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LIFE SPAN OF NONNEGATIVE SOLUTIONS TO CERTAIN QUASILINEAR PARABOLIC CAUCHY PROBLEMS

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Abstract. We consider the problem

$$\rho(x)u_t - \Delta u^m = h(x,t)u^{1+p}, \quad x \in \mathbb{R}^N,\ t > 0,$$

with nonnegative, nontrivial, continuous initial condition,

$$u(x,0) = u_0(x) \not\equiv 0, \quad u_0(x) \geq 0,\ x \in \mathbb{R}^N.$$

An integral inequality is obtained that can be used to find an exponent p_c such that this problem has no nontrivial global solution when $p \leq p_c$. This integral inequality may also be used to estimate the maximal $T > 0$ such that there is a solution for $0 \leq t < T$. This is illustrated for the case $\rho \equiv 1$ and $h \equiv 1$ with initial condition $u(x,0) = \sigma u_0(x)$, $\sigma > 0$, by obtaining a bound of the form $T \leq C_0\sigma^{-\vartheta}$.

1. Introduction

In this article, we investigate the maximal interval of existence of solutions for the problem

$$\rho(x)u_t - \Delta u^m = h(x,t)u^{1+p}, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

with nonnegative, nontrivial, continuous initial condition,

$$u(x,0) = u_0(x) \not\equiv 0, \quad u_0(x) \geq 0,\ x \in \mathbb{R}^N. \tag{1.2}$$

Fujita [3] studied this problem for the case where $m = 1$, $\rho(x) \equiv 1$ and $h(x,t) \equiv 1$ In 1966. He obtained the following, by now famous, results. When $0 < p < 2/N$ the problem fails to have a nontrivial global solution. That is to say that the maximal interval of existence of any solution is finite. When $p > 2/N$ there exists a global solution if $u_0(x) \leq Ae^{-k|x|^2}$ for some constant $k > 0$ provided that A is sufficiently small. The critical case, $p = p_c := 2/N$, was studied by Hayakawa [5], Kobayashi *et al.* [6] and Weissler [11]. They showed that there does not exist a nontrivial, nonnegative global solution in case $p = p_c$. Fujita's work has been extended and generalized by many others. In particular, we should mention that Qi [10] studied the problem

$$u_t - \Delta u^m = |x|^{\varsigma} t^r u^{1+p}.$$

He found that the critical exponent for this problem is $p_c = (m-1)(r+1) + (2 + 2r + \varsigma)/N > 0$. More references can be found, for example, in the two papers, [4]

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and [7] that motivated the present work. In the first of these, Guedda and Kirane reconfigured the test function method of Pohozaev *et al.* [8, 9] and were able to find the critical exponent for equations of the form (1.1) as well as others. The basic idea of the test function methods can be found as far back as in articles of Baras and Pierre [2] and Baras and Kersner [1]. In this article we will take the test function method, but reconfigured once again, in order to study the relationship between the size of the initial condition and the length of the maximal interval of existence. In doing this we will extend some of the results of Tzong-Yow Lee and Wei-Ming Ni [7], who obtained such information for Fujita's problem, i.e. for the case $m = 1$, $h \equiv 1$ and $\rho \equiv 1$. For example, we will show that if u is a global solution with initial condition $u(x, 0) = u_0(x)$, then an inequality of the form

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} \rho(x)u_0(x)\Phi(x/R)\,dx \le C\lambda^{\kappa}$$

must be satisfied. Here Φ is a positive eigenfunction corresponding to the principal eigenvalue of the Dirichlet problem on the unit ball, B_1, and normalized such that $\int_{B_1} \Phi(\xi)\,d\xi = 1$. The numbers C and κ depend on N, m, p, h, and ρ. When $m = 1$, $h \equiv 1$, and $\rho \equiv 1$, then $C = 1$ and $\kappa = 1/p$, a result obtained in [7]. We also obtain a bound for the maximal interval of existence. Suppose u_σ is a solution corresponding to a nontrivial, nonnegative initial condition $u(x, 0) = \sigma u_0(x)$. Let $[0, T_\sigma)$ be its maximal interval of existence. We obtain a bound of the form $T_\sigma \le C\sigma^{-\vartheta}$. When $m \ge 1$, $h \equiv 1$, and $\rho \equiv 1$ then $\vartheta = p + 1 - m$.

