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ON THE COMPOSITION CONJECTURES

MOHAMAD A. M. ALWASH

ABSTRACT. We describe a class of polynomials that satisfy the composition conjecture for the moments. We also show that the composition conjecture for the moments is not weaker than the composition conjecture for a center. The problem is related to the centers of Abel differential equation.

1. Introduction

The continuous functions $a(t)$ and $b(t)$ satisfy the *composition condition* if

$$a(t) = s'(t)a_1(s(t)), \quad \text{and} \quad b(t) = s'(t)b_1(s(t))$$

for some continuous functions a_1 and b_1, and a differentiable function s with $s(-1) = s(1)$. The differential equation

$$\dot{z} = a(t)z^3 + b(t)z^2 \tag{1.1}$$

has a *center* at $z = 0$ if all the solutions $z(t)$, starting near the origin, satisfy $z(-1) = z(1)$. The interval $[-1, 1]$ can be replaced by any closed interval. The *composition conjecture for the center problem* is that the composition condition is equivalent to that of the differential equation having a center at $z = 0$. The conjecture first appeared in [1]. It was shown in [2] that this conjecture is not true if $a(t)$ and $b(t)$ are polynomial functions in $\cos t$ and $\sin t$. The problem is motivated by the classical center-focus problem for vector fields in the plane. The conjecture was considered recently in the case that the functions $a(t)$ and $b(t)$ are polynomials in t (see [3, 4]). The problem is discussed from several angles with other versions and many particular cases.

Let

$$A(t) = \int_{-1}^{t} a(u)du, \quad B(t) = \int_{-1}^{t} b(u)du$$

and let

$$m_0 = \int_{-1}^{1} b(t)dt, \quad m_k = \int_{-1}^{1} B^k(t)a(t)dt.$$

The *composition conjecture for the moments* is that $a(t)$ and $b(t)$ satisfy the composition condition if and only if $m_k = 0$ for all $k \geq 0$. We refer the reader to [3] for

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details. This conjecture is motivated by the fact that the moments m_k are zero if and only if for all ε near 0, $z = 0$ is a center for

$$\dot{z} = \varepsilon a(t) z^3 + b(t) z^2, \tag{1.2}$$

(see [4]). We show in Section 2 that this conjecture is not true if $a(t)$ and $b(t)$ are trigonometric polynomials. The conjecture is not true also when $a(t)$ and $b(t)$ are polynomials in t (see [5]). It is still interesting to construct classes for which the conjecture is true. It has been proven recently in [4] that the conjecture is true if $b(t)$ is of degree one. The method of proof in [4] and [5] involves results from the algebra of polynomials under composition and the theory of algebraic curves. In Section 2, we give a simple and short proof of this result. We show that the moments stabilize after $\frac{1}{2}$(degree of a) steps. We also demonstrate how the method can be generalized for other classes of equations. In Section 3, we show that vanishing all the moments does not imply that $z = 0$ is a center for (1.1). This means that the composition conjecture for the moments is not weaker than the composition conjecture for a center.

2. Moments

Theorem 2.1. *Suppose that $a(t)$ and $b(t)$ are of degree d and 1, respectively. The polynomials $a(t)$ and $b(t)$ satisfy the composition condition if and only if $m_k = 0$ for $0 \le k \le [\frac{1}{2}d]$, where $[\frac{1}{2}d]$ is the largest integer that is less than or equal to $\frac{1}{2}d$.*

Proof. If the composition condition is satisfied then the integrals of $b(t)$ and $B^k(t)a(t)$ are functions of $s(t)$. Hence, $m_k = 0$ for $k \ge 0$. To prove the other part, let $P_n(t) = ((t^2-1)^n)^{(n)}$ be the n-th degree Legendre polynomials. Since $m_0 = 0$, we write $B(t) = k(t^2 - 1)$, for some nonzero constant k. We also write $a(t) = \sum_0^d k_i P_i(t)$. The result follows from the following lemma. □

Lemma 2.2. *For $k \ge 0$, let $h_k = \int_{-1}^1 B^k(t) P_n(t) dt$. If n is odd or if n is even and $n > 2k$ then $h_k = 0$. If $n = 2k$ then $h_k \neq 0$.*

Proof. If n is odd, then h_k is an integral of an odd polynomial over $[-1, 1]$; hence it is zero. For the case that n is even, we consider the integral

$$\int_{-1}^1 (t^2-1)^k ((t^2-1)^n)^{(n)} dt.$$

When $n > 2k$, we integrate by parts $2k$ times. In $\int u dv = uv - \int v du$, we take at step i

$$dv = ((t^2-1)^n)^{(n-i+1)} dt$$

The integral reduces to

$$[(t^2-1)Q(t) + K((t^2-1)^n)^{(n-2k-1)}]_{t=-1}^{t=1},$$

where K is a constant and Q is a polynomial. Each term of $((t^2-1)^n)^{(n-2k-1)}$ has the factors $t-1$ and $t+1$. Therefore, $h_k = 0$.

