for tractability. At present it produces upper bounds only and further numerical work is required to assess how sharp these are. It's tractability derived from the fact that the upper bounds produced by solving a one dimensional parabolic variational inequality require only a few seconds on a Pentium 2 PC.

However preliminary numerical results indicate that the comparison principle derived in this paper does not produce bounds that are not sharp enough to be of interest in practice. Thus the present paper should be seen as a first attempt to adapt the method of symmetrization to the American option problem and the method will need to be refined in the future (in progress).

To obtain our bounds we will use the method of symmetrization to estimate the solutions of the options on multiple assets in terms of the solution $V_k(r,\tau) : B_k \times [0,T]$ of a spatially one dimensional parabolic variational inequality

$$\begin{aligned}(V_k)_\tau - \lambda_{co}^2(\tau)(V_k)_{rr} - \frac{(n-1)\lambda_{co}^2(\tau)}{r}(V_k)_r + D(\tau)(V_k)_r + C(\tau)V_k - F(r,\tau))&\\ = 0 \quad \text{on } \{V_k > 0\}&\\ \geq 0 \quad \text{on } B_k\,,&\end{aligned} \tag{1.1}$$

where the coefficients $\lambda_{co}(\tau), D(\tau), C(\tau)$ and source term F are determined by the original volatilities, interest rate and dividend rates by a recipe that we will describe in section 3, and where k corresponds to the standard cut-off wherein the problem is localized to a ball of radius k. This equation has time dependent coefficients, with the exception of the term $1/r$ preceding the first order derivative $(V_k)_r$, familiar in physics in deriving the Laplacian in *spherical* coordinates. Because of its appearance in a variety of physical contexts, the case of *equations* of the form (1.1) has been studied numerically by mathematical physicists and efficient algorithms can be adapted to deal with the variational inequality.

The method of symmetrization was introduced by Schwarz citeSc, Steiner [42], and Hardy-Littlewood-Polya [27]. It's close connection with isoperimetric inequalities was realized by Polya and Szego , and summarized in their book. Bandle [9] and Talenti [43] pioneered the introduction of symmetrization and rearrangement techniques in the area of partial differential equations. Kawohl [31] and Mossino [38] described the state of the art and found many refinements in their books. The technique has since been substantially developed in work by Alvino, Lions and Trombetti [1], Gustafsson and Mossino [26], Diaz and Mossino [23], Ferone and Volpicelli [25], and Kesavan [33], to mention only a few. In recent work by Kinateder and Mac Donald [34] the closely connected problem of the distribution of first exit times in its dependence on the domain is considered.

The symmetrization method will be applied to the parabolic variational inequalities modeling the multidimensional American option problem. The theoretical framework for this was provided by Jaillet, Lamberton and Lapeyre Ja-La-La who showed how to adapt the theory described in Bensoussan-Lions Be-Li theory to the present setting.

Our results also apply in the case of an American option on a single asset, when the parameters are state and time dependent, and bound the price of the option by the solution of a variational inequality with purely time dependent coefficients. In light of the recent work of Broadie, Detemple, Ghysels and Torres [16] on the profound effect that stochastic volatility and dividends can have on the price of the American option on one dividend paying option, it is possible that our comparison principle might be of interest also in this setting. To help illustrate what follows in a simple case, in **Figure 1**, we illustrate the value of a put on one dividend paying asset, in the original variables, before the logarithmic transformation is introduced, at a given time t prior to expiration. The payoff function is $(K-S)^+$. The value function is tangent to the payoff at one and only one point which corresponds to the free boundary $S(t)$. In **Figure 2**, we show the same Figure but after the logarithmic change of the independent variable and the change of dependent variable $u \to u/K$. In the new variables the payoff function is $(1-e^x)^+$, so this change of variables also has the effect of making the transition between 'in the money' and out of the money, occur the value at $x = 0$. In **Figure 3** we computed the difference between the option value and the payoff in the new variables. Note that in these new variables, v is nearly *symmetric* around the y axis. It will *not* in general be exactly symmetric, but note that the inverse image of any value $y = c, 0 < c < \max v$ consists exactly of two points. It turns out that this implies that the bounds obtained are sharper if $r \geq d$, then when $r < d$ (and the opposite is true for calls). The reason for this is that v satisfies in the new variables an equation of the form

$$v_\tau - \sigma^2 v_{xx} - (r - d - \sigma^2)v_x + rv = \delta_{\{x=0\}} + (de^x - r)\mathbf{1}_{x<0} \tag{1.2}$$

Note that right hand side of (1.2) is the sum of a delta function at the value $x = 0$ and a function which is monotonically increasing *if and only if* $r \geq d$.

This paper is organized as follows: Section 2 – Formulation of the variational inequalities, background material on symmetrization and isoperimetric inequalities. Section 3 – Parameters of symmetrized comparison problem. Section 4 – Statement of Main Results. Section 5 – Proof of Main Results. Section 6 – Explicit form for regularized inhomogeneous term.

2. Formulation of the Problem

The american option problem is an optimal stopping problem for a vector of stocks $S_t = (S_t^1, \cdots S_t^n)$ which follow a diffusion process

$$dS_t^i = (r(S_t,t) - d_i(S,t))S_t^i dt + \sum_{ij} S_t^i \tilde{\sigma}_{ij}(S,t) dZ_t^j \tag{2.1}$$

where r and d_i are respectively the short rate and the continuously compounded dividend rate of the i-th stock, $\tilde{\sigma}_{ij}(S,t), i = 1, \cdots, n, j = 1, \cdots, n$ is the $n \times n$ dimensional volatility matrix, and $(Z_t)_{t\geq 0}$ is a standard $\mathbb{R}^n$ valued Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with respect to the measure P, where P is the so-called risk neutral measure. Throughout this paper we will frequently use the Einstein summation convention wherein the summation sign is omitted when summing over repeated indices,

The value process u of the american option problem time is a solution of the following problem

$$\tilde{u}(S,t) = \sup_{\tau \in \mathcal{T}} E[e^{-\int_t^\tau r(S_u,u)du} \psi(S_\tau) : S_t = S] \tag{2.2}$$

where the stopping time τ varies over all $\mathcal{F}_t$ adapted random variables and $\psi(S)$ is the option payoff. Here $(\mathcal{F}_t)_{t\geq 0}$ denotes the P completion of the natural filtration associated to $(Z_t)_{t\geq 0}$. Intuitively the optimal stopping problem consists in finding the stopping strategy τ that maximizes the expected gain to the holder of the option. Changing variables as follows

$$\begin{aligned} x_i &= ln(S_i/K), \quad i = 1, \cdots n \\ u &= \tilde{u}/K \\ \tau &= T - t\,, \end{aligned} \tag{2.3}$$

where K is the strike of the option and let $\mathcal{C}_t$, $\mathcal{E}_t$, denote respectively the continuation and exercise region for the option, with $\mathcal{C}_t \cup \mathcal{E}_t = \mathbb{R}^n$. It can then be shown that u is a weak solution (in a sense made precise below) of **Problem** (γ_1):

$$u_\tau - \sigma_{ij}(x,\tau)\frac{\partial^2 u}{\partial x_i \partial x_j} - (r - d_i - \sigma_{ii})\frac{\partial u}{\partial x_i} + ru = 0 \tag{2.4}$$

$$x \in \mathcal{C}_\tau, \quad \tau \in [0,T]$$

$$u = \psi \quad \text{on } \mathcal{E}_\tau \tag{2.5}$$

$$u = \psi \quad \text{on } \partial\mathcal{C}_\tau$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \psi}{\partial \nu}\ quad\text{on } \partial\mathcal{C}_t \tag{2.6}$$

The free boundary condition $\frac{\partial u}{\partial \nu} = \frac{\partial \psi}{\partial \nu}$ on $\partial\mathcal{C}_t$ has a meaning only at regular points of the free boundary and, to our knowledge, no complete analysis of the regularity of the free boundary is presently available, especially in the case $n \geq 2$.

The spatial part of the operator in (2.4) is denoted $\mathcal{L}_S$

$$\mathcal{L}_S = -\sigma_{ij}(x,\tau)\frac{\partial^2}{\partial x_i \partial x_j} - (r - d_i - \sigma_{ii})\frac{\partial}{\partial x_i} + r, \tag{2.7}$$

where

$$\sigma_{ij} = \frac{1}{2K^2}\sum_k \tilde{\sigma}_{ik}\tilde{\sigma}_{jk}\,. \tag{2.8}$$

The rigorous weak formulation of the problem is formulated in terms of variational inequalities and was detailed by Jaillet Lamberton and Lapeyre [29] based on earlier work by Bensoussan and Lions and is subject to the following hypotheses:

- (H1) $r(x,\tau)$ and $d_j(x,\tau), j = 1, ..n$ are bounded C^1 functions from $\mathbb{R}^n \times [0,T]$ to $\mathbb{R}$, with bounded derivatives and $r \geq 0, d_j \geq 0$.
- (H2) The entries $\tilde{\sigma}_{ij}, i, j = 1, \cdots n$, in the matrix $\underline{\tilde{\sigma}}(x,\tau)$, are bounded C^1 functions from $\mathbb{R}^n \times [0,T]$ to $\mathbb{R}$. $\underline{\tilde{\sigma}}$ admits continuous second partial derivatives with respect to x satisfying a Holder condition in x uniformly with respect to (x,τ) in $\mathbb{R}^n \times [0,T]$.
- (H3) The matrix $\underline{\sigma} = \frac{1}{2}\underline{\tilde{\sigma}} \cdot \underline{\tilde{\sigma}}^t$ satisfies the following property: There exists $\eta > 0$ such for all $(x,t) \in [0,T] \times \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$,

$$\sum_{1\leq i,j\leq n} \sigma_{i,j}(x,t)\xi_i\xi_j \geq \eta \sum_{i=1}^n \xi_i^2 \tag{2.9}$$

(H4) The option payoffs ψ considered depend only on x and in addition satisfy the condition: There exists $M > 0$ such for all $x \in \mathbb{R}^n$

$$|\psi(x)| + \sum_{j=1}^{n} |\frac{\partial \psi}{\partial x_j}(x)| \leq M e^{M|x|}\,. \tag{2.10}$$

All option payoffs of interest in practice satisfy this growth condition.