2. The test function method

Suppose that u is a solution of (1.1)-(1.2) on $\mathbb{R}^N \times [0, t_*)$. Let $B_R := \{x \in \mathbb{R}^N : |x| < R\}$. We assume that

$$0 < m < p + 1,$$

and that there exists a continuous function h_0 defined on $B_1 \times [0, \infty)$, and real constants β and $\mu \ge 0$ such that for each $T > 0$ and $R > R_0$ we have

$$h(R\xi, R^{\beta}\tau) \ge R^{\mu} h_0(\xi, \tau) \quad \forall \xi \in B_1\ \forall \tau \in [0, T], \tag{2.1}$$

where

$$\int_0^T \int_{B_1} h_0(\xi, \tau)^{-\alpha}\,d\xi\,d\tau < \infty$$

for $\alpha = 1/p$ and for $\alpha = m/(p + 1 - m)$. The simplest examples of functions satisfying these hypotheses are those of the form $h(x, t) = A|x|^{\varsigma} t^r$ where A is a positive constant and ς and r are sufficiently small: $\varsigma < Np$, $\varsigma < N(p + 1 - m)/m$, $r < p$, and $r < (p + 1 - m)/m$.

We assume that there exists a continuous function ρ_0 defined on $B_1 \times [0, \infty)$, and a positive constant ω such that for each $R > R_0$ we have

$$\rho(R\xi) \le R^{\omega}\rho_0(\xi) \quad \forall \xi \in B_1, \tag{2.2}$$

where

$$\int_{B_1} \rho_0(\xi)^{(p+1)/p}\,d\xi < \infty.$$

Let λ_R be the principal eigenvalue for the Dirichlet problem on the ball of radius R:

$$\begin{gathered}-\Delta w(x) = \lambda w(x), \quad x \in B_R,\\ w(x) = 0 \quad x \in \partial B_R.\end{gathered}$$

We note that $\lambda_R = \lambda_1/R^2$. Let Φ denote the unique nonnegative eigenfunction corresponding to the principal eigenvalue λ_1 such that

$$\int_{B_1} \Phi(x)\,dx = 1.$$

Of course Φ is radially symmetric: $\Phi(x) = \Phi_0(|x|)$. For $0 \le S < T$ we define

$$\psi(t) := \begin{cases} 1 & \text{if } t < S\\ (1-(t-S)/(T-S))^\theta & \text{if } S \le t \le T\\ 0 & \text{if } t > T.\end{cases}$$

We also define

$$\zeta(x,t) := \psi(t/R^\beta)\Phi(x/R),$$

and, for $TR^\beta < t_*$,

$$J_R(S,T) := \int_{SR^\beta}^{TR^\beta} \int_{B_R} h(x,t)u^{(1+p)}\zeta(x,t)\,dx\,dt.$$

Using (1.1) and (1.2) and integration by parts we have

$$\begin{aligned}J_R(0,T) &= \int_0^{TR^\beta} \int_{B_R} [\rho(x)u_t - \Delta u^m]\psi(t/R^\beta)\Phi(x/R)\,dx\,dt\\ &= -\int_{B_R} \rho(x)u_0(x)\Phi(x/R)\,dx - \int_0^{TR^\beta}\int_{B_R} R^{-\beta}u\rho\psi'(t/R^\beta)\Phi(x/R)\,dx\,dt\\ &\quad + \int_0^{TR^\beta}\int_{\partial B_R} [-\frac{\partial u^m}{\partial\nu}\psi(t/R^\beta)\Phi(x/R) + u^m\psi(t/R^\beta)R^{-1}\Phi_0'(|x|/R)]\,dS\,dt\\ &\quad + \int_0^{TR^\beta}\int_{B_R} u^m\psi(t/R^\beta)R^{-2}\lambda_1\Phi(x/R)\,dx\,dt\,.\end{aligned}$$

Note that by the Maximum Principle, u cannot attain the value zero in $\mathbb{R}^N \times (0,\infty)$ and consequently the surface integral must be negative. Using the notation

$$V_R := \int_{B_R} \rho(x)u_0(x)\Phi(x/R)\,dx,$$

since $\psi'(t)=0$ except on (S,T), we have

$$\begin{aligned}
&J_R(0,T)+V_R\\
&<+\int_{SR^\beta}^{TR^\beta}\int_{B_R} u[h\psi(t/R^\beta)\Phi(x/R)]^{\frac{1}{p+1}}\rho R^{-\beta}\\
&\quad\times[-\psi'(t/R^\beta)\psi(x/R^\beta)^{-\frac{1}{p+1}}]h^{-\frac{1}{p+1}}\Phi(x/R)^{\frac{p}{p+1}}\,dx\,dt\\
&\quad+\int_0^{TR^\beta}\int_{B_R} u^m[h\psi(t/R^\beta)\Phi(x/R)]^{\frac{m}{p+1}}R^{-2}\lambda_1\\
&\quad\times h^{-\frac{m}{p+1}}\psi(t/R^\beta)^{\frac{p+1-m}{p+1}}\Phi(x/R)^{\frac{p+1-m}{p+1}}\,dx\,dt\\
&\le +J_R(S,T)^{\frac{1}{p+1}}R^{-\beta}\Big[\int_{SR^\beta}^{TR^\beta}\int_{B_R}\rho^{\frac{p+1}{p}}\\
&\quad\times\Big[[-\psi'(t/R^\beta)]^{\frac{p+1}{p}}\psi(x/R^\beta)^{-1/p}\Big]h^{-\frac{1}{p}}\Phi(x/R)\,dx\,dt\Big]^{p/(p+1)}\\
&\quad+J_R(0,T)^{\frac{m}{p+1}}\lambda R^{-2}\Big[\int_0^{TR^\beta}\int_{B_R}h^{-\frac{m}{p+1-m}}\psi(t/R^\beta)\Phi(x/R)\,dx\,dt\Big]^{\frac{p+1-m}{p+1}}.
\end{aligned}$$

