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GEOMETRIC PROPERTIES OF SOLUTIONS TO MAXIMIZATION PROBLEMS

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Abstract. We investigate the geometric configuration of the maxima of some functionals associated with solutions to Dirichlet problems for special elliptic equations. We also discuss the symmetry breaking and symmetry preservation of the solutions in some particular cases.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and let $D \subset \Omega$ be Lebesgue measurable. Consider the Dirichlet problem

$$\begin{gathered} -\Delta u(x) = \chi_D(x) \quad \text{in } \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\chi_D(x) = 1$ if $x \in D$ and $\chi_D(x) = 0$ if $x \in \Omega \setminus D$. Since $\chi_D(x)$ is not continuous, (1.1) is understood in the weak sense. By standard results on elliptic equations, problem (1.1) has a unique solution $u \in H^2(\Omega) \cap C^1(\Omega) \cap C^0(\overline{\Omega})$ [14]. Of course, the solution does not change if D is replaced by a new set which differs from D by a subset of measure zero.

In a previous paper [8], we have introduced and discussed the maximization problem

$$\max_{|G|=\alpha, |D|=\beta} \int_\Omega \chi_G u_D dx, \tag{1.2}$$

where $G \subset \Omega$, $0 < \alpha \leq |\Omega|$, $0 < \beta < |\Omega|$ and u_D is the solution to problem (1.1). The sets D and G are defined apart subsets of measure zero. In [8] we have found a result of existence for general Ω, α, β, and a result of uniqueness in case Ω is a ball or, for general Ω, in case $\alpha = |\Omega|$. We also have proved that when $\alpha = \beta$ problem (1.2) reduces to the maximization of the energy integral

$$\max_{|D|=\beta} \int_\Omega |\nabla u_D|^2 dx, \tag{1.3}$$

extensively investigated in [1,5,6,7,9,11].

In section 2 of the present paper, we shall prove that if $\beta \leq \alpha$ and (D, G) is a solution to problem (1.2) then $D \subset G$. As a consequence, for $\beta = \alpha$ we must have

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$D = G$. This special case has been investigated in [8] by using a different argument. The interest of our result relies in that the solution of problem (1.2) is not unique in general.

In section 3 we shall consider a special example to show that if Ω is symmetric, a solution (D, G) to problem (1.2) may not be symmetric. This phenomenon, known as symmetry breaking, was already observed in [6,9,11] for problem (1.3). Of course, in this situation we have multiple solutions.

In section 4 we prove that if Ω is Steiner symmetric and if (D, G) is a solution to problem (1.2) then both D and G are Steiner symmetric. This fact was already observed in [8] by using a result described in [2]. Here we use a different approach which may have independent interest.

Several open problems remain. One is the uniqueness of the solution to (1.2) for larger classes of domains Ω and general α and β. We think that the convexity of Ω should be sufficient to have uniqueness. Another problem is the investigation of the shape of the optimal pair (D, G) in case uniqueness holds. We believe that D and G are convex when Ω is convex.

A physical model of problem (1.2) is described in [8]. Many others models leading to equation (1.1) and its generalizations are discussed in [10].

2. Geometric properties

Problem (1.2) with $\alpha = |\Omega|$ reduces to

$$\max_{|D|=\beta} \int_\Omega u_D dx = \max_{|D|=\beta} \int_D w\, dx,$$

where $w = w(x)$ is the solution to the Saint-Venant problem

$$\begin{gathered} -\Delta w(x) = 1 \quad \text{in } \Omega, \\ w(x) = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

The maximizing domain D is unique in this case and can be expressed as $D = \{x \in \Omega : w(x) > t\}$ for a particular t (see [8]). Therefore, from now on we consider $\alpha < |\Omega|$. We state now our main result of this section.

