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DEFORMATION FROM SYMMETRY FOR SCHRÖDINGER EQUATIONS OF HIGHER ORDER ON UNBOUNDED DOMAINS

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Abstract. By means of a perturbation method recently introduced by Bolle, we discuss the existence of infinitely many solutions for a class of perturbed symmetric higher order Schrödinger equations with non-homogeneous boundary data on unbounded domains.

1. Introduction and main results

Let Ω be an unbounded domain in $\mathbb{R}^N$ with $\partial\Omega$ smooth, $N \geq 2K$ and $K \geq 1$, φ and V two functions in suitable spaces. The main goal of this paper is to study the existence of multiple solutions for the polyharmonic Schrödinger equation

$$\begin{gathered}
(-\Delta)^K u + V(x)u = g(x,u) + \varphi \quad \text{in } \Omega \\
\Big(\frac{\partial}{\partial\nu}\Big)^j u = \phi_j \quad \text{on } \partial\Omega \\
D^j u(x) \to 0 \quad \text{as } |x| \to \infty \\
j = 0, \dots, K-1,
\end{gathered} \tag{1.1}$$

where ν is the unit outward normal to $\partial\Omega$, the functions ϕ_j belong to $C^{K-j-1}(\partial\Omega)$ and g is a nonlinearity of power type.

Many papers have been written on the existence and multiplicity of solutions for second order elliptic problems with Dirichlet boundary data, especially by means of variational methods. In particular, problem (1.1) with $K = 1$, $\phi_0 = 0$, $V = 0$ and Ω bounded has been studied by many researchers in the last decades. If $\varphi = 0$ and $g(x, \cdot)$ is odd, the problem is symmetric and multiplicity results can be proved in a standard fashion for any subcritical g (see [21] and references therein). On the contrary, if $\varphi \neq 0$ the symmetry is lost and a natural question is whether the multiplicity is preserved under perturbation of g. Partial answers have been given in [2, 3, 4, 15, 20, 22] where existence of infinitely many solutions was obtained via techniques of critical point theory, provided that suitable restrictions on the growth rate of g are assumed. Roughly speaking, if $g(x, u) \simeq |u|^{p-2}u$ the exponent p is required to be greater than 2 but not too large, that is $2 < p < 2 + \frac{2}{N-2}$.

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The success in looking for solutions of these problems made quite interesting to study the case where, in general, the boundary datum ϕ_0 may differs from zero. This introduces a higher order loss of symmetry, since the functional associated with the problems contains two terms which fail to be even. In this situation some multiplicity achievements have been proved in [8, 9, 10, 12, 11] under suitable stronger restrictions on the growth of g and on the regularity of Ω; in particular, if $g(x,u) \simeq |u|^{p-2}u$, infinitely many solutions have been found if $2 < p < 2 + \frac{2}{N-1}$. Finally, for unbounded domains and $V \neq 0$, some results have been recently established by one of the authors in [18, 19] under more involved assumptions on the growth of g.

Now, a natural question is whether these results for the second order case extend to the higher order. If Ω is bounded, some results have been recently obtained in [13]; in this case, the "critical" exponent for the problem becomes $2 + \frac{2K}{N}$ if $\phi_j \neq 0$ for some j while it is $2+\frac{2K}{N-2K}$ (the natural extension of $2+\frac{2}{N-2}$ to the case $K > 1$) if $\phi_j = 0$ for any j. In the case where Ω is unbounded and $K > 1$ no multiplicity result for (1.1) is, to our knowledge, known. We will show (see Theorems 1.1 and 1.2) that results similar to the second order case hold; in particular, if the Lebesgue measure of Ω is finite, we find again the results of [13]. In Corollary 1.3, we give a multiplicity result when $\Omega = \mathbb{R}^N$. We recall that, in a variational setting, if Ω is unbounded, these problems also present a lack of compactness due to the failure of compactness of the Sobolev embedding $H^K(\Omega) \hookrightarrow L^2(\Omega)$. In order to overcome this problem we assume that the function V has a "good" behaviour at infinity so that the Schrödinger operator $(-\Delta)^K + V$ on $L^2(\Omega)$ admits a discrete spectrum (this may fail, in general, if $V(x) \to 0$ as $|x| \to \infty$) and the Palais-Smale condition can be recovered. If $V = 0$ our condition reduces to a "shrinking" assumption on Ω at infinity which again allow us to regain compactness and also the discreteness of the spectrum of $(-\Delta)^K$. Notice that this last property may fail on general domains of $\mathbb{R}^N$. If for instance Ω is connected with smooth boundary such that

$$\big\{x \in \mathbb{R}^N : x_1 = \cdots = x_{N-1} = 0,\, x_N > 0\big\} \subset \Omega \subset \big\{x \in \mathbb{R}^N : x_N > 0\big\}$$

and $\partial x_N/\partial\nu \leq 0$, then $-\Delta$ admits a purely continuous spectrum (see [16]).

We will actually obtain multiplicity results for a class of higher order operators more general than $(-\Delta)^K$. To achieve this we apply a method recently developed by Bolle, Ghoussoub and Tehrani for dealing with problems with perturbed symmetry (see [7, 8] for the abstract framework and [11] for some recent generalizations).

To state the main results, let $G(x,s) = \int_0^s g(x,t)\,dt$ and assume that the following conditions hold:

- (G1) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous map and $g(x,\cdot)$ is odd.
- (G2) There exists $\bar{s} \neq 0$ such that $\inf_\Omega G(x,\bar{s}) > 0$.
- (G3) There exists $\mu > 2$ such that for every $x \in \Omega$ and every $s \in \mathbb{R}$, $s \neq 0$, $0 < \mu G(x,s) \leq s g(x,s)$.
- (G4) There exist $c > 0$ and $2 < p < \mu + 1$, $p < K_*$ if $N > 2K$, such that for every $(x,s) \in \Omega \times \mathbb{R}$ $|g(x,s)| \leq c|s|^{p-1}$ where $K_* := 2N/(N-2K)$ is the critical Sobolev exponent.