Making the change of variables $\xi=x/R$ and $\tau=t/R^\beta$, and using (2.1) and (2.2), we have

$$\begin{aligned}
&J_R(0,T)+V_R\\
&<J_R(S,T)^{\frac{1}{p+1}}R^{s_1}\\
&\quad\times\Big[\int_S^T\int_{B_1}\rho_0(\xi)^{\frac{p+1}{p}}(-\psi'(\tau))^{\frac{p+1}{p}}\psi(\tau)^{-1/p}h_0(\xi,\tau)^{-1/p}\Phi(\xi)\,d\xi\,d\tau\Big]^{p/(p+1)}\\
&\quad+J_R(0,T)^{\frac{m}{p+1}}\lambda R^{s_2}\Big[\int_0^T\int_{B_1}h_0(\xi,\tau)^{-\frac{m}{p+1-m}}\psi(\tau)\Phi(\xi)\,d\xi\,d\tau\Big]^{\frac{p+1-m}{p+1}},
\end{aligned}$$

where

$$s_1:=\omega+\frac{Np-\mu-\beta}{p+1},\quad s_2:=-2+N+\beta-\frac{(N+\beta+\mu)m}{p+1}.$$

Defining

$$\begin{gathered}
A(S,T):=\int_S^T\int_{B_1}\rho_0(\xi)^{\frac{p+1}{p}}(-\psi'(\tau))^{\frac{p+1}{p}}\psi(\tau)^{-1/p}h_0(\xi,\tau)^{-1/p}\Phi(\xi)\,d\xi\,d\tau,\\
B(T):=\lambda\int_0^T\int_{B_1}h_0(\xi,\tau)^{-\frac{m}{p+1-m}}\psi(\tau)\Phi(\xi)\,d\xi\,d\tau,
\end{gathered}$$

for $R>R_0$ we have

$$J_R(0,T)+V_R<J_R(S,T)^{\frac{1}{p+1}}R^{s_1}A(S,T)^{\frac{p}{p+1}}+J_R(0,T)^{\frac{m}{p+1}}\lambda R^{s_2}B(T)^{\frac{p+1-m}{p+1}}. \quad (2.3)$$

Next we choose β such that $s_1=s_2$:

$$\beta:=\frac{(p+1)(\omega+2)+(m-1)(\mu+N)}{p+2-m}, \quad (2.4)$$

so that $s_1=s_2=s$ where

$$s:=\frac{(N+\omega)(p+1-m)-\mu-2}{p+2-m}. \quad (2.5)$$

It is our objective to use (2.3) to obtain information on the relationship between the initial condition and the length of the maximum interval of existence. However, it does also provide a proof to the following result of Guedda and Kirane:

Theorem 2.1. *If $s \le 0$, that is to say*
$$p \le p_c := m - 1 + \frac{2+\mu}{N+\omega},$$
then problem (1.1)–(1.2) *has no global solution except for* $u \equiv 0$.

Proof. When $s < 0$ we take the limit as R tends to infinity on both sides of (2.3) and obtain
$$\int_0^\infty \int_{\mathbb{R}^N} h(x,t)u^{(1+p)}\zeta(x,t)\,dx\,dt + \int_{\mathbb{R}^N} \rho(x)u_0(x)\Phi(0)\,dx = 0, \tag{2.6}$$
so that $u \equiv 0$ is the only global solution. If $s = 0$ we first note that $J_R(0,T)$ is uniformly bounded for all R. This means that we can make $J_R(S,T)$ arbitrarily small by choosing S large enough and hence we can make the first term on the right hand side of (2.3) arbitrarily small, provided we keep $T - S$ bounded. Next we can make the second term arbitrarily small by making $|T-S|$ sufficiently small. Once again we have (2.4). □

It should be noted that the choice of β depends on the value of μ and that these quantities are already related by hypothesis (2.1). This means, that in order to apply this result one needs to compute μ and β simultaneously. We illustrate this with the following example.