Theorem 2.1. *Let $|\Omega| > \alpha \geq \beta > 0$ and let (D, G) be a solution to problem* (1.2). *Then $D \subset G$. Moreover, there are positive numbers $t \leq \tau$ and positive functions $u_D(x) \leq u_G(x)$ such that $D = \{x \in \Omega : u_G(x) > \tau\}$ and $G = \{x \in \Omega : u_D(x) > t\}$ up to sets of measure zero.*

Proof. Let (D, G) be a solution to problem (1.2) and let u_D and u_G satisfy

$$\begin{gathered} -\Delta u_D(x) = \chi_D(x) \quad \text{in } \Omega, \\ u_D(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

$$\begin{gathered} -\Delta u_G(x) = \chi_G(x) \quad \text{in } \Omega, \\ u_G(x) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.2}$$

By [8] we know that

$$\{u_D(x) > t\} \subset G \subset \{u_D(x) \geq t\}, \tag{2.3}$$

$$\{u_G(x) > \tau\} \subset D \subset \{u_G(x) \geq \tau\} \tag{2.4}$$

for some non negative t, τ. Here and in the sequel, we denote by $\{u(x) > t\}$ the set $\{x \in \Omega : u(x) > t\}$.

Since $|D| > 0$ (and Ω is connected) we have $u_D(x) > 0$ in Ω. If we had $t = 0$ then, by (2.3), we would have $G = \Omega$. But this contradicts the hypothesis $\alpha < |\Omega|$. Similarly, one shows that $\tau > 0$.

Let us prove that

$$\int_\Omega |\nabla(u_G - u_D)|^2 dx \le (\tau - t)(\alpha - \beta). \tag{2.5}$$

Indeed, subtracting equation (2.1) from equation (2.2), multiplying by $(u_G - u_D)$ and integrating we find

$$\begin{aligned}\int_\Omega |\nabla(u_G - u_D)|^2 dx &= \int_\Omega (u_G - u_D)(\chi_G - \chi_D) dx\\ &= \int_{G\setminus D} (u_G - u_D) dx + \int_{D\setminus G} (u_D - u_G) dx.\end{aligned}$$

Using (2.3) and (2.4) we find

$$\int_{G\setminus D} (u_G - u_D) dx \le \int_{G\setminus D} (\tau - t) dx = (\tau - t)|G \setminus D|$$

and

$$\int_{D\setminus G} (u_D - u_G) dx \le \int_{D\setminus G} (t - \tau) dx = (t - \tau)|D \setminus G|.$$

Since $|G \setminus D| = |G| - |D \cap G|$ and $|D \setminus G| = |D| - |D \cap G|$, inequality (2.5) follows. Recalling that $\alpha \ge \beta$, (2.5) implies that $t \le \tau$.

Introduce the subsets of Ω

$$\begin{aligned}\Omega_1 &= \{u_G(x) - u_D(x) > \tau - t\},\\ \Omega_2 &= \{u_G(x) - u_D(x) = \tau - t\},\\ \Omega_3 &= \{u_G(x) - u_D(x) < \tau - t\}.\end{aligned}$$

Of course, $\Omega_1 \cup \Omega_2 \cup \Omega_3 = \Omega$. By (2.4), $u_G(x) \le \tau$ outside D, and by (2.3), $u_D(x) \ge t$ in G. Hence, $u_G(x) - u_D(x) \le \tau - t$ in $G \setminus D$. Therefore,

$$G \setminus D \subset \Omega_2 \cup \Omega_3 = \Omega \setminus \Omega_1.$$

The last inclusion yields

$$\Omega_1 \subset D \cup (\Omega \setminus G). \tag{2.6}$$

On the other side, using equations (2.1)-(2.2) we find

$$-\Delta(u_G - u_D) = \chi_G - \chi_D \le 0 \quad \text{in } D \cup (\Omega \setminus G). \tag{2.7}$$

By (2.6), inequality (2.7) holds in Ω_1. Since $u_G(x) - u_D(x) = \tau - t$ on the boundary of Ω_1, by the maximum principle, we get $u_G(x) - u_D(x) \le \tau - t$ in Ω_1. Recalling the definition of Ω_1, we conclude that this set must be empty.

By (2.4), $u_G(x) \ge \tau$ in D, and by (2.3), $u_D(x) \le t$ outside G. Hence, $u_G(x) - u_D(x) \ge \tau - t$ in $D \setminus G$. Therefore, since Ω_1 is empty,

$$D \setminus G \subset \Omega_1 \cup \Omega_2 = \Omega_2.$$

On Ω_2, $u_G(x) - u_D(x) = \tau - t$, therefore $\Delta(u_G - u_D) = 0$ almost everywhere in $D \setminus G$. On the other side, by using equations (2.1)-(2.2) once more, we get $\Delta(u_G - u_D) = 1$ on $D \setminus G$. We conclude that the measure of $D \setminus G$ must be zero, hence, $D \subset G$ up to a set of measure zero. The first assertion of the theorem is proved.