(H5) The interest rate $r(x,\tau)$ is bounded below by some positive constant r_0.

The variational inequalities considered are formulated in certain function spaces which we now describe. Let m be a non negative integer and suppose that $1 \leq p \leq \infty$ and $0 < \mu < +\infty$. $W^{m,p,\mu}$ denotes the space of all functions u whose distributional derivatives of order less than or equal to m lie in $L^p(\mathbb{R}^n, e^{-\mu|x|}dx)$. For brevity we will use the notation H_μ to denote the space $W^{0,2,\mu}(\mathbb{R}^n)$ and V_μ to denote the space $W^{1,p,\mu}$. The inner product on H_μ is denoted $(\cdot,\cdot)_\mu$. Define a a bilinear form on V_μ for each $t \in [0,T]$ as follows: For all $u, v \in V_\mu$,

$$\begin{aligned} a^\mu(\tau,u,v) &= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \sigma_{i,j}(x,\tau) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} e^{-\mu|x|} dx \\ &\quad - \sum_{i}^{n} \int_{\mathbb{R}^n} \left(r_i - d_i - \sigma_{ii} - \sum_{j}^{n} (\sigma_{ij})_{x_j} \right) \frac{\partial u}{\partial x_i} v e^{-\mu|x|} dx \\ &\quad - \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \left(\sigma_{ij} \frac{x_j}{|x|} \frac{\partial u}{\partial x_i} \right) v e^{-\mu|x|} dx + \int_{\mathbb{R}^n} r(x,\tau)\, u v e^{-\mu|x|} dx \end{aligned} \tag{2.11}$$

Coerciveness. There exist constants $\alpha > 0$ and $\rho > 0$ such that for all $\tau \in [0,T]$ and for all $u \in V_\mu$

$$a^\mu(\tau,u,u) + \rho |u|_\mu^2 \geq \alpha \|u\|_\mu^2\,, \tag{2.12}$$

where we use single bars $|\cdot|$ to denote the norm in H_μ and double bars $\|\cdot\|$ to denote the norm in V_μ. The coerciveness is ensured by hypotheses (H3).

Under the regularity and non-degeneracy assumptions on the coefficients, i.e. conditions (H1)–(H5) we have the following result

Theorem 2.1. *If $\psi \in V_\mu$, there exists a unique solution to the following parabolic variational inequalities defined on $[0,T] \times \mathbb{R}^n$*

$$\begin{gathered} u \in L^2([0,T]; V_\mu), \quad \frac{\partial u}{\partial \tau} \in L^2([0,T]; H_\mu) \\ u \geq \psi \quad a.e \text{ in } \mathbb{R}^n \times [0,T], \quad u(0) = \psi \\ \forall v \in V_\mu \quad v \geq \psi \Rightarrow (\frac{\partial u}{\partial \tau}, v - u)_\mu + a^\mu(\tau, u, v-u) \geq 0 \end{gathered} \tag{2.13}$$

where the bilinear form was defined in (2.11).

Remarks 1) The terms $\sum_{i=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_i}(x,\tau)$ and $\sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij}(x,\tau) \frac{x_i}{|x|} \frac{\partial u}{\partial x_j}$ in (2.11), arise from an integration by parts, by bringing the derivative respectively on the volatility coefficient σ_{ij} and on the exponential damping factor $e^{-\mu|x|}$.

2) Unlike Jaillet-Lamberton-Lapeyre, we will work in the backward variable $\tau = T - t$.

The proof of Theorem 2.1 is outlined in Jaillet, Lamberton and Lapeyre. It is based on the treatment in Bensoussan -Lions [12, Chapter 3, Section 4].

Class of Payoffs considered. The payoffs of practical interest for American options on several underlying assets, fall into two principal categories.

Payoffs Class A: These payoffs are illustrated by payoffs on a basket or on a spread. After normalizing the strike to be equal to one with the change of variables (2.3) these payoffs may be written

$$\eta(x) = (x)^+ \tag{2.14}$$

$$\psi_C = \eta(\Phi_C), \quad \Phi_C = \sum_{i=1}^{n} (w_i e^{x_i} - 1) \tag{2.15}$$

$$\psi_P = \eta(\Phi_P), \quad \Phi_P = \sum_{i=1}^{n} (1 - w_i e^{x_i}) \tag{2.16}$$

where for an index option all constants w_i's are positive and for a spread some of the w_i are positive and others negative. The simplest example of an index option is an option on the average of two assets where $\psi_C = (\frac{1}{2}(e^{x_1} + e^{x_2}) - 1)^+$ for a spread on two assets $\psi_C = (e^{x_1} - e^{x_2} - 1)^+$.

Payoffs Class B: Consider $\eta(\max_{i=1}^{n}(\Phi_i))$ where Φ_i is a smooth function. In the present paper *we will limit the discussion to payoffs of class A*. Option payoffs of class B require an additional regularization in our treatment due to the presence of both positive part and max in their definition. This will likely decrease the tightness of the upper bounds and is best addressed in the context of a different kind of symmetrization.

Payoff functions of Class A are only Lipschitz and so, for technical reasons, we will need to regularize them. This is achieved by approximating the payoff ψ by a function ψ_ϵ which lies in $W^{2,p,\mu}(\mathbb{R}^n)$ and such that $\psi_e \to \psi$ uniformly in $\mathbb{R}^n$. The explicit form of this regularization will not play a role until §6, see (6.6). Denoting for brevity, the solution of obstacle problem for given ψ by $u[\psi]$, our strategy below will be to obtain estimates for $u[\psi_\epsilon]$ and to then carry over these estimates to $u(\psi)$ using the following result: Under the same conditions as those in Theorem 2.1, we have that

$$\|u[\psi] - u[\psi_\epsilon]\|_{L^\infty([0,T]\times\mathbb{R}^n])} \le \|\psi - \psi_\epsilon\|_{L^\infty(\mathbb{R}^n)} \tag{2.17}$$

for the proof of which we refer to Bensoussan-Lions and and Jaillet-Lamberton-Lapeyre.

In the results below, we will frequently work with $u = u[\psi_\epsilon]$. We use the following result, which shows that the problem on all of $\mathbb{R}^n$ can be approximated by a sequence of problems on balls $B_k = \{x : |x| < k\}$ with $k \to \infty$. Let

$$B_k = \{x \in \mathbb{R}^n : |x| < k\}, \quad \partial B_k = \{x \in \mathbb{R}^n : |x| = k\},$$
$$H_k = L^2(B_k), \quad V_k = \{f \in H_k, \nabla f \in H_k\}$$

and define a bilinear form on V_k for each $t \in [0, T]$ as follows: For $u, v \in V_k$,

$$\begin{aligned} a_k(t; u, v) = &\sum_{i,j=1}^{n} \int_{B_k} \sigma_{ij}(x,t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \\ &- \sum_{i=1}^{n} \int_{B_k} \left(\sum_{j=1}^{n} (r - d_i - \sigma_{ii} - \sum_{j=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_j} \right) \frac{\partial u}{\partial x_i} v \, dx + \int_{B_k} ruv \, dx \end{aligned} \tag{2.18}$$

Theorem 2.2. *Under assumptions (H1)–(H5), there exists a unique solution u_k of the variational inequality*

$$
\begin{gathered}
u_k \in L^2([0,T];V_k), \quad \frac{\partial u_k}{\partial t} \in L^2([0,T];H_k) \\
u_k \geq \psi \quad a.e.\ in \quad [0,T]\times B_k \\
\forall v \in V_k \quad if \quad v \geq \psi \quad then \quad (\frac{\partial u_k}{\partial t}, v-u_k)_k + a_k(t,u,v-u) \geq 0 \\
u_k = \psi \quad if \quad x \in \partial B_k \\
u_k(0) = \psi
\end{gathered}
$$

Moreover the approximate solutions u_k have the property that, for $t \in [0,T]$ they converge uniformly to u as $k \to \infty$ on a ball of radius $k/2$. This result is implicitly contained in the research report by Jaillet, Lamberton and Lapeyre [1990] which is a long version of their 1988 article. Since this report is not easily available, for the reader's convenience, the salient points of the arguments are described in Appendix 1. The choice of a ball of radius $k/2$ is arbitrary. Any radius $c(k)$ such that $\lim_{k\to\infty}(k-c(k)) = +\infty$ is possible, as is clear from (7.4).