Example. Suppose that $h(x,t) = |x|^\varsigma t^r$, where we assume that $p \ne p_* := (r+1)*(m-1)-1$. Then $\mu = \varsigma + r\beta$. Solving this equation and equation (2.4) simultaneously for β and μ we obtain
$$\mu = \frac{(p+1)(\omega r + 2r + \varsigma) + (m-1)(Nr - \varsigma)}{p+1+(r+1)(1-m)},$$
$$\beta = \frac{(\omega+2)(p+1) + (m-1)(N+\varsigma)}{p+1+(r+1)(1-m)}.$$
We also compute
$$s = \frac{(N+\omega)(p - rm + 1 - m) + rN - 2r - 2 - \varsigma}{p+1+(r+1)(1-m)}.$$
We may solve the above equation for p when $s = 0$ in order to see that the critical exponent is
$$p_c = (m + rm - 1) + \frac{-rN + 2 + \varsigma + 2r}{N+\omega},$$
which agrees with the result in [10] when $\omega = 0$. Since $p_c > p_*$, the restriction $p \ne p_*$ does not affect the determination of the critical exponent.

3. Life span of a solution

For the rest of this article, we assume that $S = 0$ and that the value of β is given by (2.4). Suppressing arguments and subscripts (2.3) becomes
$$J + V < J^{\frac{1}{p+1}} R^s A^{\frac{p}{p+1}} + J^{\frac{m}{p+1}} \lambda R^s B^{\frac{p+1-m}{p+1}}. \tag{3.1}$$

We will use this to obtain an estimate for V. First we state some facts whose elementary proofs we leave to the reader.

Lemma 3.1. *Suppose that a, b, r, and q are positive constants. Define the functions $F(x) := ax^q - bx^r$, $G(x) := ax^{-q} + bx^r$ on $0 < x < \infty$. Then*

$$\max_{x>0} F(x) = (1 - q/r)a^{\frac{r}{r-q}}\big(\frac{q}{br}\big)^{\frac{q}{r-q}},$$
$$\min_{x>0} G(x) = (1 + q/r)a^{\frac{r}{r+q}}\big(\frac{br}{q}\big)^{\frac{q}{r+q}},$$

Lemma 3.2. *Let $0 < \omega_1, \omega_2 < 1$, $\omega_1 \neq \omega_2$. On $[0, \infty)$ define*

$$\Upsilon(x) := \max(x^{\omega_1}, x^{\omega_2}).$$

Let η be an arbitrary positive number, then

$$\Psi(\omega_1, \omega_2; \eta) := \max_x(\eta\Upsilon(x) - x) = \max_i \left((1 - \omega_i)\omega_i^{\frac{\omega_i}{1-\omega_i}}\eta^{\frac{1}{1-\omega_i}}\right).$$

For η sufficiently large

$$\Psi(\omega_1, \omega_2; \eta) = (1 - \overline{\omega})\overline{\omega}^{\frac{\overline{\omega}}{1-\overline{\omega}}}\eta^{\frac{1}{1-\overline{\omega}}}, \tag{3.2}$$

where $\overline{\omega} = \max(\omega_1, \omega_2)$.

Proof. The function $\eta\Upsilon(x) - x$ has at most three critical points: the cusp at $x = 1$ and the points where the functions $\eta x^{\omega_1} - x$ and $\eta x^{\omega_2} - x$ attain their maxima. It is easy to see that $\eta\Upsilon(x) - x$ cannot attain its maximum at the cusp. Applying the previous lemma, we see that the maximum value of $\eta\Upsilon(x) - x$ must be the larger of the two values

$$(1 - \omega_i)\omega_i^{\frac{\omega_i}{1-\omega_i}}\eta^{\frac{1}{1-\omega_i}}.$$

The last assertion is obvious. $\square$

We will use the notation $\overline{m} := \max(1, m)$ and

$$J_{\overline{m}} := (1 - \overline{m}/(p + 1))\big(\frac{\overline{m}}{p + 1}\big)^{\frac{\overline{m}}{p+1-\overline{m}}}.$$

Then, for η sufficiently large

$$\Psi(\frac{1}{p + 1}, \frac{m}{p + 1}, \eta) = J_{\overline{m}}\,\eta^{\frac{p+1}{p+1-\overline{m}}}.$$

Theorem 3.3. *If u is a nonnegative solution of (1.1)-(1.2) on $B_{R_*} \times [0, t_*)$, s is given by (2.5). Let*

$$A(T) := \int_0^T \int_{B_1} \rho_0(\xi)^{\frac{p+1}{p}}(-\psi'(\tau))^{\frac{p+1}{p}}\psi(\tau)^{-1/p}h_0(\xi, \tau)^{-1/p}\Phi(\xi)\,d\xi\,d\tau,$$
$$B(T) := \int_0^T \int_{B_1} h_0(\xi, \tau)^{-\frac{m}{p+1-m}}\psi(\tau)\Phi(\xi)\,d\xi\,d\tau.$$