We have $\Delta u_G = 0$ almost everywhere on the set $\{u_G(x) = \tau\} \cap D$. On the other side, $\Delta u_G = -1$ in G. Since $D \subset G$, the set $\{u_G(x) = \tau\} \cap D$ must have measure zero. Therefore, by (2.4), $D = \{u_G(x) > \tau\}$ up to a set of measure zero.

Decompose the set $\{u_D(x) = t\} \cap G$ into $E_1 = \{u_D(x) = t\} \cap G \cap D$, $E_2 = \{u_D(x) = t\} \cap G \cap (\partial D)$, $E_3 = \{u_D(x) = t\} \cap G \cap (\Omega \setminus \overline{D})$. E_1 has measure zero because $\Delta u_D = -1$ in D, therefore u_D cannot be constant on a set of positive measure. E_2 has measure zero because $u_G(x) = \tau$ on ∂D and $\Delta u_G = -1$ on G. E_3 has measure zero because the function $u_D(x)$ is harmonic (and positive) in the open set $\Omega \setminus \overline{D}$ and $u = 0$ on $\partial\Omega$. Therefore, by (2.3), we must have $G = \{u_D(x) > t\}$ up to a set of measure zero. The theorem is proved. $\square$

Remarks. By (2.1), (2.2) and Theorem 2.1, the functions $u = u_D$ and $v = u_G$ satisfy the equations

$$-\Delta u = H(v - \tau), \quad u|_{\partial\Omega} = 0, \tag{2.8}$$

$$-\Delta v = H(u - t), \quad v|_{\partial\Omega} = 0, \tag{2.9}$$

where $H(s) = 0$ for $s \le 0$ and $H(s) = 1$ for $s > 0$. The system (2.8)-(2.9) may have solutions different from u_D, u_G even when $\alpha = \beta$. Indeed, if $\alpha = \beta$ then $u_D = u_G = u$, and u satisfies

$$-\Delta u = H(u - t), \quad u|_{\partial\Omega} = 0. \tag{2.10}$$

If Ω is a thin annulus and β is small enough then problem (1.3) has a non radial solution $u = u_D$ which satisfies (2.10) [9,11]. Let $w(x)$ be the (radial) solution to the Saint-Venant problem associated with Ω. Using the method of monotone operators (starting from w) we find a radial solution $v = v(x)$ to (2.10) with $u_D(x) \le v(x) \le w(x)$. Of course, $u_D(x) \ne v(x)$ because $u_D(x)$ is non radial.

3. Symmetry breaking

In [6,9,11] it was shown the symmetry breaking of the solution to problem (1.2) in case $\alpha = \beta$. Now, we examine an example to discuss the case $\alpha \ne \beta$. Recall that if Ω is a ball then the maximum of (1.2) is reached when D and G are balls concentric with Ω [8]. Let B_1 and B_2 be open unit balls in $\mathbb{R}^2$ centered at $(-2, 0)$ and $(2, 0)$ respectively, and let $\Omega = B_1 \cup B_2$. Let $D = D_1 \cup D_2$ with D_1 a ball concentric with B_1 and radius R, and D_2 a ball concentric with B_2 and radius S. Similarly, let $G = G_1 \cup G_2$ with G_1 a ball concentric with B_1 and radius T, and G_2 a ball concentric with B_2 and radius Q. We have $|\Omega| = 2\pi$, $|D| = \pi(R^2 + S^2)$ and $|G| = \pi(T^2 + Q^2)$. Assume

$$R^2 + S^2 = b, \quad T^2 + Q^2 = a, \quad b \le a \le 2.$$

We study the problem

$$\max_{|D|=\pi b,\ |G|=\pi a} \int_G u_D\,dx \tag{3.1}$$

with $b/2 \le R^2 \le \min[1, b]$ and $a/2 \le T^2 \le \min[1, a]$. Since $b \le a$, by Theorem 2.1, the maximum in (3.1) is attained when $D \subset G$. Therefore, we may suppose $R \le T$. If $u = u_D$ is the corresponding solution to problem (1.1) we find

$$u(r) = \begin{cases} \frac{R^2}{4} - \frac{r^2}{4} - \frac{R^2}{2}\log R & 0 \le r < R \\ -\frac{R^2}{2}\log r & R \le r < 1 \end{cases} \quad \text{in } B_1$$