We also assume

- (A1) The operator A is a formally selfadjoint elliptic differential operator of order $2K$ with constant coefficients and there exists $\gamma > 0$ such that for all

$u \in C_c^\infty(\Omega)$:

$$\int_\Omega Au \cdot u \geq \gamma \begin{cases} \int_\Omega |\Delta^m u|^2 & \text{if } K = 2m \\ \int_\Omega |\nabla \Delta^m u|^2 & \text{if } K = 2m+1. \end{cases} \tag{1.2}$$

(V1) The function $V \in C(\Omega)$ is such that $\inf_\Omega V > 0$ and

$$\lim_{|x|\to\infty} \int_{S(x)\cap\Omega} \frac{1}{V(\xi)} d\xi = 0\,,$$

where $S(x)$ is the unit ball of $\mathbb{R}^N$ centered at x.

Assumption (V1) has been used by Benci and Fortunato in [5] for proving some compact embedding theorems for weighted Sobolev spaces. It is easy to see that (V1) holds in particular if V is a continuous function on $\mathbb{R}^N$ which goes to infinity as $|x| \to \infty$. As we will see, this assumption implies that the spectrum of $A+V(x)$ with Dirichlet boundary conditions in $L^2(\Omega)$ is discrete (see Proposition 2.1); from now on we will denote by (λ_n) the divergent sequence of its eigenvalues (counted with their multiplicity). We denote by $L^2(\Omega, V)$ a suitable weighted L^2 space (see Section 2 for further details) and by μ' the conjugate exponent of μ. Moreover, $|B|$ stands for the Lebesgue measure of the set B in $\mathbb{R}^N$.

Let us now set

$$\kappa(p,K,N,\mu) := \frac{2K\mu(p-2)}{(\mu-p+1)(2Kp-(p-2)N)}, \tag{1.3}$$

$$\overline{\kappa}(p,K,N,\mu) := \frac{2K\mu(p-2)}{(\mu-1)(2Kp-(p-2)N)}. \tag{1.4}$$

The following are the main results in the case $V \neq 0$.

Theorem 1.1. *Assume that conditions (G1)–(G4), (A1) and (V1) hold. Let $\varphi \in L^{\mu'}(\Omega)$ and $\phi_j \in C^{K-j-1}(\partial\Omega)$ for $j = 0, \dots, K-1$. Moreover, suppose that*

$$\lim_{n\to\infty} \frac{\lambda_n}{n^{\kappa(p,K,N,\mu)}} = \infty. \tag{1.5}$$

Then, the boundary-value problem

$$\begin{gathered} Au + V(x)u = g(x,u) + \varphi \quad \text{in } \Omega \\ \Big(\frac{\partial}{\partial\nu}\Big)^j u = \phi_j \quad \text{on } \partial\Omega \\ D^j u(x) \to 0 \quad \text{as } |x| \to \infty \\ j = 0, \dots, K-1 \end{gathered} \tag{1.6}$$

admits an unbounded sequence of solutions $(u_n) \subset H^K(\Omega) \cap L^2(\Omega, V)$. Moreover, the same conclusion holds provided that in place of (1.5) we have

$$\frac{\mu}{\mu - p + 1} < \frac{2Kp}{N(p-2)}$$

under the additional assumption that $|\Omega| < \infty$.

If $\phi_j = 0$ for every $j = 0, \dots, K-1$, then the previous result can be improved.

Theorem 1.2. *Assume that conditions (G1)–(G4), (A1) and (V1) hold. Moreover, let $\varphi \in L^{\mu'}(\Omega)$ and suppose that*

$$\lim_{n\to\infty} \frac{\lambda_n}{n^{\overline{\kappa}(p,K,N,\mu)}} = \infty. \tag{1.7}$$

Then, the boundary-value problem

$$
\begin{gathered}
Au + V(x)u = g(x,u) + \varphi \quad \text{in } \Omega \\
\Big(\frac{\partial}{\partial\nu}\Big)^j u = 0 \quad \text{on } \partial\Omega \\
D^j u(x) \to 0 \quad \text{as } |x| \to \infty \\
j = 0, \dots, K-1
\end{gathered} \tag{1.8}
$$

admits an unbounded sequence of solutions $(u_n) \subset H_0^K(\Omega) \cap L^2(\Omega, V)$. *Moreover, the same conclusion holds provided that in place of* (1.7) *we have*

$$
\frac{\mu}{\mu - 1} < \frac{2Kp}{N(p-2)}
$$

under the additional assumption that $|\Omega| < \infty$.

In general, conditions (1.5) and (1.7) are verified if V has a fast growth at infinity and p is greater than 2 but not too large. Notice that in the case $|\Omega| < \infty$ the potential does *not* affect the multiplicity range anymore. We now give an application of Theorem 1.2 when $\Omega = \mathbb{R}^N$ by considering a more particular class of potentials V such that the growth of the eigenvalues (λ_n) of $A + V(x)$ can be explicitely estimated.

Corollary 1.3. *Assume that conditions (G1)–(G4), (A1) and (V1) hold with* $\Omega = \mathbb{R}^N$. *Moreover, assume that there exists* $\alpha \geq 1$ *such that*

$$
\limsup_{\lambda\to\infty} \frac{\big|\{x \in \mathbb{R}^N : V(x) < \lambda\}\big|}{\lambda^{\frac{N}{\alpha}}} < \infty. \tag{1.9}
$$

Then, for every $\varphi \in L^{\mu'}(\mathbb{R}^N)$ *the problem*

$$
\begin{gathered}
Au + V(x)u = g(x,u) + \varphi \quad \text{in } \mathbb{R}^N \\
D^j u(x) \to 0 \quad \text{as } |x| \to \infty \\
j = 0, \dots, K-1
\end{gathered} \tag{1.10}
$$

admits un unbounded sequence of solutions $(u_n) \subset H^K(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, V)$ *provided that*

$$
\frac{\alpha}{2K + \alpha} > \frac{\mu(p-2)N}{(\mu - 1)(2Kp - (p-2)N)}.
$$

In particular, if $\mu = p$, *then* (1.10) *admits infinitely many solutions for any* $p \in]2, \bar{p}[$ *where* $\bar{p}$ *is the largest root of the quadratic equation*

