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SELF-ADJOINTNESS OF SCHRÖDINGER-TYPE OPERATORS WITH SINGULAR POTENTIALS ON MANIFOLDS OF BOUNDED GEOMETRY

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Abstract. We consider the Schrödinger type differential expression

$$H_V = \nabla^*\nabla + V,$$

where ∇ is a C^∞-bounded Hermitian connection on a Hermitian vector bundle E of bounded geometry over a manifold of bounded geometry (M, g) with metric g and positive C^∞-bounded measure $d\mu$, and $V = V_1 + V_2$, where $0 \le V_1 \in L^1_{\text{loc}}(\operatorname{End} E)$ and $0 \ge V_2 \in L^1_{\text{loc}}(\operatorname{End} E)$ are linear self-adjoint bundle endomorphisms. We give a sufficient condition for self-adjointness of the operator S in $L^2(E)$ defined by $Su = H_V u$ for all $u \in \operatorname{Dom}(S) = \{u \in W^{1,2}(E)\colon \int \langle V_1 u, u\rangle\, d\mu < +\infty \text{ and } H_V u \in L^2(E)\}$. The proof follows the scheme of T. Kato, but it requires the use of more general version of Kato's inequality for Bochner Laplacian operator as well as a result on the positivity of $u \in L^2(M)$ satisfying the equation $(\Delta_M + b)u = \nu$, where Δ_M is the scalar Laplacian on M, $b > 0$ is a constant and $\nu \ge 0$ is a positive distribution on M.

1. Introduction and main result

Let (M, g) be a C^∞ Riemannian manifold without boundary, with metric g, $\dim M = n$. We will assume that M is connected. We will also assume that M has bounded geometry. Moreover, we will assume that we are given a positive C^∞-bounded measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \dots, x^n$ there exists a strictly positive C^∞-bounded density $\rho(x)$ such that $d\mu = \rho(x)dx^1dx^2 \dots dx^n$.

Let E be a Hermitian vector bundle over M. We will assume that E is a bundle of bounded geometry (i.e. it is supplied by an additional structure: trivializations of E on every canonical coordinate neighborhood U such that the corresponding matrix transition functions $h_{U,U'}$ on all intersections $U \cap U'$ of such neighborhoods are C^∞-bounded, i.e. all derivatives $\partial_y^\alpha h_{U,U'}(y)$, where α is a multiindex, with respect to canonical coordinates are bounded with bounds C_α which do not depend on the chosen pair U, U').

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We denote by $L^2(E)$ the Hilbert space of square integrable sections of E with respect to the scalar product

$$(u,v)=\int_M \langle u(x),v(x)\rangle\,d\mu(x). \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle$ denotes the fiberwise inner product in E_x.

In what follows, $C^\infty(E)$ denotes smooth sections of E, and $C_c^\infty(E)$ denotes smooth compactly supported sections of E. Let

$$\nabla\colon C^\infty(E)\to C^\infty(T^*M\otimes E)$$

be a Hermitian connection on E which is C^∞-bounded as a linear differential operator, i.e. in any canonical coordinate system U (with the chosen trivializations of $E|_U$ and $(T^*M\otimes E)|_U$), ∇ is written in the form

$$\nabla=\sum_{|\alpha|\le 1} a_\alpha(y)\partial_y^\alpha,$$

where α is a multiindex, and the coefficients $a_\alpha(y)$ are matrix functions whose derivatives $\partial_y^\beta a_\alpha(y)$ for any multiindex β are bounded by a constant C_β which does not depend on the chosen canonical neighborhood.

We will consider a Schrödinger type differential expression of the form

$$H_V=\nabla^*\nabla+V,$$

where V is a linear self-adjoint bundle map $V\in L^1_{\rm loc}(\operatorname{End}E)$. Here

$$\nabla^*\colon C^\infty(T^*M\otimes E)\to C^\infty(E)$$

is a differential operator which is formally adjoint to ∇ with respect to the scalar product (1.1).

If we take $\nabla=d$, where $d\colon C^\infty(M)\to\Omega^1(M)$ is the standard differential, then $d^*d\colon C^\infty(M)\to C^\infty(M)$ is called the scalar Laplacian and will be denoted by Δ_M.

We make the following assumption on V.

(A1) $V=V_1+V_2$, where $0\le V_1\in L^1_{\rm loc}(\operatorname{End}E)$ and $0\ge V_2\in L^1_{\rm loc}(\operatorname{End}E)$ are linear self-adjoint bundle maps (here the inequalities are understood in the sense of operators $E_x\to E_x$).