and

$$u(s)=\begin{cases}\frac{S^2}{4}-\frac{s^2}{4}-\frac{S^2}{2}\log S & 0\le s<S\\ -\frac{S^2}{2}\log s & S\le s<1\end{cases}\quad\text{in } B_2,$$

with $r^2=(x_1+2)^2+x_2^2$ and $s^2=(x_1-2)^2+x_2^2$. Hence,

$$\begin{aligned}\int_G u_D\,dx=2\pi\Big[&\int_0^R\Big(\frac{R^2}{4}-\frac{r^2}{4}-\frac{R^2}{2}\log R\Big)r\,dr-\frac{R^2}{2}\int_R^T r\log r\,dr\\ &+\int_0^S\Big(\frac{S^2}{4}-\frac{s^2}{4}-\frac{S^2}{2}\log S\Big)s\,ds-\frac{S^2}{2}\int_S^Q s\log s\,ds\Big].\end{aligned}$$

If we put $J(T^2,R^2)=\frac{4}{\pi}\int_G u_D dx$, we find

$$\begin{aligned}J(T^2,R^2)&=R^2\Big[-\frac{R^2}{2}-T^2\log T^2+T^2\Big]\\ &\quad+(b-R^2)\Big[-\frac{b-R^2}{2}-(a-T^2)\log(a-T^2)+(a-T^2)\Big].\end{aligned}$$

We look for solutions to (3.1) which are symmetric with respect to the line $x_1=0$. The symmetric configuration corresponds to $T^2=a/2$ and $R^2=b/2$. Easy computation yields

$$J(a/2,b/2)=b\Big[\frac{a}{2}-\frac{a}{2}\log\frac{a}{2}-\frac{b}{4}\Big].$$

For $b\le a\le 1$, $T^2=a$ and $R^2=b$ (non symmetric configuration) we have

$$J(a,b)=b\Big[a-a\log a-\frac{b}{2}\Big].$$

For $b\le 1<a$ (non symmetric configuration) we find

$$J(1,b)=b\Big[1-\frac{b}{2}\Big].$$

Define

$$z(a)=\begin{cases}a & 0<a\le\sqrt{e}/2\\ 2a\log\frac{e}{2a} & \sqrt{e}/2<a\le 1\\ 2\big[2-a\log\frac{2e}{a}\big] & 1<a<2.\end{cases}$$

For $0<a\le 1$ and $0<b<z(a)$ we have $J(a/2,b/2)<J(a,b)$, and for $1<a<2$ and $0<b<z(a)$ we have $J(a/2,b/2)<J(1,b)$. Hence, for $0<b<z(a)$ the symmetric configuration is not optimal.

Now, connect B_1 with B_2 by a straight channel of width h. Arguing as in [9] and using our previous result one can prove that for h small enough and $0<b<z(a)$ the optimal configuration cannot be symmetric.

4. Symmetry preservation

Let Ω be bounded, connected and Steiner symmetric with respect to a hyperplane Π. In [8] we have proved that problem (1.2) has a solution (D,G) such that D and G are Steiner symmetric with respect to the same hyperplane. We now describe a new method for proving that all solutions to problem (1.2) are Steiner symmetric.

Theorem 4.1. *Let Ω be a bounded domain, Steiner symmetric with respect to a hyperplane Π and let u and v be positive solutions to the problems*

$$-\Delta u=f(v),\quad u|_{\partial\Omega}=0,\tag{4.1}$$

$$-\Delta v = g(u), \quad v|_{\partial\Omega} = 0, \tag{4.2}$$

where f and g are increasing in $[0,\infty)$ and vanish in some interval $[0,\kappa]$. Then, both u and v are symmetric with respect to the hyperplane Π.

Proof. Since f and g may have discontinuities, equations (4.1) and (4.2) are satisfied in a weak sense. However, by standard results on elliptic equations, u and v must belong to $C^1(\Omega)\cap C^0(\overline{\Omega})$.

The method of moving planes for $C^2(\Omega)$ solutions of scalar equations is described in [13,3]. The method has been extended to $C^1(\Omega)$ solutions in the book [12]. We follow very closely the approach described in [12] to get symmetry results for solutions to the system of equations (4.1)-(4.2).