$$
2(KN + \alpha(N-K))p^2 - (\alpha(5N - 2K) + 4KN)p + 2\alpha N = 0.
$$

Proof. Denote by $\mathcal{N}(\lambda, A + V(x), \mathbb{R}^N)$ the number of the eigenvalues of $A + V(x)$ in $L^2(\mathbb{R}^N)$ which are less or equal than λ. As proved in [17, Theorem 3], there exists a constant $B_{N,K} > 0$ such that

$$
\mathcal{N}(\lambda, A + V(x), \mathbb{R}^N) \leq B_{N,K} \int_{\mathbb{R}^N} ((\lambda - V(x))_+)^{N/2K}
$$

for every λ. Clearly, by virtue of the positivity of V and (1.9), for λ sufficiently large we have

$$\begin{aligned}\int_{\mathbb{R}^N} \big((\lambda - V(x))_+\big)^{N/2K} &= \int_{\{V(x)<\lambda\}} (\lambda - V(x))^{N/2K} \\ &\le \lambda^{N/2K} \big|\{V(x) < \lambda\}\big| \\ &\le M\lambda^{N(2K+\alpha)/2K\alpha}\end{aligned}$$

for some positive constant M depending on N and α. Therefore, by choosing in the previous inequality

$$\lambda = \Big(\frac{n}{M}\Big)^{2K\alpha/N(2K+\alpha)}$$

for $n \in \mathbb{N}$ sufficiently large we have

$$\lambda_n \ge C_{N,K,\alpha}\, n^{2K\alpha/N(2K+\alpha)}$$

being $C_{N,K,\alpha}$ a suitable positive constant depending on N, K and α. The assertion now follows immediately by applying Theorem 1.2. $\square$

To give an idea of the amplitude of the range $]2, \bar{p}[$, in the following table we list the values of $\bar{p}$ for $K = 1, \dots, 10$ corresponding to the dimensions $N = 2K + 1$.

$\alpha \ge 1$	$K = 1$	$K = 2$	$K = 3$	$\cdots$	$K = 10$
$\alpha = 1$	$\bar{p} = 2.2310$	$\bar{p} = 2.1688$	$\bar{p} = 2.1284$	$\cdots$	$\bar{p} = 2.0463$
$\alpha = 2$	$\bar{p} = 2.3494$	$\bar{p} = 2.2895$	$\bar{p} = 2.2319$	$\cdots$	$\bar{p} = 2.0901$
$\alpha = 3$	$\bar{p} = 2.4201$	$\bar{p} = 2.3786$	$\bar{p} = 2.3161$	$\cdots$	$\bar{p} = 2.1314$
$\alpha = 4$	$\bar{p} = 2.4668$	$\bar{p} = 2.4466$	$\bar{p} = 2.3854$	$\cdots$	$\bar{p} = 2.1704$
$\alpha = 5$	$\bar{p} = 2.5000$	$\bar{p} = 2.5000$	$\bar{p} = 2.4432$	$\cdots$	$\bar{p} = 2.2072$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\alpha = \infty$	$\bar{p} \simeq 2.6929$	$\bar{p} \simeq 2.9314$	$\bar{p} \simeq 3.0514$	$\cdots$	$\bar{p} \simeq 3.2816$

Table 1. Values of $\bar{p}$ varying α when $K = 1, \dots, 10$ and $N = 2K + 1$.

Observe that condition (1.9) holds for example if V is a positive function verifying $V(x) = |x|^\alpha$ for some $\alpha \ge 1$ and for all $x \in \mathbb{R}^N$, $|x|$ large. If, in particular, V grows exponentially fast and $\mu = p$, then Corollary 1.3 yields infinitely many solutions for any $p \in]2, p_\infty[$, being p_∞ the largest root of the equation $2(N - K)p^2 - (5N - 2K)p + 2N = 0$ (see the last row of Table 1). If $N = 2K + 1$, we get $p_\infty \to \sqrt{2} + 2$ as $K \to \infty$.

Note that the condition (V1) holds if and only if for every $b > 0$

$$\lim_{|x|\to\infty} \big|S(x) \cap \Omega_b\big| = 0, \quad \text{where} \quad \Omega_b = \big\{x \in \Omega : \ 0 < V_0 \le V(x) \le b\big\}. \tag{1.11}$$

Therefore, in the case where the potential function V is identically equal to zero, we are led to consider the following assumption, which corresponds to a "shrinking" condition at infinity:

(D1) The domain Ω is unbounded and

$$\lim_{|x|\to\infty} |S(x) \cap \Omega| = 0.$$

This condition is a necessary and sufficient condition for the embedding of a space "like" $H^K(\Omega)$ in $L^2(\Omega)$ to be compact; moreover it implies that the spectrum of A with Dirichlet boundary data consists of a sequence (μ_n) of eigenvalues (with finite multiplicity) having ∞ as the only accumulation point (see Proposition 2.2).

Let κ and $\bar{\kappa}$ be defined as in (1.3) and (1.4). The following corollaries complement the results of [13] dealing with bounded domains.

Corollary 1.4. *Assume that Ω is a domain satisfying (D1) and that conditions (G1)–(G4) and (A1) hold. Let $\varphi \in L^{\mu'}(\Omega)$ and $\phi_j \in C^{K-j-1}(\partial\Omega)$ for $j = 0, \dots, K-1$. Moreover, suppose that*

$$\lim_{n\to\infty} \frac{\mu_n}{n^{\kappa(p,K,N,\mu)}} = \infty. \tag{1.12}$$

Then, the boundary-value problem

$$\begin{gathered} Au = g(x,u) + \varphi \quad \text{in } \Omega \\ \big(\frac{\partial}{\partial\nu}\big)^j u = \phi_j \quad \text{on } \partial\Omega \\ D^j u(x) \to 0 \quad \text{as } |x| \to \infty \\ j = 0, \dots, K-1 \end{gathered} \tag{1.13}$$

admits an unbounded sequence of solutions $(u_n) \subset H^K(\Omega)$. Moreover, the same conclusion holds provided that in place of (1.12) *we have*

$$\frac{\mu}{\mu - p + 1} < \frac{2Kp}{N(p-2)}$$

under the additional assumption that $|\Omega| < \infty$.

If $\phi_j = 0$ for every $j = 0, \dots, K-1$, a stronger result holds.