Transformation to a inhomogeneous equation in the continuation region. We transform the solution of the problem guaranteed by Theorem 2.2 to an equivalent problem with *zero payoff* and a *non-zero source term* in the continuation region, by making the transformation $u_k \to v_k = u_k - \psi$. For brevity let

$$
L_k^2 = L^2(B_k)\,, \quad V_0^k = W_0^{1,2}(B_k)\,. \tag{2.19}
$$

We then obtain the following problem

$$
\begin{gathered}
v_k \in L^2([0,T];V_0^k)\,, \quad \frac{\partial v_k}{\partial t} \in L^2([0,T];L_2^k) \\
v_k \geq 0 \quad \text{a.e. in } [0,T]\times B_k \\
\forall\, w_k \in V_0^k \quad \text{with} \quad w_k \geq 0 \\
(\frac{\partial v_k}{\partial t}, w_k - v_k) + a_k(t,v_k,w_k-v_k) \geq -a_k(t,\psi_\epsilon,w_k-v_k) \\
v_k(0) = 0
\end{gathered}
\tag{2.20}
$$

In later sections the explicit form of the inhomogeneous term $-\mathcal{L}_S\psi$, (see (2.7)), where ψ is the payoff function of a basket option, will be useful, and is given below. The calculation is most easily carried out in the original variables S and then transfered to the new variables. This form for the right hand side can be derived in two ways. One is to regularize the payoff function ψ from above and pass to the limit. The other is to apply Federer's coarea formula directly to the Lipschitz function ψ. In the case of a call on an index or a call on a spread we get

Calls:

$$
\begin{aligned}
-\mathcal{L}_S(\psi_C) &= \sigma_{ij} w_i w_j e^{x_i+x_j} \frac{1}{|\nabla\Phi_C|_{\Phi_C=0^+}} \delta_{\{\Phi_C=0\}} \\
&\quad + (r-d_i) w_i e^{x_i} \mathbf{1}_{\{\sum w_i e^{x_i}>1\}} (\sum w_i e^{x_i} - 1)^+ \\
&= \sigma_{ij} w_i w_j e^{x_i+x_j} \frac{1}{|\nabla\Phi_C|_{\Phi_C=0^+}} \delta_{\{\Phi_C=0\}} + (r - w_i d_i e^{x_i}) \mathbf{1}_{\{\sum w_i e^{x_i}>1\}}\,.
\end{aligned}
\tag{2.21}
$$

For puts we have:
Puts

$$-\mathcal{L}_S(\psi_P) = \sigma_{ij} w_i w_j e^{x_i+x_j} \frac{1}{|\nabla\Phi_P|_{\Phi_P=0^+}} \delta_{\{\Phi_P=0\}} + (w_i d_i e^{x_i} - r)\mathbf{1}_{\{1>\sum w_i e^{x_i}\}} \quad (2.22)$$

where $\delta_{\{\Phi=0\}}$ is shorthand for the $n-1$ dimensional Hausdorff measure restricted to the $n-1$ dimensional surface $\{\Phi = 0\}$, ie. $\mathcal{H}_{n-1}\lfloor\{\Phi = 0\}$. Note that the well known term $rK\mathbf{1}_{\sum w_i e^{x_i}>1}$ that usually appears in the right hand side in our case has become $r\mathbf{1}_{\sum w_i e^{x_i}>1}$, since we work in the normalized variables.

Semi-discretization of the regularized problem. The estimates obtained below, using symmetrization techniques on the regularized, localized problem, require the technique of semi-discretization, also known as Rothe's method. On the domain B_k consider the following sequence of approximating problems. Let $0 \le \tau_1 \le \tau_2 \cdots, \tau_n = T$ be the partition associated to the n-th approximating problem where

$$\Delta\tau_n = \frac{T}{n} \quad (2.23)$$

Define an approximating elliptic variational problem by letting

$$(V_0^k)^+ = \{u \in H_0^1(B_k) : u \ge 0\} \quad (2.24)$$

(recall that V_0^k was defined earlier in (2.19)). and

$$a_k^{m,n}(u,v) = \frac{1}{\Delta\tau_n}\int_{m\Delta\tau_n}^{(m+1)\Delta\tau_n} a_k(s,u,v)ds, \qquad u,v \in (V_0^k)^+, \quad (2.25)$$

where a_k was defined in (2.18). Let

$$G_\epsilon^D = -\mathcal{L}_S\psi_\epsilon^D, \qquad D = C \quad \text{or} \quad P \quad (2.26)$$

Recall that ψ_ϵ is a regularization of ψ and then define

$$(G_\epsilon^D)^{m,n} = \frac{1}{\Delta\tau_n}\int_{m\Delta\tau_n}^{(m+1)\Delta\tau_n} G_\epsilon^D(s)ds\,. \quad (2.27)$$

When we average the bilinear form $a_k(\cdot,u,v)$ over the time interval $[m\Delta\tau_n,(m+1)\Delta\tau_n]$, we average coefficients of the operator , eg. $r(x,\cdot),\sigma_{ij}(x,\cdot),d(x,\cdot)$.

Now for fixed n, let $v_k^{m,n} \in (V_0^k)^+, m = 1,..,n$ be the solution to the following elliptic variational problem, defined recursively by $v_k^{0,n} = 0$ and for all $w \in (V_0^k)^+$,

$$a_k^m(v_k^{m,n}, w - v_k^{m,n}) - ((G_\epsilon^D)^{m,n}, w - v_k^{m,n}) \ge \Big(\frac{v_k^{m-1,n} - v_k^{m,n}}{\Delta\tau_n}, w - v_k^{m,n}\Big)\,. \quad (2.28)$$

Rewrite this expression as

$$\begin{aligned} &a_k^m(v_k^{m,n}, w - v_k^{m,n}) + \frac{1}{\Delta\tau_n}(v_k^{m,n}, w - v_k^{m,n}) \\ &\ge \Big(\frac{1}{\Delta\tau_n}v_k^{m-1,n} + ((G_\epsilon^D)^{m,n}, w - v_k^{m,n})\Big), \quad \forall w \in (V_0^k)^+ \end{aligned} \quad (2.29)$$

Writing the problem in this form, makes clear that the coefficient of the zero-th order term is increased by a factor proportional to $\frac{1}{(\Delta\tau_n)}$ and so by well known results our elliptic variational problem is solvable when $(\Delta\tau_n)$ is small enough, for arbitrary inhomogeneity $\frac{1}{(\Delta\tau_n)}v_k^n + (G_\epsilon^D)^{m,n}$. Moreover, one has the convergence result in Bensoussan-Lions [12] which shows that the solution of this problem converges weakly in $L^2([0,T]:H_0^1(B_k))$ and weak * in $L^\infty([0,T]:L^2(B_k))$ to a solution of problem (2.20).

This completes our discussion of introduction to the optimal stopping problem and its rigorous formulation. In the next section we give some background material on symmetrization so that we may then introduce the radially symmetric (in spatial variables) comparison problem.

2.1. **Background material on Symmetrization .** We recall some background material on rearrangements and symmetrization. If $\phi \in L^1(\Omega)$ we let

$$\mu_\phi(t) = |\{x \in \Omega : \phi(x) > t\}|, \quad t \in \mathbb{R}, \tag{2.30}$$

where, if A is a Lebesgue measurable set, $|A|$ denotes the n dimensional Lebesgue measure of A. $\mu_\phi(t)$ is called the distribution function of ϕ. Also we define the monotone decreasing rearrangement of ϕ by

$$\phi^*(s) = \sup\{t : \mu_\phi(t) > s : s \in [0, |\Omega|]\} \tag{2.31}$$

The increasing rearrangement of ϕ is defined by

$$\phi_*(s) = \phi^*(|\Omega| - s) \quad s \in [0, |\Omega|] \tag{2.32}$$

We also let Ω^* be the solid ball with the same volume as Ω and define the Schwartz symmetrization of ϕ by

$$\phi^\sharp(x) = \phi^*(\omega_n |x|^n) \quad x \in \Omega^*, \tag{2.33}$$

where ω_n is the volume of he unit sphere in $\mathbb{R}^n$.

If μ_ϕ is strictly decreasing and continuous ϕ^* is the smallest decreasing function from $[0,\Omega]$ such that $\phi^*(\mu_\phi(t)) \geq t$ for every $t \in \mathbb{R}$. A basic property of ϕ^* , $\phi^\sharp$ is that ϕ and ϕ^* and $\phi^\sharp$ have the same distribution function. This implies that for any Borel function F we have

$$\int_\Omega F(\phi) = \int_{-\infty}^{+\infty} F(t)\,d\mu_\phi(t) = \int_{\Omega^\sharp} F(\phi^\sharp) = \int_0^{|\Omega|} F(\phi^*) \tag{2.34}$$

In §4 we will use the notation $\phi^{*,k}$ and $\phi^{\sharp,k}$ to emphasize that we are considering symmetrizations relative to the domain B_k.

Properties of rearrangements. The following properties of rearrangements will be frequently used in the sequel.

(i) (a) For a constant c, $(cf)^* = cf^*$ and
 (b) $(f+c)^* = f^* + c$.