For $x \in \Omega$, we put $x = (x_1, y)$, with $y = (x_2, \cdots, x_N)$. We may assume that the hyperplane Π has equation $x_1 = 0$. Let

$$M = \sup_{x\in\Omega} x_1, \quad \mathrm{d}(x) = \mathrm{dist}(x, \partial\Omega).$$

Our assumptions on u and v imply that

$$\exists h > 0 : \mathrm{d}(x) < h \Rightarrow u(x) < \kappa \text{ and } v(x) < \kappa. \tag{4.3}$$

For such h and for $\mu \in [0, M)$ we define

$$\Sigma(\mu) = \{x \in \Omega : x_1 > \mu\}, \quad \Sigma_h(\mu) = \Sigma(\mu) \cap \{x \in \Omega : \mathrm{d}(x) < h\}. \tag{4.4}$$

Let $x^\mu = (2\mu - x_1, y)$. If $x \in \Sigma(\mu)$ then $x^\mu \in \Omega$ because Ω is Steiner symmetric. For $x \in \Sigma(\mu)$ define

$$w(x) = u(x) - u(x^\mu), \qquad z(x) = v(x) - v(x^\mu).$$

We claim that these two functions satisfy, in a weak sense, the inequalities

$$\Delta w \geq 0 \quad \text{and} \quad \Delta z \geq 0 \quad \forall x \in \Sigma_h(\mu). \tag{4.5}$$

For the proof of (4.5), take $\phi \in C_0^\infty(\Sigma_h(\mu))$, $\phi(x) \geq 0$. Using equation (4.1) we find

$$\int_{\Sigma_h(\mu)} \nabla u(x)\cdot\nabla\phi\,dx = \int_{\Sigma_h(\mu)} f(v(x))\phi\,dx. \tag{4.6}$$

Denote by $\Sigma_h^-(\mu)$ the symmetric image of $\Sigma_h(\mu)$ with respect to the line $x_1 = \mu$. Of course, $\Sigma_h^-(\mu) \subset \Omega$. For $x \in \Sigma_h^-(\mu)$, define $\psi(x) = \phi(x^\mu)$. By equation (4.1) we also have

$$\int_{\Sigma_h^-(\mu)} \nabla u(x)\cdot\nabla\psi\,dx = \int_{\Sigma_h^-(\mu)} f(v(x))\psi\,dx.$$

Using the change of variables $(x_1, y) \to (2\mu - x_1, y)$, the last equation becomes

$$\int_{\Sigma_h(\mu)} \nabla u(x^\mu)\cdot\nabla\phi\,dx = \int_{\Sigma_h(\mu)} f(v(x^\mu))\phi\,dx. \tag{4.7}$$

Subtracting (4.7) from (4.6) we find

$$\int_{\Sigma_h(\mu)} \nabla w\cdot\nabla\phi\,dx = \int_{\Sigma_h(\mu)} \big[f(v(x)) - f(v(x^\mu))\big]\phi\,dx. \tag{4.8}$$

Recalling that $0 < v(x) < \kappa$ in $\Sigma_h(\mu)$, that $f(t)$ vanishes on $[0,\kappa]$ and that $f(t) \geq 0$ in $(0,\infty)$, (4.8) yields

$$\int_{\Sigma_h(\mu)} \nabla w\cdot\nabla\phi\,dx \leq 0 \qquad \forall\phi \in C_0^\infty(\Sigma_h(\mu)), \quad \phi(x) \geq 0.$$

Hence, $\Delta w \geq 0$ in $\Sigma_h(\mu)$. The same proof holds for z, therefore (4.5) is proved.

Observe that equation (4.8) also holds for $\phi \in C_0^\infty(\Sigma(\mu))$. Hence, if we know that $v(x) \le v(x^\mu)$ on $\Sigma(\mu)$ then, using the monotonicity of f, we get $\Delta w \ge 0$ in $\Sigma(\mu)$. A similar remark holds for z. We summarize this fact as

$$\Delta w \ge 0 \quad \text{and} \quad \Delta z \ge 0 \quad \text{whenever} \quad z(x) \le 0 \quad \text{and} \quad w(x) \le 0 \quad \forall x \in \Sigma(\mu). \tag{4.9}$$

Apply now the method of moving planes. Assume h small enough so that (4.3) holds. For μ such that $M-\mu \le h$, we apply the maximum principles ([12], Theorem 2.19 and Theorem 2.13) to the inequality $\Delta w \ge 0$ in $\Sigma(\mu)$. Since $u(x) = 0$ on $\partial\Omega$ and $u(x) > 0$ in Ω, we have $w(x) \le 0$ on $\partial\Sigma(\mu)$ with $w(x) < 0$ at some point of the boundary of each connected component. We conclude that $w(x) < 0$ on $\Sigma(\mu)$ for such values of μ. The same conclusion holds for $z(x)$. Recall that $w(x)$ and $z(x)$ depend on μ.