Corollary 1.5. *Assume that Ω is a domain satisfying (D1). Let $\varphi \in L^{\mu'}(\Omega)$ and assume conditions (G1)–(G4) and (A1). Moreover, suppose that*

$$\lim_{n\to\infty} \frac{\mu_n}{n^{\bar{\kappa}(p,K,N,\mu)}} = \infty. \tag{1.14}$$

Then, the boundary-value problem

$$\begin{gathered} Au = g(x,u) + \varphi \quad \text{in } \Omega \\ \big(\frac{\partial}{\partial\nu}\big)^j u = 0 \quad \text{on } \partial\Omega \\ D^j u(x) \to 0 \quad \text{as } |x| \to \infty \\ j = 0, \dots, K-1 \end{gathered} \tag{1.15}$$

admits an unbounded sequence of solutions $(u_n) \subset H^K_0(\Omega)$. Moreover, the same conclusion holds provided that in place of (1.14) *we have*

$$\frac{\mu}{\mu - 1} < \frac{2Kp}{N(p-2)}$$

under the additional assumption that $|\Omega| < \infty$.

In the case where $\mu = p$ and Ω has finite measure, we compare in the following table some of the existence ranges for nonzero and zero boundary data when $N = 2K + 1$.

$K \geq 1$	$\exists j : \phi_j \neq 0$	$\forall j : \phi_j = 0$
$K = 1$	$p < 2.6666$	$p < 4$
$K = 2$	$p < 2.8000$	$p < 6$
$K = 3$	$p < 2.8571$	$p < 8$
$\vdots$	$\vdots$	$\vdots$
$K = 10$	$p < 2.9523$	$p < 22$

TABLE 2. Comparison between zero and nonzero boundary data

By comparing Table 1 with the second column of Table 2, note how the situation $|\Omega| = \infty$ has a bad influence, with respect to the case $|\Omega| < \infty$, on the existence ranges. Finally we stress that under suitable additional regularity assumptions on Ω, ϕ_j and φ, by virtue of a special Pohožaev type identity, in the case $K = 1$ Theorem 1.1 and Corollary 1.4, can be improved (see [8, 19]). Unfortunately, it seems that a similar identity cannot be easily obtained when $K > 1$. On the other hand, if $\phi_j = 0$ for every j, the "critical" exponent for our problem seems to be $2 + \frac{2K}{N-2K}$, then the results contained in Theorem 1.2 and Corollary 1.5 coincide with those already stated for $K = 1$ (see [19, Corollary 1.6] if $|\Omega| = \infty$ and [3, 8, 22] if $|\Omega| < \infty$).

2. The variational framework

Let $K \geq 1$ and $B \subset \mathbb{R}^N$ smooth. We endow the spaces $L^s(B)$ and $H^K(B)$ with the norms

$$\|u\|_{L^s(B)} = \Big(\int_B |u|^s \Big)^{1/s}, \quad \|u\|_{H^K(B)} = \Big\{ \int_B |u|^2 + \sum_{|\mu|=K} \int_B |D^\mu u|^2 \Big\}^{1/2}.$$

We recall that, by [1, Corollary 4.16], the norm $\|\cdot\|_{H^K(B)}$ is equivalent to the standard norm of $H^K(B)$. Moreover, let $H_0^K(B)$ be the completion of $C_c^\infty(B)$ with respect to $\|\cdot\|_{H^K(B)}$. Now, we endow $H_0^K(B)$ with another norm equivalent to $\|\cdot\|_{H^K(B)}$. We say that a function u on B is in $\widetilde{H}^K(B)$ if it is the restriction to B of a function in $H^K(\mathbb{R}^N)$. We set

$$\|u\|_{\widetilde{H}^K(B)} = \inf\big\{ \|v\|_{H^K(\mathbb{R}^N)} : v \in H^K(\mathbb{R}^N),\, v = u \text{ on } B \big\}.$$

It is possible to prove that $\widetilde{H}^K(B)$ is a Banach space, $H_0^K(B)$ is continuously embedded in $\widetilde{H}^K(B)$ and that the norms $\|\cdot\|_{\widetilde{H}^K(B)}$ and $\|\cdot\|_{H^K(B)}$ are equivalent in $H_0^K(B)$ (see [6]). For the sake of simplicity, if $B = \Omega$ we will write $\|u\|_s$, $\|u\|_{K,2}$ and $\widetilde{\|u\|}_{K,2}$ in place of $\|u\|_{L^s(\Omega)}$, $\|u\|_{H^K(\Omega)}$ and $\|u\|_{\widetilde{H}^K(\Omega)}$. ¿From now on, we assume that the function V satisfies condition (V1). Then, we can consider the weighted L^2 space

$$L^2(\Omega, V) = \Big\{ u \in L^2(\Omega) : \int_\Omega V(x) u^2 < \infty \Big\}$$

equipped with the inner product $\int_\Omega V(x)uv$ and the Sobolev space

$$H^K(\Omega, V) = H^K(\Omega) \cap L^2(\Omega, V)$$

endowed with the weighted inner product

$$(u,v)_V = \int_\Omega V(x)uv + \sum_{|\mu|=K} \int_\Omega D^\mu u D^\mu v. \tag{2.1}$$

We denote by $\|\cdot\|_V$ the norm induced by (2.1). On the other hand, we can consider the Banach space

$$\widetilde{H}^K(\Omega,V) = \widetilde{H}^K(\Omega) \cap L^2(\Omega,V)$$

endowed with the corresponding norm $\|\cdot\widetilde{\|}_V = \|\cdot\widetilde{\|}_{K,2} + \int_\Omega V(x)u^2$ and the subspace $H_0^K(\Omega,V) = H_0^K(\Omega) \cap L^2(\Omega,V)$, where $\|\cdot\|_V$ and $\|\cdot\widetilde{\|}_V$ are equivalent.

To overcome the lack of compactness of the problem, the following Propositions are needed.