For the case that $n = 2k$, we integrate by parts $2k$ times. The value of h_k reduces to $K \int_{-1}^1 (t^2-1)^n dt$, where K is a nonzero constant. It is clear that this integral is nonzero.

Now, the conditions in the statement of the Theorem imply that

$$\begin{gathered} k_0 = 0 \\ c_{11}k_2 = 0 \\ c_{21}k_2 + c_{22}k_4 = 0 \\ c_{31}k_2 + c_{32}k_4 + c_{33}k_6 = 0 \\ \vdots \\ c_{k1}k_2 + c_{k2}k_4 + \cdots + c_{kk}k_{2k} = 0 \end{gathered}$$

where, c_{ij} are constants and $c_{ii} \neq 0$. Solving these equations, recursively, imply that $k_{2i} = 0$ for $i \geq 0$. Hence $a(t)$ has only odd powers of t. Therefore, the composition condition is satisfied with $s(t) = t^2 - 1$. □

Using the same method of proof, it is possible to generalize Theorem 2.1. We state the following theorem; its proof is similar to that of Theorem 2.1.

Theorem 2.3. *Suppose that:*

I. *The function $b(t)$ satisfies $B(-1) = B(1)$.*
II. *There is a sequence of functions $q_0(B(t)), q_1(B(t)), q_2(B(t)), \dots,$ with $\int_{-1}^{1} B^k(t)q_i(t)dt = 0$ if i is even and $i > 2k$; $\int_{-1}^{1} B^k(t)q_{2k}(t)dt \neq 0$.*
III. *The function $a(t)$ is a linear combination of $q_0, Bq_1, q_2, Bq_3, \dots$.*

Then the composition condition is satisfied if and only if $m_k = 0$ for all $k \geq 0$.

Now, we show that the composition conjecture for the moments is not true if $a(t)$ and $b(t)$ are trigonometric polynomials in $\cos t$ and $\sin t$; here we take the interval $[0, 2\pi]$. Let

$$\begin{aligned} f(t) &= h\cos^3 t + 3\cos^2 t \sin t + (6k + 3h)\cos t \sin^2 t - \sin^3 t \\ g(t) &= \cos^3 t + (5k + 2h)\cos^2 t \sin t - 3\cos t \sin^2 t - k\sin^3 t. \end{aligned}$$

We take $a(t) = -f(t)g(t)$ and $b(t) = g'(t) - f(t)$. The solution is a center for equation (1.1) if $2k^2 + hk + 1 = 0$; this follows from the center conditions of a related two-dimensional quadratic system (see, for example, [4]). It is easy to check that

$$m_0 = m_1 = m_2 = m_3 = 0, m_4 = \int_0^{2\pi} B^4(t)a(t)dt = \frac{5\pi}{24}(h + k)^5 \neq 0$$

This proves the following statement.

Theorem 2.4. *If $a(t)$ and $b(t)$ are given as above, then $z = 0$ is a center for* (1.1) *but it is not a center for* (1.2), *with ε near* 0.

3. The center

Let $a(t) = T_2'(t) + T_3'(t)$ and $b(t) = T_6'(t)$, where $T_n(t) = \cos(n \arccos t)$ is the n-th degree Chebyshev polynomial. Here, we take the interval $[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$. For this class of polynomials, $m_k = 0$ for $k \geq 0$; this follows from the following properties of Chebyshev polynomials:

$$\begin{gathered} T_6(t) = T_3(T_2(t)) = T_2(T_3(t)), \\ T_n(-\frac{\sqrt{3}}{2}) = T_n(\frac{\sqrt{3}}{2}), \quad n = 2, 3, 6. \end{gathered}$$

However, the composition condition is not satisfied. This is the simplest of the counterexamples given in [5]. With these $a(t)$ and $b(t)$, we show that $z = 0$ is not a center for equation (1.1). The first necessary conditions for a center are given in [3]; we list the first five conditions.

$$c_1 = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} b(t)dt$$

$$c_2 = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} a(t)dt$$

$$c_3 = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} B(t)a(t)dt$$

$$c_4 = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} B^2(t)a(t)dt$$

$$c_5 = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} (2B^3(t)a(t) - b(t)A^2(t))dt\,.$$

Direct computations, give $c_1 = c_2 = c_3 = c_4 = 0$ and $c_5 = -\frac{864\sqrt{3}}{385}$. This proves the following statement.

Theorem 3.1. *Let* $a(t) = T_2'(t) + T_3'(t)$ *and* $b(t) = T_6'(t)$. *Over the interval* $[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$, *the solution* $z = 0$ *is a center for equation* (1.2) *but it is not a center for equation* (1.1).

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Mohamad A. M. Alwash

Department of Mathematics, West Los Angeles College, 9000 Overland Avenue, Los Angeles, CA 90230-3519, USA

E-mail address: `alwashm@wlac.edu`