(ii) If h is a monotone increasing function then $(h(f))^* = h(f^*)$

(iii) If $\mathbf{1}_A$ is the characteristic function of a set A then

$$(\mathbf{1}_A)^* = \mathbf{1}_{A^*} \tag{2.35}$$

(iv) The Hardy-Littlewood inequalities

$$\int fg \leq \int f^* g^* \tag{2.36}$$

$$\int fg \geq \int f^* g_* \tag{2.37}$$

(v)

$$\int_0^\mu (f+g)^* \leq \int_0^\mu f^* + \int_0^\mu g^*$$

(vi)

$$\int_{v>t} f \le \int_{v^*>t} f^*$$

(vii) If $h \ge 0$ is a non-increasing function,

$$\int_0^\mu (f+g)^* h(s)ds \le \int_0^\mu (f^*+g^*)h(s)ds$$

(viii) For non negative f and g, Chong and Rice [21],

$$\int_0^\mu (fg)^* \le \int_0^r f^* g^* \,. \tag{2.38}$$

Background material on Minkowski-Buseman isoperimetric inequality. Let D be a closed set (not necessarily convex) with boundary satisfying certain Lipschitz conditions . Given a symmetric non-negative quadratic form $Q(x,x) = \sum Q_{ij}x_ix_j$, consider the action of Q on the unit vector $\mathbf{n}$ to the surface ∂D and integrate it's square root over the surface

$$\bar\Lambda = \int_{\partial D} \sqrt{Q(\mathbf{n},\mathbf{n})}\, dS \tag{2.39}$$

Since Q is a convex function (see for instance Bonnessen and Fenchel [13]) the weight $\zeta := Q^{1/2}(x,x)$, which is homogeneous of degree one, can be used to determine the boundary ∂C^Q of a convex set C^Q as follows

$$\partial C^Q = \{\zeta^{-1}(u)u : u \in S^{n-1}\} \tag{2.40}$$

Note that C^Q is precisely that convex set determined by the condition $y : \zeta(y) \le 1$. Indeed, for $u \in S^{n-1}$ $\zeta(\frac{u}{\zeta(u)}) = 1$, since ζ is homogeneous of degree one.Then we consider $(C^Q)^0$ the convex set polar to C^Q. This is defined in terms of the support function

$$S(u) = \max\{(u,x) : x \in C^Q\} = \max_{\xi\in\mathbb{R}^n\neq 0} \frac{(u,\xi)}{\zeta(\xi)}, \tag{2.41}$$

of the convex set as follows

$$(C^Q)^0 = \{y \in \mathbb{R}^n : S(u) \le 1\} \tag{2.42}$$

We are now in a position to state a special case of the Minkowski-Buseman inequality [19].

$$\bar\Lambda \ge n|D|^{\frac{n-1}{n}} |(C^Q)^0|^{1/n} \tag{2.43}$$

In the case of the convex sets considered in the present paper, this inequality can be obtained from a scaling argument using the principle axes of the quadratic form Q.

This inequality holds for any convex function ζ (ie. not necessarily of the form $Q^{1/2}$) which is homogeneous of degree one. Note that this inequality is always stronger than that obtained by applying the standard isoperimetric inequality in conjunction with the lower bound on the quadratic form Q.

$$\zeta(x) = Q(x,x)^{1/2} \ge \alpha^{1/2}|x|\,, \tag{2.44}$$

where α is the smallest eigenvalue of the positive quadratic form Q, and where the classical isoperimetric inequality reads

$$\text{surface area}\,(\partial D) \ge n\omega_n^{1/n}|D|^{\frac{n-1}{n}} \tag{2.45}$$

Indeed, merely combining (2.44) and (2.45), would yield

$$\bar{\Lambda} \geq \alpha^{1/2} n \omega_n^{1/n} |D|^{\frac{n-1}{n}},$$

and the inequality (2.43) is stronger because we always have

$$|(C^Q)^0|^{1/n} \geq \alpha^{1/2} \omega_n^{1/n}$$

To see this, it suffices to note that the dual convex set $(C^Q)^0$ always contains the ball of radius $\alpha^{1/2}$. Indeed for the support function $S(u)$ of C^Q we have

$$S(u) = \sup_{\xi \in S^{n-1}} \frac{< u, \xi >}{\zeta(\xi)} \leq \frac{1}{\alpha^{1/2}} |u|,$$

since for $\xi \in S^{n-1}$ $Q(\xi, \xi) \geq \alpha$. Therefore if $|u| \leq \alpha^{1/2}$ then $S(u) \leq 1$, and so $u \in (C^Q)^0$ as required. The parameters of the symmetrized problem are defined in the next section.

3. Parameters of symmetrized one dimensional-in-space parabolic variational inequality

Diffusion coefficient. Let $\Lambda(\tau)$ be the largest purely time dependent, symmetric, positive definite matrix smaller than $\underline{\sigma}(x, \tau)$, ie.

$$\Lambda(\tau) \leq \underline{\sigma}(x, \tau) \quad \forall x \in \mathbb{R}^n$$

and if $\tilde{\Lambda}(\tau)$ is any other purely time dependent matrix that is smaller (in the sense of matrices, ie. the difference is a positive matrix) than $\underline{\sigma}(x, \tau)$ (which exists since $\underline{\sigma}(x, t)$ is a smooth symmetric matrix) then

$$\tilde{\Lambda}(\tau) < \Lambda(\tau)$$

Let $Q_\Lambda(\xi, \xi)$ be the associated quadratic form. Let

$$\lambda_{co}(\tau) = \Big\{ \frac{|\{y \in \mathbb{R}^n : (Q_\Lambda y, y) \leq 1)\}^o|}{\omega_n} \Big\}^{1/n}, \tag{3.1}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. The definition of the polar convex set was given in (2.42). In the present case it coincides with the set $\{z \in \mathbb{R}^n : (\Lambda^{-1}(\tau) z, z) \leq 1\}$, where $\Lambda^{-1}(\tau)$ is the inverse of $\Lambda(\tau)$. As an example, when $n = 2$ and $\lambda_{11} = a^2$, $\lambda_{22} = b^2$ and $\lambda_{12} = \lambda_{21} = 0$ the convex set C is the ellipse $a^2 x^2 + b^2 y^2 \leq 1$ and the polar set C^0 is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, i.e., the eccentricities are switched. The volume of the polar convex set is πab, so the volume divided by the volume of the unit sphere is ab and $\sqrt{ab}$ is larger than the square root of the minimum eigenvalue, $\min(\sqrt{a^2}, \sqrt{b^2})$, of the matrix Λ. The consideration of the polar convex set is quite natural in cases where the volatility matrix has eigenvalues that are *substantially different* in magnitude.

In the context of partial differential equations it's usefulness was pointed out and illustrated in Alvino, Ferone, Trombetti and Lions [3]. In such cases it gives a considerably sharper estimate than the one that would be obtained by using simply the ellipticity constant of the matrix, which corresponds to the minimum of the eigenvalues.

Drift term. First define

$$\bar{\mathbf{D}}_{ij}(x,\tau) = r(x,t) - d_i(x,t) - \sigma_{ii}(x,t) + \sum_{j=1}^{n} (\sigma_{ij})_{x_j}(x,t) \tag{3.2}$$

Remark It is important to note that the definition of $D(x,\tau)$ involves the partial derivative of the volatilities with respect to the logarithm of the stocks. This means that if we attempt to use the model to obtain upper bounds for options in a given market, we build into our implementation of the model a guess at the size and (certainly) of the *sign* of these partial derivatives (a generalization to the multi-factor model of incorporating information about the so-called "smile") and input this information into our "effective parameters", i.e. the parameters of our parabolic variational inequality with one spatial variable.

Denote by $\hat{\lambda}_{ij}(\tau), i,j = 1,\cdots n$, the entries of the matrix $\underline{\Lambda}^{-1}(\tau)$ and let, for fixed $\tau \in [0,T]$

$$D(\tau) = \lambda_{co}(\tau) \left(\max_{x\in\mathbb{R}^n} \hat{\lambda}_{ij}(\tau) \bar{D}_i(x,\tau) \bar{D}_j(x,\tau) \right)^{1/2} \tag{3.3}$$

Zero-th order term. The next two input functions of the one dimensional problem are defined in terms of those in the multi-dimensional problem as follows:

$$C(\tau) = \min_{x\in\mathbb{R}^n} r(\tau,x) \tag{3.4}$$

ϵ family of inhomogeneous terms.

$$F_\epsilon = (-\mathcal{L}_S(\psi_\epsilon))^{\sharp,k}$$

where $\mathcal{L}_S$ is given by (2.7)) and $f^{\sharp,k}$ denotes the Schwartz symmetrization of f, defined in §2.1.

4. Main Results

Our main results are the following comparison results.

Theorem 4.1 (Purely time dependent bounds for v_k)). *Let $v_k(x,\tau)$, for $n \geq 1$ be the unique solution in $L^2([0,T];V_0^k]$ with $v_t \in L^2([0,T];L_k^2]$, of (2.20) on a large ball B_k, where ψ is the payoff of a put or call on an index or a spread, and let $W_k = W_k^\epsilon$ be the unique weak radial solution, in the same space, of the following problem:*

$$\begin{gathered} (W_k)_\tau - \lambda_{co}^2(\tau)(W_k)_{rr} - \frac{n-1}{r}\lambda_{co}^2(\tau)(W_k)_r + D(\tau)(W_k)_r + C(\tau)W_k = F_\epsilon \\ \text{on } \{W_k > 0\} \\ (W_k)_r(0,\tau) = 0 \quad W_k(r,0) = 0 \end{gathered} \tag{4.1}$$

where,

$$F_\epsilon = -(\mathcal{L}_S\psi_\epsilon)^{\sharp,k},$$

and where the parameters $\lambda_{co}(\tau)$, $C(\tau)$, $D(\tau)$ were defined in §3, in the case $n \geq 2$ and in the case $n = 1$, $\lambda_{co}^2(\tau) = \min_{x\in\mathbb{R}^n} \sigma(x,\tau)$ (σ as in (2.8)), then the following comparison principle holds

$$\int_0^V \frac{1}{e(\mu,\tau)} v_k^{*,k}(\mu,\tau)d\mu \leq \int_0^V \frac{1}{e(\mu,\tau)} W_k^{*,k}(\mu,\tau)d\mu \tag{4.2}$$

where, $\mu = \omega_n r^n$, $V \in [0, |B_k|]$ *and*

$$e(\mu,\tau) = \exp\left(\frac{D(\tau)}{(\lambda_{co}(\tau))^2}\,\mu^{1/n}\right) \tag{4.3}$$

Note the *coincidence region* for the radial problem corresponds to the set $\{W_k = 0\} = B_k \setminus \{W_k > 0\}$.