Proposition 2.1. *Assume that Ω is an unbounded domain in $\mathbb{R}^N$ and let V a function satisfying assumption (V1). Then the embedding $\widetilde{H}^K(\Omega,V) \hookrightarrow L^2(\Omega)$ is compact. It follows that the embedding $H_0^K(\Omega,V) \hookrightarrow L^s(\Omega)$ is compact for every $s \in [2, K_*[$ and the spectrum of the selfadjoint realization of $A + V(x)$ with homogeneous Dirichlet boundary conditions in $L^2(\Omega)$ is discrete.*

Proof. If $\Omega = \mathbb{R}^N$, in [5, Theorem 3.1] it has been proved that the space $H^K(\mathbb{R}^N, V)$ is compactly embedded in $L^2(\mathbb{R}^N)$. Small modifications in their proof allow to extend this result to a general unbounded domain Ω for the space $\widetilde{H}^K(\Omega,V)$. For the sake of completeness we sketch the proof. Let (u_n) be a sequence in $\widetilde{H}^K(\Omega,V)$ such that $u_n \rightharpoonup 0$ in $\widetilde{H}^K(\Omega,V)$. Then, there exists $M \in \mathbb{R}^+$ such that $\int_\Omega V(x)u_n^2 \leq M$ and $u_n|_A \rightharpoonup 0$ in $\widetilde{H}^K(A)$ for every $A \in \sigma_0$, where

$$\begin{aligned} \sigma_0 &= \{A \in \sigma_m : \ A \text{ is open}\}, \\ \sigma_m &= \{A \subset \Omega : \ A \text{ is measurable and } |A \cap S(x)| \to 0 \text{ for } |x| \to \infty\}. \end{aligned}$$

By virtue of Theorem 2.8 of [6] we have

$$\forall A \in \sigma_0 : \quad \widetilde{H}^K(A) \text{ is compactly embedded in } L^2(A)$$

and then for all $A \in \sigma_0$,

$$u_n|_A \to 0 \quad \text{in } L^2(A). \tag{2.2}$$

Our aim is to prove that $\|u_n\|_2 \to 0$. Taking any $\varepsilon > 0$, we have

$$\|u_n\|_2^2 = \int_\Omega \frac{1}{V(x)} V(x) u_n^2 \leq \varepsilon M + \int_{\Omega_{1/\varepsilon}} u_n^2,$$

where, by (1.11), it is $\Omega_{1/\varepsilon} \in \sigma_m$. A slightly modified version of Lemma 3.2 in [5] implies that there exists $A_{1/\varepsilon}$ in σ_0 such that $\Omega_{1/\varepsilon} \subset A_{1/\varepsilon}$. Hence for all $n \in \mathbb{N}$,

$$\|u_n\|_2^2 \leq \varepsilon M + \int_{A_{1/\varepsilon}} u_n^2$$

so that, by (2.2), $\|u_n\|_2 \to 0$. Therefore, $\widetilde{H}^K(\Omega,V)$ is compactly embedded in $L^2(\Omega)$. Moreover, by the Sobolev embedding we have $H^K(\Omega) \hookrightarrow L^s(\Omega)$ for any $s \in [2, K_*]$. Now, by the Gagliardo-Nirenberg interpolation inequality, for any u which belongs to $L^2(\Omega) \cap L^{K_*}(\Omega)$ it results $u \in L^s(\Omega)$ and $\|u\|_s \leq \|u\|_2^{1-\ell} \|u\|_{K_*}^\ell$ with $\frac{1}{s} = \frac{1-\ell}{2} + \frac{\ell}{K_*}$ and $0 \leq \ell \leq 1$. Hence, the embedding of $H_0^K(\Omega,V)$ in $L^s(\Omega)$ is compact for any $s \in [2, K_*[$. Finally, since the operator $A + V(x)$ with homogeneous

Dirichlet boundary conditions is essentially selfadjoint on $C_0^\infty(\Omega)$, the discreteness of the spectrum follows arguing as in [5, Theorem 4.1]. $\square$

Proposition 2.2. *Assume that Ω is an unbounded domain in $\mathbb{R}^N$. Then the embedding $\widetilde{H}^K(\Omega) \hookrightarrow L^s(\Omega)$ is compact for every $s \in [2, K_*[$ if and only if Ω satisfies the assumption (D1). In particular, if (D1) holds, $H_0^K(\Omega)$ is compactly embedded in $L^s(\Omega)$ and the spectrum of A with homogeneous Dirichlet boundary conditions in $L^2(\Omega)$ is discrete.*

Proof. For the first part, see [6, Theorem 2.8]. Then, since A with Dirichlet boundary data is essentially selfadjoint on $C_0^\infty(\Omega)$, the discreteness of the spectrum follows by repeating the argument in [5, Theorem 4.1]. $\square$

Now, let us denote by Φ a solution (which exists by standard minimum arguments) of the following linear problem

$$\begin{gathered} Au + V(x)u = 0 \quad \text{in } \Omega \\ \big(\frac{\partial}{\partial\nu}\big)^j u = \phi_j \quad \text{on } \partial\Omega \\ D^j u(x) \to 0 \quad \text{as } |x| \to \infty \\ j = 0, \dots, K-1. \end{gathered}$$

Note that the function Φ belongs to $L^t(\Omega)$ for every $t \geq 2$. Indeed, if we set $\Omega_0 = \{x \in \Omega : \Phi(x) > 1\}$, Ω_0 is bounded since Φ goes to zero at infinity. Moreover by the regularity of $\partial\Omega$, A, V and the data ϕ_j, we have $\Phi \in L^t(K)$ for each compact $K \subset\subset \Omega$ (use the interior regularity estimates of [14]). Therefore we have

$$\int_\Omega |\Phi(x)|^t \leq \int_{\overline{\Omega}_0} |\Phi(x)|^t + \int_{\Omega\setminus\Omega_0} |\Phi(x)|^2 < \infty.$$

Then, the original problem (1.6) can be reduced to

$$\begin{gathered} Aw + V(x)w = g(x, w+\Phi) + \varphi \quad \text{in } \Omega \\ \big(\frac{\partial}{\partial\nu}\big)^j w = 0 \quad \text{on } \partial\Omega \\ D^j w(x) \to 0 \quad \text{as } |x| \to \infty \\ j = 0, \dots, K-1. \end{gathered}$$

More precisely, a function u is a weak solution of (1.6) if and only if $u \in H^K(\Omega)$, $u = w + \Phi$ and the function $w \in H_0^K(\Omega)$ is such that for all $\eta \in H_0^K(\Omega)$,

$$\int_\Omega Aw \cdot \eta + \int_\Omega V(x)w\eta = \int_\Omega g(x, w+\Phi)\eta + \int_\Omega \varphi\eta.$$