Then for all $(R, T) \in \{(\rho, \tau) : R_0 \le \rho \le R_*,\ 0 \le \tau \le t_*\rho^{-\beta}\}$, we have

$$\int_{B_R} \rho(x)u_0(x)\Phi(x/R)\,dx < \Psi(\frac{1}{p + 1}, \frac{m}{p + 1}; ([A(T)^{\frac{p}{p+1}} + \lambda B(T)^{\frac{p+1-m}{p+1}}]R^s). \tag{3.3}$$

In particular, if u is a global nonnegative solution then

$$\limsup_{R\to\infty} R^{-S}\int_{B_R}\rho(x)u_0(x)\Phi(x/R)\,dx \le J_{\overline{m}}\inf_T\Big[A(T)^{\frac{p}{p+1}}+\lambda B(T)^{\frac{p+1-m}{p+1}}\Big]^{\frac{p+1}{p+1-\overline{m}}}, \tag{3.4}$$

where

$$S:=\frac{s(p+1)}{p+1-\overline{m}}=\frac{(p+1)[(N+\omega)(p+1-m)-\mu-2]}{(p+1-\overline{m})(p+2-m)}.$$

Proof. For the sake of convenience we define

$$\Theta(T)=A(T)^{\frac{p}{p+1}}+\lambda B(T)^{\frac{p+1-m}{p+1}}.$$

From (3.1) we see that $V\le\Upsilon(J)\Theta(T)R^s-J$, where

$$\Upsilon(\sigma):=\max\{\sigma^{\frac{1}{p+1}},\sigma^{\frac{m}{p+1}}\}.$$

Then by Lemma 2, we have (3.3). For R sufficiently large we can use equation (3.2) to conclude the validity of (3.4). □

Corollary 3.4. *Suppose that there exist positive constants ρ_c and h_c such that for $R>R_0$,*

$$h(R\xi,R^\beta\tau)\ge h_cR^\mu,\quad\text{and}\quad \rho(R\xi)\le\rho_cR^\omega,$$

where β is given by (2.4). Suppose that u is a nonnegative global solution. Then

$$\limsup_{R\to\infty}R^{-S}\int_{B_R}\rho(x)u_0(x)\Phi(x/R)\,dx\le J_{\overline{m}}\,K_m^{\frac{p+1}{p+1-\overline{m}}}\lambda^{\frac{p+1}{(p+2-m)(p+1-\overline{m})}}$$

where

$$K_m:=(p+2-m)\Big(\frac{\rho_c^{(p+1-m)}}{(p+1-m)^{(p+1-m)}h_c}\Big)^{1/(p+2-m)}. \tag{3.5}$$

Proof. We easily obtain

$$A(T)\le A_0\equiv\frac{\rho_c^{\frac{p+1}{p}}h_c^{-\frac{1}{p}}\theta^{\frac{p+1}{p}}}{(\theta-1/p)T^{\frac{1}{p}}},\quad\text{and}\quad B(T)\le B_0\equiv\frac{h_c^{-\frac{m}{p+1-m}}T}{\theta+1}.$$

Then

$$V<R^s\big(J^{\frac{1}{p+1}}A_0^{\frac{p}{p+1}}+J^{\frac{m}{p+1}}\lambda B_0^{\frac{p+1-m}{p+1}}\big)-J\le R^s\Theta_0(T)\Upsilon(J)-J,$$

where

$$\Theta(T)\le\Theta_0(T):=\alpha_0T^{-\frac{1}{p+1}}+\beta_0T^{\frac{p+1-m}{p+1}}$$

with

$$\alpha_0:=\frac{\rho_ch_c^{-1/(p+1)}\theta}{(\theta-1/p)^{p/(p+1)}},\quad \beta_0=\frac{\lambda h_c^{-m/(p+1)}}{(\theta+1)^{(p+1-m)/(p+1)}}.$$

By Lemma 1

$$\begin{aligned}\Theta_{00}&:=\min(\Theta_0(T))\\&=\big[(p+1-m)^{-1}\alpha_0\big]^{(p+1-m)/(p+2-m)}\beta_0^{1/(p+2-m)}[p+2-m]\\&=\frac{(p+2-m)(p+1-m)^{-\frac{p+1-m}{p+2-m}}\rho_c^{\frac{p+1-m}{p+2-m}}h_c^{-\frac{1}{(p+2-m)}}\lambda^{\frac{1}{p+2-m}}\theta^{\frac{p+1-m}{p+2-m}}}{(\theta-1/p)^{p(p+1-m)/[(p+1)(p+2-m)]}[\theta+1]^{(p+1-m)/[(p+1)(p+2-m)]}}.\end{aligned}$$