As a consequence of (4.2), we have the following statement.

Corolary 4.2. *Under the same conditions of Theorem 4.1, we have*

$$\max_{B_k} v_k(x,\tau) \le \max_{B_k} W_k(x,\tau) \quad 0 \le t \le T \tag{4.4}$$

For the proof: Divide (4.2) by V and let $V \to 0$.

Corolary 4.3 (Purely time dependent bound for original problem)**.** *Let u be the value of the American option, whose payoff $\psi(x)$ is a call or put on an index or on a spread, then given $\delta > 0$ arbitrarily small there exists a $k = K(\delta, n, r, d_i, \sigma_{i,j})$ such that on the ball of radius $\frac{k}{2}$ we have*

$$|u(x,\tau) - \psi(x)| \le \max_{x\in B_k} W_k^\delta + 2\delta$$

Remark The dependence on δ is complicated to express but at its root is the inequality

$$|u-\psi| \le |u-u_\epsilon| + |u_\epsilon - \psi_\epsilon| + |\psi_\epsilon - \psi|$$

where u_ϵ is the solution of (2.13) with ψ_ϵ replacing by ψ. The next step is to estimate $u_\epsilon - \psi_\epsilon$ on $B_{k/2}$ by v_k , the solution on B_k of (2.20), using the results of Appendix 1, and then to estimate v_k by W_k using the result of Theorem 4.1.

The results above can be complemented with the following result.

Theorem 4.4 (time and state dependent bounds for v_k)**.** *Let $v_k(x,\tau)$, for $n \ge 1$ be the solution of (2.20) on a large ball B_k , let $z = (\sum_{i=1}^n |w_i| e^{x_i} e^{-(r-d_i)\tau} + 1)$ and let $W_k^z \ge 0$ be the solution of the spatially one dimensional equation*

$$\begin{aligned}
&(W_k^z)_\tau - \lambda_{co}^2(\tau) V_{rr}^z - \frac{n-1}{r}\lambda_{co}^2(\tau) V_r^z + D^z(\tau)(W_k^z)_r - C^z(\tau) W_k^z \\
&= (-\frac{\mathcal{L}_S \psi_\epsilon}{z})^{\sharp,k} \quad \text{in the region} \quad \{W_k^z > 0\} \\
&\frac{\partial W_k^z}{\partial r}(0,\tau) = 0, \quad W_k^z(r,0) = 0
\end{aligned}$$

where the coefficients $D^z(\tau)$ and $C^z(\tau)$ are defined by the same algorithm (3.3) and (3.4) as in section 3, but now applied to the new effective drift and zero-th order terms

$$(\bar{D}_i(x,\tau) + \frac{2}{z}\sigma_{ij}\frac{\partial z}{\partial x_j})$$

and new zero order term

$$\frac{1}{z}(\mathcal{L}_S z + C(x,\tau))$$

and new forcing term

$$F_\epsilon^z = (\sum_{i=1}^n |w_i| e^{x_i} e^{-d_i t} + 1) F_\epsilon$$

then we have the same estimates as in Theorem 4.1, with $v_k^z = \frac{v_k}{z}$ replacing v_k and with W_k^z replacing W_k.

Corolary 4.5 (Time and state dependent bounds). *Let $u[\psi]$ be the solution of the American option problem, with payoff ψ that is either a call or put on an index or on a spread, then under the same conditions as in Theorem 4.4, for all $x \in B_k$*

$$|u(x,\tau) - \psi(x)| \leq \frac{\max_{x\in B_k}(W_k^z)^\delta) + 2\delta}{\sum_{i=1}^n |w_i| e^{x_i} e^{-(r-d_i)\tau} + 1} \tag{4.5}$$

Remark This second class of bounds can be thought of as deriving a comparison principle for the price measures in a special set of units, i.e. choosing a *numeraire*. The particular numeraire used above is convenient but by no means the only possible one.

5. Proof of the main results

Estimates for the elliptic variational inequalities associated to the time-discretized problems. In this section we provide estimates for the elliptic variational inequalities associated to the time-discretized, and regularized parabolic variational inequalities. Estimates for the elliptic variational inequalities then lead to estimates for the parabolic one, using the method of Vasquez [45], as developed by Ferone and Volpicelli [25] In carrying over the estimates for the elliptic problem to the parabolic one, we wish to allow the volatility and drift parameters of the symmetrized problem to be time dependent. This can however be accommodated by a simple extension of their argument.

For this, let us further simplify the notation by letting

$$\bar{D}_i^{m,n} = (r - d_i - \sigma_{ii} - (\sum_{j=1}^n (\sigma_{ij})_{x_j})^{m.n} \tag{5.1}$$

We will lighten the notation by dropping, provided the context is clear, the superscript "n, m" in the presentation below. We also will denote by G^-, the part of the inhomogeneous term that results from solving the elliptic variational problem in the preceding interval, ie.

$$G^- = \frac{1}{\Delta\tau} v_k^{m,n} \tag{5.2}$$

In this simplified notation, the localized and regularized problem on the domain B_k can then be written in the form, find $v_k \in V_0^k$ such that for all $w_k \in V_0^k$,

$$\int_{B_k} (\sigma_{ij}(v_k)_{x_i}(w_k)_{x_j} - \bar{D}_i(v_k)_{x_i} w_k + (r + \frac{1}{\Delta\tau}) v_k\, w_k) \geq \int_{B_k} (G_\epsilon^D + G^-) w_k \tag{5.3}$$

and we let

$$a_k(v,w) = \int_{B_k} (\sigma_{ij}(v)_{x_i}(w)_{x_j} - \bar{D}_i(v)_{x_i} w + (r + \frac{1}{\Delta\tau}) v\, w) \tag{5.4}$$

We next follow closely the steps in Alvino-Matarasso-Trombetti, 1992, and make the necessary adjustment to incorporate the use of the Buseman-Minkowski inequality in the treatment of the principal part and of the drift terms.

In (5.3), use the test function

$$\phi_h(x) = \begin{cases} h & t+h < v_k(x) \\ v_k(x) - t & t < v_k(x) \le t+h \\ 0 & v_k(x) \le t \end{cases} \tag{5.5}$$

with $h \geq 0$ and $t \in]0, \sup \mathrm{u}[$. Since $v_k \pm \phi_h \geq 0$ we can replace the test function w_k in V_0^k, by the functions $v_k \pm \phi_h$. We thus obtain

$$\frac{1}{h} a_k(v_k, \phi_h) = \frac{1}{h} \int_{B_k} (G_\epsilon^D + G^-)\phi_h \tag{5.6}$$

where, in the present simplified notation, $a_k(v_k, \phi_h)$ is given by (5.4).

Dividing by h and taking the limit as $h \to 0$, leads in a standard way (see Alvino-Lions-Trombetti [1]) to the equality (using $w_k = \pm\phi_h$)

$$-\frac{d}{dt} \int_{v_k>t} \sigma_{ij}(x,\tau)(v_k)_{x_i}(v_k)_{x_j} = \int_{v_k>t} \bar{D}_i (v_k)_{x_i} - \int_{v_k>t} (r + \frac{1}{\Delta\tau})\, v_k + \int_{v_k>t} (G_\epsilon^D + G^-) \tag{5.7}$$

Case 1: $n \geq 2$ The estimates are now carried out in the following steps:
Estimate from below of quadratic term, using Minkowski-Buseman inequality We estimate from below

$$-\frac{d}{dt} \int_{B_k} \sigma_{ij}(x,\tau)(v_k)_{x_i}(v_k)_{x_j} \tag{5.8}$$

From Schwarz's inequality and an argument due to Talenti [34, page 711-713],

$$-\frac{d}{dt} \int_{v_k>t} \sqrt{\sigma_{ij} v_{x_i} v_{x_j}} \leq \Big\{ -\frac{d}{dt} \int_{v_k>t} \sigma_{ij} v_{x_i} x_{x_j} \Big\}^{1/2} \Big\{ -\frac{d}{dt} \mu_{v_k}(t) \Big\}^{1/2} \tag{5.9}$$