By $W^{1,2}(E)$ we denote the completion of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_1$ defined by the scalar product

$$(u,v)_1:=(u,v)+(\nabla u,\nabla v)\quad u,v\in C_c^\infty(E).$$

By $W^{-1,2}(E)$ we will denote the dual of $W^{1,2}(E)$.

2. Quadratic forms

In what follows, all quadratic forms are considered in the Hilbert space $L^2(E)$.

By h_0 we denote the quadratic form

$$h_0(u)=\int|\nabla u|^2\,d\mu \tag{2.1}$$

with the domain $\mathrm{D}(h_0)=W^{1,2}(E)\subset L^2(E)$. Clearly, h_0 is a non-negative, densely defined and closed form.

By h_1 we denote the quadratic form

$$h_1(u)=\int\langle V_1u,u\rangle\,d\mu \tag{2.2}$$

with the domain

$$\mathrm{D}(h_1) = \{u \in L^2(E) : \int \langle V_1 u, u\rangle \, d\mu < +\infty\}. \tag{2.3}$$

Clearly, h_1 is a non-negative, densely defined, and closed form.

By h_2 we denote the quadratic form

$$h_2(u) = \int \langle V_2 u, u\rangle \, d\mu \tag{2.4}$$

with the domain

$$\mathrm{D}(h_2) = \{u \in L^2(E) : \int |\langle V_2 u, u\rangle| \, d\mu < +\infty\}. \tag{2.5}$$

Clearly, h_2 is a densely defined form. Moreover, h_2 is symmetric (but not semi-bounded below).

We make the following assumption on h_2.

(A2) Assume that h_2 is h_0-bounded with relative bound $b < 1$, i.e.

(i) $\mathrm{D}(h_2) \supset \mathrm{D}(h_0)$

(ii) There exist constants $a \geq 0$ and $0 \leq b < 1$ such that

$$|h_2(u)| \leq a\|u\|^2 + b|h_0(u)|, \quad \text{for all } u \in \mathrm{D}(h_0), \tag{2.6}$$

where $\|\cdot\|$ denotes the norm in $L^2(E)$.

Remark 2.1. With the above assumptions on (M, g), bundle E and connection ∇, Assumption (A2) holds if $V_2 \in L^p(\operatorname{End} E)$, where $p = n/2$ for $n \geq 3$, $p > 1$ for $n = 2$, and $p = 1$ for $n = 1$. The proof is given in the last section of this article.

As a realization of H_V in $L^2(E)$, we define the operator S in $L^2(E)$ by the formula $Su = H_V u$ on the domain

$$\mathrm{Dom}(S) = \{u \in W^{1,2}(E) : \int \langle V_1 u, u\rangle \, d\mu < +\infty \text{ and } H_V u \in L^2(E)\}. \tag{2.7}$$

Remark 2.2. For all $u \in \mathrm{D}(h_0) = W^{1,2}(E)$ we have $\nabla^*\nabla u \in W^{-1,2}(E)$, and from Corollary 3.7 below it follows that for all $u \in W^{1,2}(E) \bigcap \mathrm{D}(h_1)$, we have $Vu \in L^1_{\mathrm{loc}}(E)$. Thus $H_V u$ in (2.7) is a distributional section of E, and the condition $H_V u \in L^2(E)$ makes sense.

We now state the main result.

Theorem 2.3. *Assume that (M, g) is a manifold of bounded geometry with positive C^∞-bounded measure $d\mu$, E is a Hermitian vector bundle of bounded geometry over M, and ∇ is a C^∞-bounded Hermitian connection on E. Suppose that Assumptions (A1) and (A2) hold. Then S is a semi-bounded below self-adjoint operator.*

Remark 2.4. Theorem 2.3 extends a result of T. Kato, cf. Theorem VI.4.6(a) in [8] (see also remark 5(b) in [7]) which was proven for the operator $-\Delta + V$, where Δ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric and measure, and $V = V_1 + V_2$, where $0 \leq V_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $0 \geq V_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ are as in Assumption (A1) above, and the quadratic form h_2 corresponding to V_2 is as in Assumption (A2) above.

3. Proof of Theorem 2.3

We adopt the arguments from Sec. VI.4 in [8] to our setting with the help of more general version of Kato's inequality (3.1).

We begin with the following variant of Kato's inequality for Bochner Laplacian (for the proof see Theorem 5.7 in [2]). The original version of Kato's inequality was proven in Kato [5].