Taking the limit as $\theta\to\infty$ we have $\lim_{\theta\to\infty}\Theta_{00}=K_m\lambda^{1/(p+2-m)}$. Then after substituting this into equation (3.4), the proof is complete. □

When we are dealing with the problem originally considered by Fujita ($\rho \equiv \rho_0 \equiv \rho_c \equiv 1$, $h \equiv h_0 \equiv h_c \equiv 1$, and $m = 1$), then $J_{\overline{m}} = p(p+1)^{-(p+1)/p}$ and $K_m = p^{-1}(p+1)^{(p+1)/p}$ and we see that the above inequality reduces to

$$\limsup_{R\to\infty} R^{-N+2/p} \int_{B_R} \rho(x)u_0(x)\Phi(x/R)\,dx \le \lambda^{1/p}. \tag{3.6}$$

This is precisely the result found in [7]. As done in that article we can deduce the following result.

Corollary 3.5. *When $N \ge S$, Theorem 2.1 and Corollary 3.4 remain valid if we replace*

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} \rho(x)u_0(x)\Phi(x/R)\,dx$$

by $\liminf_{|x|\to\infty} |x|^{N-S}\rho(x)u_0(x)$.

Proof. The statement of this corollary follows from the inequalities:

$$\begin{aligned}
&\lim_{R\to\infty} R^{-S} \int_{B_R} \rho(x)u_0(x)\Phi(x/R)\,dx \\
&\ge \lim_{R\to\infty} R^{-S} \int_{B_R\setminus B_k} \inf_{R\ge|x|\ge k} \big(|x|^{N-S}\rho(x)u_0(x)\big) R^{S-N}\Phi(x/R)\,dx \\
&\ge \lim_{R\to\infty} \inf_{R\ge|x|\ge k} \big(|x|^{N-S}\rho(x)u_0(x)\big) \int_{B_R\setminus B_k} R^{-N}\Phi(x/R)\,dx \\
&= \lim_{R\to\infty} \inf_{R\ge|x|\ge k} \big(|x|^{N-S}\rho(x)u_0(x)\big) \int_{B_1\setminus B_{k/R}} \Phi(\xi)\,d\xi \\
&= \inf_{|x|\ge k} \big(|x|^{N-S}\rho(x)u_0(x)\big).
\end{aligned}$$

The proof is complete by letting k tend to infinity. $\square$

Inequality (3.3) can also be used to obtain an upper bound for the length of the maximal interval of existence. Consider problem (1.1)–(1.2). By the *life span* for initial condition u_0, we mean the least upper bound of all values T such that $[0,T)$ is a maximal interval of existence of a solution to (1.1)-(1.2). Let us fix u_0, $u_0 \not\equiv 0$ and $u_0(x) \ge 0$ for all $x \in \mathbb{R}^N$. We denote by $L(\sigma)$, $\sigma > 0$, the life span corresponding to initial condition σu_0. Assume the hypotheses of Theorem 1 are satisfied, then there exists a value Λ such that

$$\Lambda V_R = \Psi(R^s\Theta(T_M)),$$

where T_M is the value of T at which $\Theta(T)$ attains its minimum value. Let Θ_L denote the restriction of Θ to the interval $[0,T_M)$. If we take $\sigma > \Lambda$, then $L(\sigma) < \infty$ and we see from (3.4) that

$$L(\sigma) \le R^\beta \Theta_L^{-1}\left(R^{-s}\Psi^{-1}(\sigma V_R)\right). \tag{3.7}$$

In the next result we use this inequality to obtain an explicit upper bound for the life span of a solution.

Theorem 3.6. *Assume the hypotheses of Corollary 3.4 Let u_0 be a nonnegative nontrivial continuous function on $\mathbf{R}^N$. There exist positive numbers Λ_m, C_1 and σ_1 so that the life span $L(\sigma)$ corresponding to the initial condition σu_0 with $\sigma > \Lambda_m$ satisfies*

$$L(\sigma) \le C_1\sigma^{-(p+1-\overline{m})}. \tag{3.8}$$

Proof. Decreasing the value of T_M to a value T_m if needed, we may assume that the function Θ_0, introduced above, is decreasing on $(0, T_m)$. We can choose Λ_m such that $\Lambda_m V_R \geq \Psi(R^s\Theta(T_m))$ and also so that $\Lambda_m V_R \geq C_3$ where C_3 is a sufficiently large constant so that whenever $\sigma > \Lambda_m$ then

$$\Psi^{-1}(\sigma V_R) = \left[(1-\overline{\omega})^{-1}\overline{\omega}^{-\frac{\overline{\omega}}{1-\overline{\omega}}}\right]^{1-\overline{\omega}}(\sigma V_R)^{\frac{p+1-\overline{m}}{p+1}}$$