Hence, our aim is to state the existence of multiple critical points of the functional

$$I_1(u) = \frac{1}{2}\int_\Omega Au \cdot u + \frac{1}{2}\int_\Omega V(x)u^2 - \int_\Omega G(x, u+\Phi) - \int_\Omega \varphi u$$

defined on the Hilbert space $X_V = H_0^K(\Omega, V)$, endowed with the equivalent inner product

$$(u,v)_{X_V} = \int_\Omega V(x)uv + \begin{cases} \int_\Omega \Delta^m u\, \Delta^m v & \text{if } K = 2m \\ \int_\Omega \nabla\Delta^m u\, \nabla\Delta^m v & \text{if } K = 2m+1. \end{cases}$$

We denote by $\|\cdot\|_{X_V}$ the corresponding norm induced by $(\cdot,\cdot)_{X_V}$. Following the abstract perturbation method that we will describe in the next section, let us consider the family of functionals $I_\vartheta = I(\vartheta,\cdot) : X_V \to \mathbb{R}$, $0 \le \vartheta \le 1$, defined by

$$I_\vartheta(u) = \frac{1}{2}\int_\Omega Au\cdot u + \frac{1}{2}\int_\Omega V(x)u^2 - \int_\Omega G(x, u+\vartheta\Phi) - \int_\Omega \vartheta\varphi u.$$

Standard arguments show that I is a C^1 functional and it satisfies

$$\frac{\partial I}{\partial\vartheta}(\vartheta,u) = -\int_\Omega g(x,u+\vartheta\Phi)\Phi - \int_\Omega \varphi u, \tag{2.3}$$

$$\begin{aligned} I'_\vartheta(u)[v] &= \frac{\partial I}{\partial u}(\vartheta,u)[v] \\ &= \int_\Omega Au\cdot v + \int_\Omega V(x)uv - \int_\Omega g(x,u+\vartheta\Phi)v - \int_\Omega \vartheta\varphi v \end{aligned} \tag{2.4}$$

for every $\vartheta\in[0,1]$ and $u,v\in X_V$.

3. Proof of the results

To apply the method introduced by Bolle for dealing with problems with broken symmetry, let us recall the main theorem as stated in [11]. Consider two continuous functions $\varrho_1,\varrho_2 : [0,1]\times\mathbb{R}\to\mathbb{R}$ which are Lipschitz continuous with respect to the second variable and with $\varrho_1\le\varrho_2$.

Let J_0 be a C^1-functional on a Hilbert space X with norm $\|\cdot\|$. We say that a C^1-functional $J : [0,1]\times X\to\mathbb{R}$ is a good family of functionals starting from J_0 and controlled by ϱ_1,ϱ_2 if $J(0,\cdot)=J_0$ and if it satisfies the conditions (H1)–(H4) below, where $J_\vartheta := J(\vartheta,\cdot)$.

- (H1) For every sequence (ϑ_n,u_n) in $[0,1]\times X$ such that $(J(\vartheta_n,u_n))$ is bounded and $\lim_n J'_{\vartheta_n}(u_n)=0$, there exists a convergent subsequence.
- (H2) For any $b>0$ there exists $C_b>0$ such that if $(\vartheta,u)\in[0,1]\times X$ then $|J_\vartheta(u)|\le b$ implies
$$\Big|\frac{\partial J}{\partial\vartheta}(\vartheta,u)\Big| \le C_b(\|J'_\vartheta(u)\|+1)(\|u\|+1)$$
- (H3) For any critical point u of J_ϑ we have
$$\varrho_1(\vartheta,J_\vartheta(u)) \le \frac{\partial}{\partial\vartheta}J(\vartheta,u) \le \varrho_2(\vartheta,J_\vartheta(u))$$
- (H4) For any finite dimensional subspace W of X it results
$$\lim_{u\in W,\,\|u\|\to\infty}\ \sup_{\vartheta\in[0,1]} J(\vartheta,u) = -\infty.$$

Setting $\bar\varrho_i(s) := \sup_{\vartheta\in[0,1]}|\varrho_i(\vartheta,s)|$ we have the following result [11, Theorem 2.1].

Theorem 3.1. *Let $\varrho_1\le\varrho_2$ be two velocity fields. Let J_0 be an even C^1-functional on X and J a good family of functionals starting from J_0 and controlled by ϱ_1,ϱ_2. Let X be a Hilbert decomposed as*

$$X = \cup_{n=0}^\infty X_n,$$

where $X_0 = X_-$ is a finite dimensional subspace and (X_n) is an increasing sequence of subspaces of X such that $X_n = X_{n-1}\bigoplus\mathbb{R}e_n$. Consider the levels

$$c_n = \inf_{h\in\mathcal{H}}\sup_{h(X_n)} J_0,$$

where

$$\mathcal{H} = \Big\{ h \in C(X, X) : h \text{ is odd and } h(u) = u \text{ for } \|u\| > R \text{ for some } R > 0 \Big\}.$$

Assume that, for n large, it is $c_n \geq B_1 + (B_2(n))^{\bar{\beta}}$ where $\bar{\beta} > 0$, $B_1 \in \mathbb{R}$, $B_2(n) > 0$ and

$$\bar{\varrho}_i(s) \leq A_1 + A_2 |s|^{\bar{\alpha}}, \quad 0 \leq \bar{\alpha} < 1, \; A_1, A_2 \geq 0.$$

Then J_1 has an unbounded sequence of critical levels if

$$\lim_{n\to\infty} \frac{(B_2(n))^{\bar{\beta}}}{n^{\frac{1}{1-\bar{\alpha}}}} = \infty.$$

Let us now return to our concrete framework. In order to apply the previous theorem to the functional $I(\vartheta, u)$, we need the following lemmas. In the sequel, we will denote by c_i some suitable positive constants.

Lemma 3.2. *Let $(\vartheta_n, u_n) \subset [0,1] \times X_V$ be such that for some $C > 0$*

$$|I(\vartheta_n, u_n)| \leq C, \quad \lim_n I'_{\vartheta_n}(u_n) = 0 \quad \text{in } X_V'.$$

Then, up to a subsequence, (ϑ_n, u_n) converges in $[0,1] \times X_V$.