Lemma 3.1. *Assume that (M, g) is a Riemannian manifold. Assume that E is a Hermitian vector bundle over M and ∇ is a Hermitian connection on E. Assume that $w \in L^1_{\mathrm{loc}}(E)$ and $\nabla^*\nabla w \in L^1_{\mathrm{loc}}(E)$. Then*

$$\Delta_M|w| \leq \mathrm{Re}\langle \nabla^*\nabla w, \operatorname{sign} w\rangle, \tag{3.1}$$

where

$$\operatorname{sign} w(x) = \begin{cases} \frac{w(x)}{|w(x)|} & \text{if } w(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we will use the following Lemma whose proof is given in Appendix B of [2].

Lemma 3.2. *Assume that (M, g) is a manifold of bounded geometry with a smooth positive measure $d\mu$. Assume that*

$$\big(\, b + \Delta_M \,\big)\, u \;=\; \nu \;\geq\; 0, \quad u \in L^2(M),$$

where $b > 0$, $\Delta_M = d^*d$ is the scalar Laplacian on M, and the inequality $\nu \geq 0$ means that ν is a positive distribution on M, i.e. $(\nu, \phi) \geq 0$ for any $0 \leq \phi \in C_c^\infty(M)$. Then $u \geq 0$ (almost everywhere or, equivalently, as a distribution).

Remark 3.3. It is not known whether Lemma 3.2 holds if M is an arbitrary complete Riemannian manifold. For more details about difficulties in the case of arbitrary complete Riemannian manifolds, see Appendix B of [2].

Lemma 3.4. *The quadratic form $h := (h_0 + h_1) + h_2$ is densely-defined, semi-bounded below and closed.*

Proof. Since h_0 and h_1 are non-negative and closed, it follows by Theorem VI.1.31 from [8] that $h_0 + h_1$ is non-negative and closed. Since h_1 is non-negative, it follows immediately from Assumption (A2) that h_2 is $(h_0 + h_1)$-bounded with relative bound $b < 1$. Since $h_0 + h_1$ is a closed, non-negative form, by Theorem VI.1.33 from [8], it follows that $h = (h_0 + h_1) + h_2$ is a closed semi-bounded below form. Since $C_c^\infty(E) \subset \mathrm{D}(h_0) \bigcap \mathrm{D}(h_1) \subset \mathrm{D}(h_2)$, it follows that h is densely defined. □

In what follows, $h(\cdot, \cdot)$ will denote the corresponding sesquilinear form obtained from h via polarization identity.

Self-adjoint operator H associated to h. Since h is densely defined, closed and semi-bounded below form in $L^2(E)$, by Theorem VI.2.1 from [8] there exists a semi-bounded below self-adjoint operator H in $L^2(E)$ such that

(i) $\mathrm{Dom}(H) \subset \mathrm{D}(h)$ and

$$h(u, v) = (Hu, v) \quad \text{for all } u \in \mathrm{Dom}(H), \text{ and } v \in \mathrm{D}(h).$$

(ii) $\mathrm{Dom}(H)$ is a core of h.

(iii) If $u \in \mathrm{D}(h)$, $w \in L^2(E)$ and $h(u,v) = (w,v)$ holds for every v belonging to a core of h, then $u \in \mathrm{Dom}(H)$ and $Hu = w$. The semi-bounded below self-adjoint operator H is uniquely determined by the condition (i).

In what follows we will use the following well-known Lemma.

Lemma 3.5. *Assume that $0 \le T \in L^1_{\rm loc}(\operatorname{End} E)$ is a linear self-adjoint bundle map. Assume also that $u \in Q(T)$, where $Q(T) = \{u \in L^2(E) \colon \langle Tu, u\rangle \in L^1(M)\}$. Then $Tu \in L^1_{\rm loc}(E)$.*

Proof. By adding a constant we can assume that $T \ge 1$ (in the operator sense). Assume that $u \in Q(T)$. We choose (in a measurable way) an orthogonal basis in each fiber E_x and diagonalize $1 \le T(x) \in \operatorname{End}(E_x)$ to get

$$T(x) = \operatorname{diag}(c_1(x), c_2(x), \dots, c_m(x)),$$

where $0 < c_j \in L^1_{\rm loc}(M)$, $j = 1, 2, \dots, m$ and $m = \dim E_x$.