with $\overline{\omega} = \overline{m}/(p+1)$. We write

$$\Psi^{-1}(\sigma V_R) = \gamma_0 V_R^{\frac{p+1-\overline{m}}{p+1}} \sigma^{\frac{p+1-\overline{m}}{p+1}},$$

where $\gamma_0 := (p+1)(p+1-\overline{m})^{-\frac{p+1-\overline{m}}{p+1}}\overline{m}^{-\frac{\overline{m}}{p+1}}$. Since

$$\Theta(T) \leq \Theta_0(T) \leq \alpha_0 T^{-\frac{1}{p+1}} + \beta_0 T_m^{\frac{p+1-m}{p+1}}$$

on $[0, T_m)$, it follows that

$$\Theta_L^{-1}(\eta) \leq \Big[\frac{\eta - \beta_0 T_m^{\frac{p+1-m}{p+1}}}{\alpha_0}\Big]^{-(p+1)}, \qquad \text{for } \eta > \beta_0 T_m^{\frac{p+1-m}{p+1}}.$$

Let $[0, T_\infty)$ be the maximal interval of existence of u and let $T = \tau R^{-\beta}$ where $0 < \tau < T_\infty)$. We define

$$G(R,\sigma) := R^\beta \alpha_0^{p+1}\Big[\gamma_0 R^{-s} V_R^{\frac{p+1-\overline{m}}{p+1}} \sigma^{\frac{p+1-\overline{m}}{p+1}} - \delta_0\Big]^{-(p+1)},$$

where $\delta_0 := \beta_0 T_m^{\frac{p+1-m}{p+1}}$. Whenever $\tau < L(\sigma)$ we have $\tau \leq G(R,\sigma)$. Therefore

$$L(\sigma) \leq G(R,\sigma). \tag{3.9}$$

It is easily seen that this implies equation (16). □

Inequality (17) must be satisfied for all $R > R_0$, However, because the domains depend on R we cannot improve our bound by merely taking the infimum over all $R \geq R_0$. Nevertheless, it is sometimes possible to do so by finding the envelope of the curves $\tau = G(R,\sigma)$. We illustrate this in the next section.

4. Application of results to the problem $u_t = \Delta u^m + u^{p+1}$

Suppose that $m \geq 1$, $\rho \equiv 1$, $h \equiv 1$, and for some nonnegative constant δ, $|x|^{-\delta}u_0$ is bounded from below by a positive constant. Let u_σ be a solution of (1.1) with initial condition $u_\sigma(x,0) = \sigma u_0(x)$. In this case

$$\beta = \frac{2(p+1) + N(m-1)}{p+2-m}, \quad s = \frac{N(p+1-m) - 2}{p+2-m}.$$

We could substitute these values into (17), obtain $G(R,\sigma)$, and then find an envelope for the R-parameterized curves $y = G(R,\sigma)$. However, the R-dependence of the domains and the fact that Ψ is piecewise defined complicate matters. So it is easier to use inequality (3.3) directly. The left side of this inequality is greater than

$$\sigma\int_{B_R} K|x|^\delta \Phi(x/R)\,dx = \sigma K R^{N+\delta}\int_{B_1} |\xi|^\delta \Phi(\xi)\,d\xi = K_1\sigma R^{N+\delta}.$$

Let $[0, T_\sigma)$ be the maximal interval of existence of u_σ. We assume that σ is sufficiently large to ensure that $T_\sigma < \infty$. We may replace Θ in right hand side of (3.3) by Θ_0 and obtain

$$K_1\sigma R^{N+\delta} \leq \Psi(\Theta_0(\tau R^{-\beta})R^s)$$

whenever $0 < \tau < T_\sigma$. Therefore, $\sigma \le \max(F_1(R;\tau), F_2(R;\tau))$, where

$$F_i(R;\tau) := C_i R^{-\delta-N}\Big[\alpha_0\tau^{-\frac{1}{p+1}}R^{\frac{\beta}{p+1}+s} + \beta_0\tau^{\frac{p+1-m}{p+1}}R^{-\beta(\frac{p+1-m}{p+1})+s}\Big]^{q_i},$$

where C_1 and C_2 are certain positive constants and $q_1 := (p+1)/p$ and $q_2 := (p+1)/(p+1-m)$. Now, we define

$\Omega_1^{(i)} := \beta/(p+1) + s - (N+\delta)/q_i, \quad \Omega_2^{(i)} := \beta(p+1-m)/(p+1) - s + (N+\delta)/q_i,$

$\omega_1 := 1/(p+1)$, and $\omega_2 := (p+1-m)/(p+1)$. Then we may write simply

$$F_i(R;\tau) = C_i\big[\alpha_0\tau^{-\omega_1}R^{\Omega_1^{(i)}} + \beta_0\tau^{\omega_2}R^{-\Omega_2^{(i)}}\big]^{q_i}.$$