Let (m, M) be the largest interval of μ such that both

$$w(x) < 0 \quad \text{and} \quad z(x) < 0$$

hold on $\Sigma(\mu)$. By contradiction, assume $m > 0$. Since w and z are continuous with respect to μ, we have

$$w(x) \le 0 \quad \text{and} \quad z(x) \le 0 \quad \forall x \in \Sigma(m).$$

Then, $\Delta w \ge 0$ and $\Delta z \ge 0$ on $\Sigma(m)$ by (4.9). The strong maximum principle ([12] Theorem 2.13) and the assumption $m > 0$ yield

$$w(x) < 0 \quad \text{and} \quad z(x) < 0 \quad \forall x \in \Sigma(m).$$

The boundary point Lemma ([12] Lemma 2.12) applied to the flat boundary $x_1 = \mu$ of $\Sigma(\mu)$, $m \le \mu \le M$ yields

$$\frac{\partial w}{\partial x_1} < 0 \quad \text{and} \quad \frac{\partial z}{\partial x_1} < 0 \quad \forall x \in \overline{\Sigma(m)} \setminus \partial\Omega.$$

Recalling the definition of w and z we must have

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{and} \quad \frac{\partial v}{\partial x_1} < 0 \quad \forall x \in \overline{\Sigma(m)} \setminus \partial\Omega. \tag{4.10}$$

Following again the argument described in [12] (pag. 97), for $\epsilon > 0$ and $\tau > 0$ small, choose a set

$$E_\epsilon = (m-\epsilon, m+\tau] \times S, \quad S \subset \mathbb{R}^{N-1}, \quad E_\epsilon \subset \Sigma(m-\epsilon)$$

as well as a compact subset $F \subset \Sigma(m)$. Using (4.10) one proves that $w(x)$ and $z(x)$ are strictly negative on E_ϵ provided $\{m\} \times S$ is a compact subset of $\{x_1 = m\} \cap \Omega$. Let $G_\epsilon = \Sigma(m-\epsilon) \setminus (E_\epsilon \cup F)$. S and F can be chosen so that, for ϵ small, $w(x) < 0$ on F and $G_\epsilon \subset \Sigma_h(m-\epsilon)$. Using the strong maximum principle again one gets $w(x) < 0$ and $z(x) < 0$ on $\Sigma(m-\epsilon)$. This contradicts the maximality of (m, M) for the negativity of $w(x)$ or $z(x)$.

We conclude that $m = 0$. Hence, $u(x_1, y) \le u(-x_1, y)$ and $v(x_1, y) \le v(-x_1, y)$ on $\Sigma(0)$. Repetition of the same proof starting from the left side of Ω leads to the inequalities $u(x_1, y) \ge u(-x_1, y)$ and $v(x_1, y) \ge v(x_1, y)$ on $\Sigma(0)$, and the theorem follows. $\square$

Remark. The result of Theorem 4.1 can be extended the the more general system

$$-\Delta u = h(u) + f(v), \quad u|_{\partial\Omega} = 0,$$
$$-\Delta v = k(v) + g(u), \quad v|_{\partial\Omega} = 0,$$

where f and g are as before, whereas h and k are locally Lipschitz continuous in $(0, \infty)$. Indeed, in this case, instead of (4.5) one finds

$$\Delta w + c_1(x, \mu) w \geq 0 \quad \text{and} \quad \Delta z + c_2(x, \mu) z \geq 0 \quad \forall x \in \Sigma_h(\mu),$$

where $c_1(x, \mu)$ and $c_2(x, \mu)$ are bounded uniformly with respect to μ. The maximum principles for thin sets apply in this situation [12].

Symmetry results for systems in case of smooth functions are discussed in [18].

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