Thus we must estimate below the term

$$-\frac{d}{dt} \int_{v_k>t} \sqrt{\sigma_{ij} v_{x_i} x_{x_j}} \tag{5.10}$$

The basic idea for doing this is to extend the Minkowski-Buseman inequality to the setting where the surface ∂D is not a Lipshitz surface, but rather is the level set of an $H_0^1(B_k)$ surface. Such a generalization was in fact obtained by Amar and Belletini [6] and Alvino, Ferone, Lions and Trombetti [3], who consider an wider class of functions generalizing those of bounded variation in the usual metric to the case of the Minkowsky metric. We illustrate the basic idea here by making stronger assumptions on v_k. Let v_k be a Lipshitz function function we by applying the co-area formula, adapted to Sobolev functions, Almgren-Lieb [4] and Ziemer [48] that expression (5.10) equals

$$\int_{v_k=t} \sqrt{\sigma_{ij} \frac{(v_k)_{x_i}}{|\nabla v_k|} \frac{(v_k)_{x_j}}{|\nabla v_k|}}\, d\mathcal{H}_{n-1} \tag{5.11}$$

so that setting

$$\nu = \frac{\nabla v_k}{|\nabla v_k|} \tag{5.12}$$

Equation (5.11) becomes

$$\int_{\{v_k=t\}} \sqrt{\sigma_{ij}\nu_i\nu_j}\, d\mathcal{H}_{n-1} \tag{5.13}$$

and taking $\partial D = \{v_k = t\}$ and $Q_{ij} = \sigma_{ij}$ in (2.43) and using the assumption $\underline{\sigma}(x,\tau) \ge \underline{\Lambda}(\tau)$ we see from the definition (3.1) that

$$\int_{v_k=t} \sqrt{\sigma_{ij}\nu_i\nu_j}\, d\mathcal{H}_{n-1} \ge n\omega_n^{1/n}\lambda_{co}(\tau)|\{v_k > t\}|^{\frac{n-1}{n}} \tag{5.14}$$

The non trivial technical questions involved in justifying these manipulations for a class of functions (an appropriate generalization of BV functions) which include those delt with here are given in Amar and Belletini and in Alvino-Ferone-Trombetti-Lions. Combining (5.14) and (5.9) we obtain

$$\Big\{-\frac{d}{dt}\int_{v_k>t}\sigma_{ij}v_{x_i}v_{x_j}\Big\}^{1/2} \ge n\omega_n^{1/n}\lambda_{co}(\tau)(\mu_{v_k>t}(t))^{1-\frac{1}{n}}(-\frac{d}{dt}\mu_{v_k>t})^{-1/2} \tag{5.15}$$

Control of the drift term Let

$$\bar{\mathbf{D}}(x,\tau) = (\bar{D}_1(x,\tau), \bar{D}_2(x,\tau), \bar{D}_3(x,\tau)) \tag{5.16}$$

The term $-\int_{v>t}\bar{D}_i(x,\tau)(v_k)_{x_i}(x,\tau)$ is estimated as follows. Recall the definition of $\underline{\Lambda}$ and it's inverse in §3 and denote the entries of the matrix by λ_{ij} and those of its inverse by $\hat{\lambda}_{ij}$. Then, using the inequality $(x,y) \le (\zeta x, \zeta^0 y)$ for the convex, positive, homogeneous function $\zeta x = \sqrt{Q(x,x)}$ $(Qx,x = \bar{\lambda}_{ij}x_ix_j$ and its conjugate, $\zeta^0 y = \sqrt{Q^{-1}(x,x)}$ where Q^{-1} is the quadratic form associated to the inverse matrix $\underline{\Lambda}^{-1}(\tau)$ of $\Lambda(\tau))$

$$\begin{aligned}
&-\int_{v>t}\bar{D}_i(x,\tau)(v_k)_{x_i}(x,\tau)\\
&=\int_{v_k>t}\Big(\hat{\lambda}_{ij}\bar{D}_i\bar{D}_j\Big)^{1/2}\big(\lambda_{ij}(v_k)_{x_i}(v_k)_{x_j}\big)^{1/2}\\
&=\int_t^{+\infty} ds -\frac{d}{ds}\Big\{\int_{v_k>s}(\hat{\lambda}_{ij}\bar{D}_i\bar{D}_j)^{1/2}\big(\lambda_{ij}(v_k)_{x_i}(v_k)_{x_j}\big)^{1/2}\Big\}\\
&\le \int_t^{+\infty} ds\Big(-\frac{d}{ds}\int_{v_k>s}\hat{\lambda}_{ij}\bar{D}_i\bar{D}_j\Big)^{1/2}\Big(-\frac{d}{ds}\int_{v_k>s}\lambda_{ij}(\tau)(v_k)_{x_i}(v_k)_{x_j}\Big)^{1/2}
\end{aligned} \tag{5.17}$$

Recall the definition (2.19) of $D(\tau)$ from §3 (see (3.3)) and note that we then clearly have

$$\Big(-\frac{d}{ds}\int_{v_k>s}\hat{\lambda}_{ij}(\bar{D}_i\bar{D}_j\Big)^{1/2} \le \frac{D(\tau)}{\lambda_{co}(\tau)}(-\mu'_{v_k}(s))^{1/2} \tag{5.18}$$

so that (5.17) can be written

$$\begin{aligned}
&|\int_{v_k>t}\bar{D}_i(x,\tau)(v_k)_{x_i}(x,\tau)dV|\\
&\le \frac{D(\tau)}{\lambda_{co}(\tau)}\int_t^{+\infty}(-\mu'_{v_k}(s))^{1/2}\Big\{-\frac{d}{ds}\int_{v_k>s}\lambda_{ij}(\tau)(v_k)_{x_i}(v_k)_{x_j}\Big\}^{1/2}
\end{aligned} \tag{5.19}$$

We use (5.15), rewritten in the form

$$\frac{-\mu'_{v_k}(s))^{1/2}\mu_{v_k}(s)^{\frac{1-n}{n}}}{n\lambda_{co}(\tau)}\Big\{-\frac{d}{ds}\int_{v_k>s}\lambda_{ij}(\tau)(v_k)_{x_i}(v_k)_{x_j}\Big\}^{1/2} \quad \ge 1 \tag{5.20}$$

and multiply (5.19) under the integrand in by (5.20), so our final estimate of the drift term is

$$
\begin{aligned}
&\Big|\int_{v_k>t} \bar{D}_i(x,\tau)(v_k)_{x_i}(x,\tau)\Big| \\
&\le \frac{D(\tau)}{n\omega_n^{1/n}(\lambda_{co}(\tau))^2}\int_t^{+\infty}\{(-\mu'_{v_k}(s))(\mu_{v_k}(s))^{\frac{1-n}{n}}(-\frac{d}{ds}\int_{v_k>s}\lambda_{ij}(\tau)(v_k)_{x_i}(v_k)_{x_j})\}ds
\end{aligned}
\tag{5.21}
$$

To estimate the zero-th order term in (5.3) use properties i)(b) and v) and vi) of the rearrangement to get

$$
-\int_{v_k>t}(r+\frac{1}{\Delta\tau})v_k \le -\int_{v_k^{*,k}>t}(r_*+\frac{1}{\Delta\tau})v_k^{*,k} \le -\int_{v_k^{*,k}>t}(C(\tau)+\frac{1}{\Delta\tau})v_k^{*,k}, \tag{5.22}
$$

where the last inequality follows from the definition (3.4) of $C(\tau)$

For the inhomogeneous term, we record the following inequality, which is needed in carrying over these estimates to the parabolic case, using the argument in Ferone-Volpicelli (see p.563-565)

$$
\int_{v_k>t} e^{-1}(\mu,\tau)(G_\epsilon^D+G^-) \le \int_{v_k^{*,k}>t} e^{-1}(\mu,\tau)((G_\epsilon^D)^{*,k}+(G^-)^{*,k}), \tag{5.23}
$$

where e is defined below (see (4.3)). The inequality follows immdediately from Property (vii) of the rearrangement, since $e^{-1}=\exp(-\frac{D(\tau)}{\lambda_{co}^2(\tau)})\mu^{-1/n}$ is, for fixed τ, a decreasing function of μ.

Combining the various inequalities, we thus have arrived at the inequality

$$
\begin{aligned}
&-\frac{d}{ds}\int_{v_k>s}\lambda_{ij}(\tau)(v_k)_{x_i}(v_k)_{x_j} \\
&\le \int_{v_k>s}\sigma_{ij}(x,\tau)(v_k)_{x_i}(v_k)_{x_j} \\
&\le \frac{D(\tau)}{n(\lambda_{co}(\tau))^2}\int_t^{+\infty} -(\mu'_{v_k}(s)(\mu_{v_k}(s))^{\frac{1-n}{n}}\Big(-\frac{d}{ds}\int_{v_k>s}\lambda_{ij}(v_k)_{x_i}(v_k)_{x_j}\Big) \\
&\quad + \int_{v_k^{*,k}>t}(G_\epsilon^D)^{*,k}+(G^-)^{*,k}) - (C(\tau)+\frac{1}{\Delta\tau})v_k^{*,k}
\end{aligned}
$$

Now we make use of the following form of Gronwall's inequality. When v satisfies

$$
v(t) \le g(t) + \int_t^{+\infty} h(\tau)v(\tau)d\tau \tag{5.24}
$$

and v is zero at $+\infty$, then

$$
v(t) \le -\int_t^{+\infty} g'(\tau)e^{\int_t^\tau h(s)ds}\,. \tag{5.25}
$$

Applying this inequality, with

$$v(t) = -\frac{d}{ds}\int_{v_k>s} \lambda_{ij}(v_k)_{x_i}(v_k)_{x_j}, \tag{5.26}$$

$$g(t) = \int_0^{\mu_{v_k}(t)} (G_\epsilon^D)^{*,k} + (G^-)^{*,k}) - (C(\tau) + \frac{1}{\Delta\tau})v_k^{*,k} := \int_0^{\mu_{v_k}(t)} (\mathcal{R}_v(s))ds, \tag{5.27}$$

$$h(t) = \frac{D(\tau)}{n(\lambda_{co}(\tau))^2}(-\mu'_{v_k}(t))(\mu_{v_k}(t))^{\frac{1-n}{n}}, \tag{5.28}$$

we obtain

$$\begin{aligned} v(t) &\le \int_t^{+\infty} (-\mu'_{v_k}(s)\mathcal{R}(s)\exp\Big(-\frac{D(s)}{n(\lambda_{co}(\tau))^2}(\int_t^s (\mu_{v_k}(s))^{\frac{1-n}{n}}\mu'_{v_k}(u)du)\Big)ds \\ &= \exp(\frac{D(\tau)}{(\lambda_{co}(\tau))^2})\,(\mu_{v_k}(t))^{1/n}) \\ &\quad\times \int_t^{+\infty} \exp(-\frac{D(\tau)}{(\lambda_{co}(\tau))^2}(\mu_{v_k}(s))^{1/n}(-\mu'_{v_k}(s)))\mathcal{R}_v(s)ds \end{aligned} \tag{5.29}$$

which introducing the notation

$$e(\mu,\tau) = \exp\Big(\frac{D(\tau)}{\omega_n^{1/n}\lambda_{co}(\tau))^2}\Big)\mu^{1/n} \tag{5.30}$$

and making a change of variables leads to

$$v(t) \le e(\mu,\tau)\int_0^{\mu(t)} e^{-1}(\mu,\tau)\,\mathcal{R}_v(\mu)d\mu \tag{5.31}$$