Proof. Since we have $(I'_{\vartheta_n}(u_n), u_n)_{X_V} = o(\|u_n\|_{X_V})$ as $n \to \infty$, for every $\rho \in \left]\frac{1}{\mu}, \frac{1}{2}\right[$, taking into account (1.2), (2.4) and (G3), (G4), for n large it results

$$\begin{aligned} C + \rho\|u_n\|_{X_V} &\geq I_{\vartheta_n}(u_n) - \rho(I'_{\vartheta_n}(u_n), u_n)_{X_V} \\ &\geq (\frac{1}{2} - \rho)\bar{\gamma}\|u_n\|^2_{X_V} + (\rho\mu - 1)\, c_1 \|u_n + \vartheta\Phi\|^\mu_\mu \\ &\quad - c_2 \int_\Omega |u_n + \vartheta\,\Phi|^{p-1}\,|\Phi| - \int_\Omega |u_n + \vartheta\,\Phi||\varphi| - c_3 \end{aligned}$$

where $\bar{\gamma} = \min\{\gamma, 1\}$. Now, by the Young inequality, for any $\varepsilon > 0$ it results

$$\begin{gathered} \int_\Omega |u_n + \vartheta\,\Phi|^{p-1}|\Phi| \leq \varepsilon \int_\Omega |u_n + \vartheta\Phi|^\mu + \beta_{\mu,p}(\varepsilon)\int_\Omega |\Phi|^s, \\ \int_\Omega |u_n + \vartheta\,\Phi||\varphi| \leq \varepsilon \int_\Omega |u_n + \vartheta\,\Phi|^\mu + \beta_\mu(\varepsilon)\int_\Omega |\varphi|^{\mu'}, \end{gathered}$$

where we have set

$$\beta_\mu(\varepsilon) = \frac{\mu - 1}{\mu}\left(\frac{1}{\varepsilon\mu}\right)^{\frac{1}{\mu-1}}, \quad \beta_{\mu,p}(\varepsilon) = \frac{\mu - p + 1}{\mu}\left(\frac{p-1}{\varepsilon\mu}\right)^{\frac{p-1}{\mu-p+1}}, \quad s = \frac{\mu}{\mu - p + 1}.$$

Then, fixed $\varepsilon > 0$ sufficiently small, it follows that

$$C + \rho\|u_n\|_{X_V} \geq (\frac{1}{2} - \rho)\bar{\gamma}\|u_n\|^2_{X_V} + ((\rho\mu - 1)\, c_1 - (c_2 + 1)\,\varepsilon)\, \|u_n + \vartheta\Phi\|^\mu_\mu - c_4,$$

which implies the boundedness of (u_n) in X_V. Up to a subsequence, it results $u_n \rightharpoonup u$ in X_V, which in view of Proposition 2.1 implies that $u_n \to u$ in $L^s(\Omega)$ for every $s \in [2, K_*[$ up to a further subsequence. Therefore, since the map

$$X_V \xrightarrow{\;\Upsilon\;} L^{\frac{K_*}{p-1}}(\Omega) \xrightarrow{(A+V(x))^{-1}} X_V, \qquad \Upsilon(u) = g(u + \vartheta\Phi)$$

is compact, a standard argument allows to prove that $u_n \to u$ in X_V. $\square$

Lemma 3.3. *For every $b > 0$ there exists $B > 0$ such that*

$$\Big|\frac{\partial}{\partial\vartheta}I(\vartheta,u)\Big| \le B(1+\|I'_\vartheta(u)\|_{X'_V})(1+\|u\|_{X_V})$$

for all $(\vartheta,u)\in[0,1]\times X_V$ with $|I_\vartheta(u)|\le b$.

The proof of the above lemma follows the arguments in [13, Lemma 4.2].

Lemma 3.4. *Let $\varrho_1,\varrho_2:[0,1]\times\mathbb{R}\to\mathbb{R}$ be functions defined as*

$$-\varrho_1(\vartheta,s)=\varrho_2(\vartheta,s)=D\big(s^2+1\big)^{\frac{p-1}{2\mu}} \tag{3.1}$$

for a suitable $D>0$. Then

$$\varrho_1(\vartheta,I_\vartheta(u)) \le \frac{\partial}{\partial\vartheta}I(\vartheta,u)\le\varrho_2(\vartheta,I_\vartheta(u))$$

at every critical point u of I_ϑ. Moreover, if

$$-\widehat{\varrho}_1(\vartheta,s)=\widehat{\varrho}_2(\vartheta,s)=D\big(s^2+1\big)^{1/(2\mu)} \tag{3.2}$$

the same holds provided that $\phi_j=0$ for every $j=0,\dots,K-1$.

Proof. Arguing as in the proof of Lemma 3.2 and choosing $\rho=\frac{1}{2}$, we find $c_5>0$ such that

$$\|u+\vartheta\Phi\|_\mu^\mu \le c_5\big(I_\vartheta^2(u)+1\big)^{1/2} \tag{3.3}$$

for every critical point u of I_ϑ. On the other hand by combining (G4) and (2.3), taking into account that $\Phi\in L^{\mu/(\mu-p+1)}(\Omega)$, we have

$$\big|\frac{\partial}{\partial\vartheta}I(\vartheta,u)\big| \le c_6\|u+\vartheta\Phi\|_\mu^{p-1}+c_7 \tag{3.4}$$

for some $c_6,c_7>0$ if $\phi_j\neq 0$ for some $j\in\{0,\dots,K-1\}$ and, analogously,

$$\big|\frac{\partial}{\partial\vartheta}I(\vartheta,u)\big| \le c_8\|u\|_\mu \tag{3.5}$$

for some $c_8>0$ if $\phi_j=0$ for every $j=0,\dots,K-1$. Therefore, putting together (3.3) with (3.4) when $\phi_j\neq 0$ and (3.3) with (3.5) when $\phi_j=0$, the assertions follow. □

Taking into account conditions (G2) and (G3), the following property can be easily shown.