If we can find functions $y = F_i(\tau)$ such that

$$F_i(R,\tau) \ge F_i(\tau) \quad \forall \tau > 0,$$

and such that for each value of τ there exists a value $R_\tau^{(i)}$ where

$$F_i(R_\tau^{(i)},\tau) = F_i(\tau)$$

then $\sigma \le F_i(R,\tau)$ for all R if and only if $\sigma \le F_i(\tau)$. We make our mission somewhat easier by making a change of variables: let $z_i := R^{\Omega_1^{(i)}+\Omega_2^{(i)}}$ and $\eta := \tau^{\omega_1+\omega_2}$, so that $F_i(R;\tau) = C_i\tau^{-\omega_1 q_i}h_i(z_i;\eta)^{q_i}$, where

$$h_i(z_i;\eta) = \alpha_0 z_i^{1-\gamma_i} + \beta_0 z_i^{-\gamma_i}\eta,$$

and $\gamma_i := \Omega_2^{(i)}/(\Omega_1^{(i)} + \Omega_2^{(i)})$. For the rest of this article, we suppress the index i. We easily find the envelope

$$y = h(\eta) := \alpha_0^\gamma\beta_0^{1-\gamma}\Big[\big(\frac{\gamma}{1-\gamma}\big)^{1-\gamma} + \big(\frac{1-\gamma}{\gamma}\big)^{\gamma}\Big]\eta^{1-\gamma},$$

which leads us to define $F(\tau) := C\tau^{-\omega_1 q}h(\eta)^q$. If we define $\eta_z := \alpha_0\beta_0^{-1}(1-\gamma)\gamma^{-1}z$, then we may write

$$h(\eta) = \big[\alpha_0 z^{1-\gamma_i}\eta_z^{\gamma-1} + \beta_0 z^{-\gamma}\eta_z^{\gamma}\big]\,\eta^{1-\gamma},$$

which immediately shows that the parameterized family of lines $y = h(z,\eta)$ are tangent to the concave curve $y = h(\eta)$ at the respective points $(\eta_z, h(\eta_z))$. Consequently $h(z,\eta) \ge h(\eta)$ for all $z > 0$ and $\eta > 0$, which implies that $F(R;\tau) \ge F(\tau)$. Tracing back through the change of variables we find that $F(R_\tau,\tau) = F(\tau)$ provided we pick $R_\tau = z^{1/(\Omega_1+\Omega_2)}$ where z is the solution of $\eta_z = \tau^{\omega_1+\omega_2}$. Going back to the use of the index i, we see that $\sigma \le \max(F_1(\tau), F_2(\tau))$ where

$$F_i(\tau) := C_i\tau^{-\omega_1 q_i}\big[h_i(\tau^{\omega_1+\omega_2})\big]^{q_i} = M_i\tau^{\theta_i},$$

for some positive constants M_1 and M_2 and with

$$\theta_i = [(1-\gamma_i)\omega_2 - \gamma_i\omega_1]\,q_i. \tag{4.1}$$

Therefore, $\sigma \le \max(M_1\tau^{\theta_1}, M_2\tau^{\theta_2})$. Suppose that the exponents θ_i are negative and let $\vartheta_i := -1/\theta_i$. Then it is clear that $\tau \le C_0\sigma^{-\vartheta}$ for some constant C_0, provided we take $\vartheta := \min(\vartheta_1, \vartheta_2)$ and provided σ is restricted to sufficiently large values. Using equation (18) we can compute the values of ϑ_i, and then obtain the following result.

Corollary 4.1. *For each $\sigma > 0$, let u_σ be a solution of the problem*

$$u_t = \Delta u^m + u^{p+1},$$
$$u(x,0) = \sigma u_0(x)$$

on $\mathbb{R}^N \times [0, T_\sigma)$ where $[0, T_\sigma)$ is its maximum interval of existence. Assume that $0 < m < p+1$ and $u_0(x) \geq K\,|x|^\delta$ for some constants δ and $K > 0$, and that the numbers ϑ_1 and ϑ_2 given below are positive:

$$\vartheta_1 = \frac{[2(p+1) + N(m-1)]p}{2(p+1) + N(m-1) + \delta p(p+2-m)}$$
$$\vartheta_2 = \frac{(2p+2+Nm-N)(p+1-m)}{2(p+1) - N(m-1)(p+1-m) + \delta(p+1-m)(p+2-m)}.$$

Then there exist positive constants C_0 and σ_0 such that

$$T_\sigma \leq C_0 \sigma^{-\vartheta}$$

for all $\sigma > \sigma_0$, where $\vartheta = \min(\vartheta_1, \vartheta_2)$.

Note that in case $m = 1$ and $\delta = 0$, ϑ is simply equal to p, agreeing with the asymptotic result in [7].

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