Combining this estimate again with (5.15), we obtain

$$(-\mu'_{v_k}(t))^{-1} \le \frac{1}{(n\omega_n^{1/n})^2\lambda_{co}(\tau)^2}\mu_{v_k}^{2-\frac{2}{n}}(t)\; e(\mu,\tau)\int_0^{\mu_{v_k}(t)} e^{-1}(\mu)\mathcal{R}_v(\mu)d\mu\,. \tag{5.32}$$

Arguing as in Talenti [43, pp. 711-713], the last inequality can be expressed in terms of the decreasing rearrangement

$$(v_k^{*,k})' \le \frac{1}{(n\omega_n^{1/n})^2\lambda_{co}^2}\mu^{\frac{2}{n}-2}(t)\; e(\mu,\tau)\int_0^{\mu_{v_k}(t)} e^{-1}(\mu,\tau)\mathcal{R}_v(\mu)d\mu, \tag{5.33}$$

with $0<\mu<|B_k|$. With this differential inequality in hand, comparison arguments for one dimensional differential inequalities, may be used, as discussed in Alvino-Matarasso-Trombetti, to establish Theorem 4.1, the key point being that when $\lambda_{co}(\tau)$ and $D(\tau)$ are defined as in §3, the differential inequality becomes an equality. The interested reader is refered to the proofs Lemma 2.3 p. 275 and Theorem 3.2 p. 277 of Alvino-Materasso-Trombetti [2].

Case 2: $n=1$ The one dimensional case lends itself to the sharpest estimates. Beginning from (5.7), after using the same test function we have

$$-\frac{d}{dt}\int_{v_k>t}\sigma_{ij}(x,\tau)(v_k)_x^2 = \frac{d}{dt}\int_{v_k>t}\bar{D}_i(v_k)_x - \int_{v_k>t}(r+\frac{1}{\delta}\tau)v_k + \int_{v_k>t}(G_\epsilon^D + G^-) \tag{5.34}$$

Letting $\lambda(\tau) = \min_{x\in\mathbb{R}} \bar\sigma(x,\tau)$, and by Shwarz's inequality we get

$$-(\lambda(\mu'_{v_k})^{-1})\frac{d}{dt}\int_{v_k>t} |(v_k)_x|^2 \le -\frac{d}{dt}\int_{v_k>t} \sigma_{ij}(x,\tau)(v_k)_x^2 \tag{5.35}$$

Combining this with the sharp relation

$$-\frac{d}{dt}\int_{v_k>t} |(v_k)_x| = \mathcal{M}(v_k)(t), \tag{5.36}$$

where $\mathcal{M}(v_k)$ is the multiplicity function of v_k, we arrive at the lower bound

$$\mathcal{M}^2(v_k(t))\lambda(\tau)(-\mu'_{v_k}(t))^{-1} \tag{5.37}$$

for the left hand side 0f (5.34). Note that since v_k is zero on the boundary, $\mathcal{M}(v_k)(t) \ge 2$ and is equal to 2 for all t if and only if the superlevel sets $\{v_k > t\}$ are all equivalent to intervals. Deriving these claims rigorously for BV functions is a bit delicate. A beautiful presentation thereof appears is Talenti [44, pp.102-105].

We estimate the drift term,

$$|\int_{v_k>t} \bar D(x,\tau)(v_k)_x| \le \max_{\mathbb{R}} |\bar D(x,\tau)|)\int_{v_k>t} |(v_k)_x| \tag{5.38}$$

and let $D(\tau) = \max_{\mathbb{R}} |\bar D(x,\tau)|$. In the remaining estimates the one dimensional Schwarz symmetrization (which coincides in the case $n = 1$ with the symmetrically decreasing rearrangement) is used and by the same estimates as in the multidimensional case we arrive at the result that v_k may be estimated above by the (even in x) solution of the one dimensional problem

$$\begin{gathered}
(W_k)_\tau(|x|,\tau) - \lambda(\tau)(W_k)_{xx}(|x|,\tau) + D(\tau)(W_k)_x(|x|,\tau) + C(\tau)W_k(|x|,\tau) \\
= F_\epsilon(|x|,\tau) \quad \in \{W_k > 0\} \\
W_k \ge 0 \quad \text{in} \quad B_k \\
W_k(|x|,0) = 0 \quad x \in B_k \\
\frac{\partial W_k}{\partial |x|}(0,\tau) = 0 \quad 0 < \tau \le T
\end{gathered}$$

where $F_\epsilon(x,\tau) = -(\mathcal{L}_S\psi_\epsilon)^{\sharp,k}$, as claimed.

6. Explicit form for regularized inhomogeneous term

For basket and spread options, in the dimensionless variables, we have

$$\begin{gathered}
\psi_C = \eta(\Phi_C), \quad \Phi_C = \sum(w_i e^{x_i} - 1), \\
\psi_P = \eta(\Phi_P), \quad \Phi_P = (1 - \sum w_i e^{x_i})\,.
\end{gathered}$$

We approximate η by a smooth function defined below and let

$$\psi_\epsilon^C = \eta_\epsilon(\Phi_C)\,, \quad \psi_\epsilon^P = \eta_\epsilon(\Phi_P) \tag{6.1}$$

In the case of calls we consider the upper approximations $\eta_\epsilon^+ : \mathbb{R} \to \mathbb{R}$ to the function η [2]

$$\frac{d^2}{dy^2}\eta_\epsilon^+(y) = \frac{1}{\epsilon}\,\mathbf{1}_{\{-2\epsilon<y<-\epsilon\}} \tag{6.2}$$

[2]Note that such smoothings of the function η are well known, but we exploit the specific form of the smoothing below to use the smoothing in conjunction with the symmetrization

Thus the derivative $(\eta_\epsilon^+)'$ grows from zero to 1 on the interval $-2\epsilon, -\epsilon$. Similarly, we define η_ϵ^- so that

$$\frac{d^2}{dy^2}\eta_\epsilon^- = \frac{1}{\epsilon}\mathbf{1}_{\{\epsilon<y<2\epsilon\}} \tag{6.3}$$

and then in all of the above examples we replace η by η_ϵ^+ in the case of calls and by η_ϵ^- in the case of puts. Since $\eta_\epsilon^\pm$ thus defined are Lipschitz with Lipschitz constant equal to 1, it is immediate that for these regularizations, we have the property

$$|\psi_\epsilon^D - \psi^D| \le \epsilon \tag{6.4}$$

and therefore we may use the comparison principle mentioned in §2. In addition the first derivatives of ψ_ϵ are well behaved but the second derivatives are not and this will be explored below.

Calculation of the smoothed source term The calculation of the action of the operator $\frac{\partial}{\partial\tau} + \mathcal{L}_S$ on the payoff ψ_ϵ is more straightforward to carry out in the original spatial variables S. It can then be easily transposed to the new variables x and new dependent variable $\frac{u}{K}$. We have

$$(\eta_\epsilon(\Phi))_{S_i} = \eta_\epsilon'(\Phi)\Phi_{S_i}$$
$$(\eta_\epsilon(\Phi))_{S_iS_j} = \eta_\epsilon''(\Phi)\Phi_{S_i}\,\Phi_{S_j} + \eta'(\Phi)\Phi_{S_iS_j}$$

We let

$$G_\epsilon^C(x) = -\mathcal{L}_S(\eta_\epsilon^+(\Phi^C)), \quad G_\epsilon^P(x) = -\mathcal{L}_S(\eta_\epsilon^-(\Phi^P))$$

So,

$$\begin{gathered} G_\epsilon^D = -S_iS_j\sigma_{i,j}\left(\eta_\epsilon'\Phi^D_{S_iS_j} + \eta_\epsilon''\Phi^D_{S_i}\Phi^D_{S_j}\right) + S_i(r-d_i)\,\eta_\epsilon'\Phi^D_{S_i} - r\psi_\epsilon^D \\ D^+ = C, \quad D^- = P, \quad \psi_\epsilon^{D^\pm} = \eta_\epsilon^\pm(\Phi^{D^\pm}) \end{gathered} \tag{6.5}$$

For the action of the operator $-\mathcal{L}_S$ on $\psi_\epsilon^{D^\pm}$, we get

$$G_\epsilon^{D^\pm}(x) = \frac{1}{\epsilon}\sum_{i,j=1}^n e^{x_i+x_j}w_iw_j\sigma_{i,j}\mathbf{1}_{\{-2\epsilon<\Phi^{D^+}<-\epsilon\}} \pm w_ie^{2x_i}(r-d_i)\eta_\epsilon'(\Phi^{D^\pm}) - r\psi_\epsilon^{D^\pm}\,. \tag{6.6}$$

Note that the only difference in the form of the right hand side for calls and puts is the call has '$+'$ and the put a '$-'$ multiplying the term involving a first derivative.