Lemma 3.5. *For every finite dimensional subspace W of X_V we have*

$$\lim_{\|u\|_{X_V}\to\infty,\,u\in W}\ \sup_{\vartheta\in[0,1]} I(\vartheta,u)=-\infty.$$

Let us introduce a suitable class of minimax values for the even functional I_0. Denote by X_V^n the subspace of X_V spanned by the first n eigenfunctions of the operator $A+V(x)$. Let us consider

$$c_n=\inf_{h\in\mathcal{H}}\sup_{h(X_V^n)} I_0,$$

where, for a suitable constant $R>0$, we have set

$$\mathcal{H}=\big\{h\in C(X_V,X_V):\ h \text{ is odd and } h(u)=u \text{ for } \|u\|_{X_V}>R\big\}.$$

Clearly, for every integer n, c_n is a critical value of I_0 and $c_n \leq c_{n+1}$. Now, we need a suitable estimate on the $c_n's$. First, let us point out that by Lemma 3.5 for all n there exist $R_n > 0$ such that if $\|u\|_{X_V} > R_n$ then $I_0(u) \leq I_0(0) = 0$. Setting

$$\mathcal{H}_n = \big\{h \in C(D_n, X_V) : \ h \text{ is odd and } h(u) = u \text{ for } \|u\|_{X_V} = R_n\big\},$$

where $D_n = \big\{u \in X_V^n : \ \|u\|_{X_V} \leq R_n\big\}$, we deduce that

$$c_n \geq \inf_{h\in\mathcal{H}_n} \sup_{h(D_n)} I_0.$$

Arguing as in [15], we have the following result.

Lemma 3.6. *There exist $\bar{b} > 0$ such that, for every $n \in \mathbb{N}$,*

$$c_n \geq \bar{b}\lambda_n^{\bar{\beta}(p,K,N)}, \quad \bar{\beta}(p,K,N) = \frac{2Kp - N(p-2)}{2K(p-2)}.$$

Proof. Fix $n \in \mathbb{N}$. By [15, Lemma 1.44] for every $h \in \mathcal{H}_n$ and $\rho \in]0, R_n[$ there exists w with

$$w \in h(D_n) \cap \partial B(0,\rho) \cap (X_V^{n-1})^{\perp}.$$

Therefore,

$$\max_{u\in D_n} I_0(h(u)) \geq I_0(w) \geq \inf_{u\in\partial B(0,\rho)\cap(X_V^{n-1})^{\perp}} I_0(u)\,. \tag{3.6}$$

Note that for every $u \in \partial B(0,\rho) \cap (X_V^{n-1})^{\perp}$ we have $I_0(u) \geq K(u)$, where

$$K(u) = \frac{1}{2}\|u\|_{X_V}^2 - \frac{c}{\mu}\|u\|_p^p.$$

By the Gagliardo-Nirenberg inequality there exists $c_9 > 0$ such that for all $u \in X_V$,

$$\|u\|_p \leq \|u\|_{K_*}^{\ell}\|u\|_2^{1-\ell} \leq c_9\|u\|_{X_V}^{\ell}\|u\|_2^{1-\ell}, \quad \ell = \frac{N(p-2)}{2Kp}.$$

Since $u \in (X_V^{n-1})^{\perp}$ implies $\|u\|_2 \leq \lambda_n^{-1/2}\|u\|_{X_V}$, one obtains

$$K(u) \geq \frac{1}{2}\rho^2 - \frac{c(c_9)^p}{\mu}\lambda_n^{-(1-\ell)p/2}\rho^p.$$

Therefore, by choosing $\rho = \rho_n = c'\lambda_n^{\frac{(1-\ell)p}{2(p-2)}}$ where c' is a suitable positive constant, we can assume $\rho_n < R_n$ and therefore the assertion follows by (3.6) and the previous inequalities. $\square$

Remark 3.7. As proved in [13, Lemma 5.2] for *bounded* domains (see also [22] in the case $K = 1$), the following sharp estimate holds for $n \in \mathbb{N}$

$$c_n \geq b\, n^{\frac{2pK}{(p-2)N}} \tag{3.7}$$

for some $b > 0$. In our framework, if v_n denotes a critical point of $K(u) = \frac{1}{2}\|u\|_{X_V}^2 - \frac{c}{\mu}\|u\|_p^p$ at the level

$$b_n = \inf_{h\in\mathcal{H}_n} \sup_{h(D_n)} K(u),$$

by using suitable Morse index estimates of v_n, we can prove the lower estimates $b_n \geq c_{10}\|v_n\|_p^p$ and $\|v_n\|_{(p-2)N/2K} \geq c_{11}\, n^{2K/(p-2)N}$. On the other hand, if $|\Omega| = \infty$, we are not able to compare these norms and get (3.7). If instead Ω has finite measure, then (3.7) holds.

We are now ready to complete the proof of the results stated in the introduction.

Proof of Theorem 1.1. By combining Lemma 3.2, Lemma 3.3, (3.1) of Lemma 3.4, Lemma 3.5 and Lemma 3.6 the assertion follows by Theorem 3.1 with $X = X_V$, $\|\cdot\| = \|\cdot\|_{X_V}$, $X_n = X_V^n = \mathrm{span}\{v_1, \dots, v_n\}$ being v_j the j-th eigenfunction of $A + V(x)$, $B(n) = \lambda_n$, $\bar{\beta} = \frac{2Kp-N(p-2)}{2K(p-2)}$ and $\bar{\alpha} = \frac{p-1}{\mu}$. If $|\Omega| < \infty$, Remark 3.7 yields the stronger conclusion. □

Proof of Theorem 1.2. It suffices to argue as in the proof of Theorem 1.1 using (3.2) of Lemma 3.4 in place of (3.1), namely $\bar{\alpha} = \frac{1}{\mu}$ □

Proof of Corollaries 1.4 and 1.5. Taking into account Proposition 2.2, it suffices to argue as for the proof of Theorems 1.1 and 1.2 using Theorem 3.1 with $X = H_0^K(\Omega)$, $\|\cdot\| = \|\cdot\|_{K,2}$ and $X_n = \mathrm{span}\{v_1, \dots, v_n\}$ being v_j the j-th eigenfunction of A. If $|\Omega| < \infty$, Remark 3.7 yields the stronger conclusion. □

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