Dependence of the solution on ϵ and k. With (6.6) we have in fact a 2-parameter family of comparison problems. In this section we address the question as to how the bounds derived depend on these parameters. In two important cases we can give an analytical elucidation of this dependence. In the other cases the question needs to be investigated numerically. The two cases where an analytical elucidation is possible are:

- An index Put option (with or without dividends), with payoff $\psi^P = (1 - \sum_i w_ie^{x_i})^+$, the dependence of k simplifies considerably. Indeed in this case the continuation region is, in the original S variables connected and bounded away from zero where the minimum distance can be estimated via the lower bound on the volatility matrix in the lognormal coordinates . This is clear intuitively and follows from the results in Broadie and Detemple [15] and Villeneuve [47]. Translated to the x variables this means that the continuation region is connected and bounded away from $-\infty$, i,e. $\min_{i=1,\cdots n}\min_{x\in\mathcal{C}} x_i = -L_1$, in all directions. Furthermore when any x_i is sufficiently large the payoff of the option is zero in the complement of the region $\mathcal{C}_n = \mathcal{C} \cap \{\sum_i w_i e^{x_i} < 1\}$ and the right hand side in (6.6) is identically zero. Let L_2 be a constant so large that $\mathcal{C}_n$ is contained in $B(0, L_2)$. The maximum value of $u - \psi^D$ is not reached on the latter set. Thus, if we choose $k_0 = \max(L_1, L_2)$ we can be sure that the maximum of $u - \psi$ will be captured in B_{k_0} and there is no need to choose a larger k_0. For fixed k the optimal bounds as a function of ϵ must be determined numerically.

• For a call or put option on one asset, as shown in Laurence 2000 (pages 49-51), it is possible to pass to the limit as $\epsilon \to 0$ and derive a limiting comparison problem for any fixed k. For a put the same considerations as above apply to find a reasonable value for k. Using put call symmetry as in Detemple 2001 we can then extend the result to calls.

7. Appendix: The Relation between the solution on all of $\mathbb{R}^n$ and the solution on B_k

In Bensoussan and Lions, 1982, it is shown that u_k, the solution of the variational problem in Theorem 2.2, is also a solution of an optimal stopping problem on B_k. Let $T_k^{t,x} = \inf\{s > t, |X_s^{t,x}| > k\}$ then u_k solves

$$u_k(x,t) = \sup_{\tau \in T_k} E[e^{-\int_t^{\tau \wedge T_k^{t,x}} r(s)ds} \psi(X^{t,x}_{\tau \wedge T_k^{t,x}})]$$

On the other hand u, the solution on all of $\mathbb{R}^n$, is also a solution of the problem

$$u(x,t) = \sup_{\tau \in T^{t,T}} E[e^{-\int_t^{\tau} r(s)ds} \psi(X^{t,x}_{\tau})]$$

So we need to estimate $|u(x,t) - u_k(x,t)|$. The key to such an estimate is to note that

$$\begin{aligned} |u(x,t) - u_k(x,t)| &\le \sup_{\tau \in T_{t,T}} E\big[\big|\psi(X^{t,x}_{\tau}) - \psi(X^{t,x}_{T_k})\big| \, \mathbf{1}_{T_k^{t,x} < \tau}\big] \\ &\le E\big[2t[2 \sup_{[0,T]} |\psi(X^{t,x}_s)| \, \mathbf{1}_{T_k^{t,x} < \tau}\big] \\ &\le 2M \sqrt{E(\exp(2M \sup_{[t,T]} |X^{t,x}_s|)} \sqrt{P(T_k^{t,x} < T)} \end{aligned}$$

To estimate the difference between u and u_k, we thus need to estimate the two terms

$$E(\exp(2M \sup_{[t,T]} |X^{t,x}_s|), \tag{7.1}$$

$$P(T_k^{t,x} < T) \tag{7.2}$$

for x restricted to a ball of radius $R(k) < k$. The dependence of $R(k)$ on k is clarified below. Details as to how these terms may be estimated are given in the report by Jaillet-Lamberton-Lapeyre [28, pages 107-109]. We recall the form of the final estimate and review its dependence on the constants. For (7.1) this takes the form

$$\sup_{(x,t) \in [0,T] \times \{x \in \mathbb{R}^n : |x| \le k\}} E\big[\exp(2M \sup_{[t,T]} |X^{t,x}_s|)\big] \le C < \infty,$$

where C depends on a pointwise bound on the magnitudes of the drift coefficient and of the volatility matrix, as well as on T and R. For (7.2) let

$$\hat{\sigma} = \sup \Big(\sum_{i=1}^{n} \sum_{j,k=1}^{n} \sigma_{ij}\sigma_{ik} \Big)^{1/2}$$

i.e. a bound for the trace of $\sigma^*\sigma$.

$$P(T_K^{t,x} < T) = P\Big(\sup_{s\in[t,T]} |X_s^{t,x}| > k \Big) \tag{7.3}$$

and then estimate the right hand side below

$$P\Big(\sup_{s\in[t,T]} |X_s^{t,x}| > k \Big) \le nP\Big\{ \sup_{s\in[0,\hat{\sigma}^2 T]} |B_s| > (k - R - DT) \times \frac{1}{n} \Big\}$$

where B_s is a standard one dimensional Brownian motion. Thus we get

$$\begin{aligned} P(\sup_{s\in[0,\hat{\sigma}^2 T]} |B_s| > k - R - DT) &\le 2P\Big(\sup_{s\in[0,\hat{\sigma}^2 T]} B_s > k - R - DT \Big) \\ &= 2(1 - N(k - R - DT)) \end{aligned} \tag{7.4}$$

where N is the cumulative normal distribution function. Therefore in order to obtain an estimate for (7.2) that becomes small as $k \to \infty$ we must choose k to be large compared to R ie choose $k - R - DT \gg 1$.

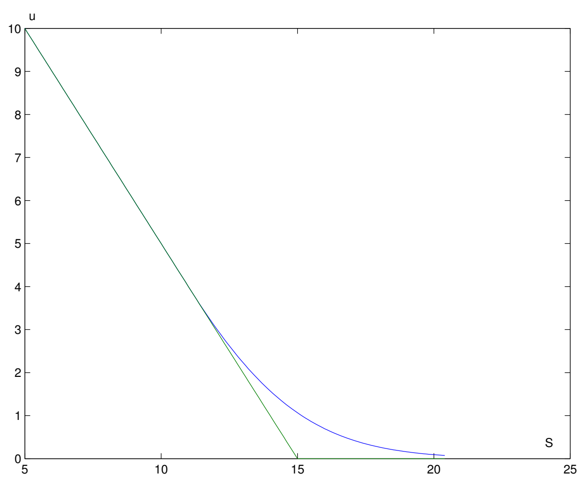

FIGURE 1. An American put option with parameters $K = 15$, $r = 0, 5$, $d = .03$. Here $\sigma = 0.2$ is plotted against spot one year from expiration. The S coordinate of the point of tangency between the payoff option value corresponds to the position of the free boundary at that time

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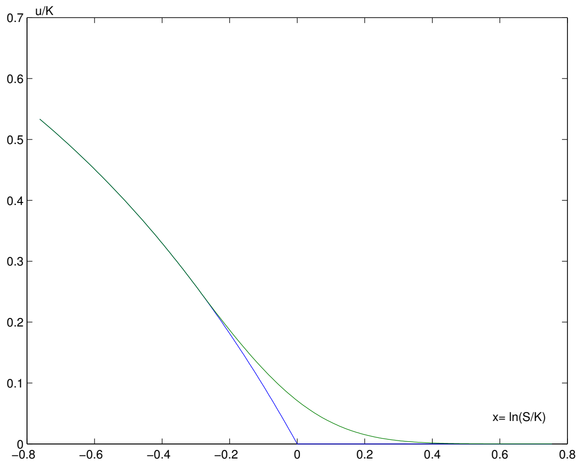

FIGURE 2. The American put option with the same parameters as in Figure 1, but now in the normalized logarithmic variable $log(S/K)$ and normalized payoff $(1-e^x)^+$

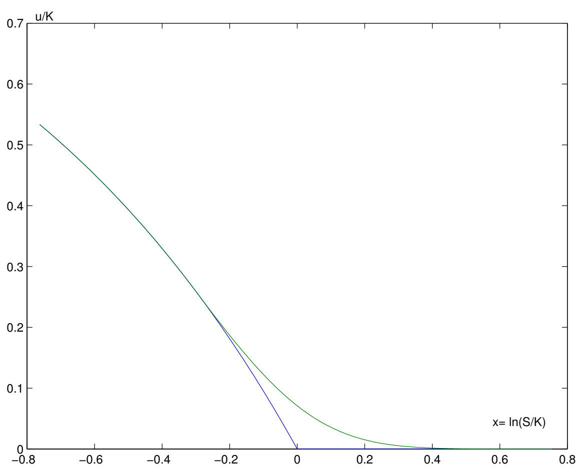

FIGURE 3. The difference between the option's value and the payoff's value, in the normalized variables

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