Let $u_j(x)$ $(j = 1, 2, \dots, m)$ be the components of $u(x) \in E_x$ with respect to the chosen orthogonal basis of E_x. Then for all $x \in M$

$$\langle Tu, u\rangle = \sum_{j=1}^{m} c_j(x)|u_j(x)|^2.$$

Since $u \in Q(T)$, we know that $0 < \int \langle Tu, u\rangle \, d\mu < +\infty$. Since $c_j > 0$, it follows that $c_j|u_j|^2 \in L^1(M)$, for all $j = 1, 2, \dots, m$.

Now, for all $x \in M$ and $j = 1, 2, \cdots, m$

$$2|c_j u_j| = 2|c_j||u_j| \le |c_j| + |c_j||u_j|^2, \tag{3.2}$$

The right hand side of (3.2) is clearly in $L^1_{\rm loc}(M)$. Therefore $c_j u_j \in L^1_{\rm loc}(M)$. But $(Tu)(x)$ has components $c_j(x)u_j(x)$ $(j = 1, 2, \dots, m)$ with respect to chosen bases of E_x. Therefore $Tu \in L^1_{\rm loc}(E)$, and the Lemma is proven. $\square$

The following corollary follows immediately from Lemma 3.5.

Corollary 3.6. *If $u \in D(h_1)$, then $V_1 u \in L^1_{\rm loc}(E)$.*

Corollary 3.7. *If $u \in D(h)$, then $Vu \in L^1_{\rm loc}(E)$.*

Proof. Let $u \in \mathrm{D}(h) = \mathrm{D}(h_0) \bigcap \mathrm{D}(h_1)$. By Assumption (A1) we have $V = V_1 + V_2$, where $0 \le V_1 \in L^1_{\rm loc}(\operatorname{End} E)$ and $0 \ge V_2 \in L^1_{\rm loc}(\operatorname{End} E)$. By Corollary 3.6 it follows that $V_1 u \in L^1_{\rm loc}(E)$ and since $\mathrm{D}(h) \subset \mathrm{D}(h_2)$, by Lemma 3.5 we have $-V_2 u \in L^1_{\rm loc}(E)$. Thus $Vu \in L^1_{\rm loc}(E)$, and the corollary is proven. $\square$

Lemma 3.8. *The following operator relation holds: $H \subset S$.*

Proof. We will show that for all $u \in \mathrm{Dom}(H)$, we have $Hu = H_V u$. Let $u \in \mathrm{Dom}(H)$. By property (i) of operator H we have $u \in \mathrm{D}(h)$, hence by Corollary 3.7 we get $Vu \in L^1_{\rm loc}(E)$. Then, for any $v \in C_c^\infty(E)$, we have

$$(Hu, v) = h(u, v) = (\nabla u, \nabla v) + \int \langle Vu, v\rangle \, d\mu, \tag{3.3}$$

where $(\cdot, \cdot)$ denotes the L^2-inner product.

The first equality in (3.3) holds by property (i) of operator H, and the second equality holds by definition of h.

Hence, using integration by parts in the first term on the right hand side of the second equality in (3.3) (see, for example, Lemma 8.8 from [2]), we get

$$(u, \nabla^*\nabla v) = \int \langle Hu - Vu, v\rangle \, d\mu, \quad \text{for all} \quad v \in C_c^\infty(E). \tag{3.4}$$

Since $Vu \in L^1_{\rm loc}(E)$ and $Hu \in L^2(E)$, it follows that $(Hu - Vu) \in L^1_{\rm loc}(E)$, and (3.4) implies $\nabla^*\nabla u = Hu - Vu$ (as distributional sections of E). Therefore,

$$\nabla^*\nabla u + Vu = Hu,$$

and this shows that $Hu = H_V u$ for all $u \in \mathrm{Dom}(H)$.

Now by definition of S it follows that $\mathrm{Dom}(H) \subset \mathrm{Dom}(S)$ and $Hu = Su$ for all $u \in \mathrm{Dom}(H)$. Therefore $H \subset S$, and the Lemma is proven. □

Lemma 3.9. *$C_c^\infty(E)$ is a core of the quadratic form $h_0 + h_1$.*

Proof. We need to show that $C_c^\infty(E)$ is dense in the Hilbert space $\mathrm{D}(h_0 + h_1) = \mathrm{D}(h_0) \bigcap \mathrm{D}(h_1)$ with the inner product

$$(u, v)_{h_0+h_1} := h_0(u, v) + h_1(u, v) + (u, v),$$

where $(\cdot, \cdot)$ is the inner product in $L^2(E)$.

Let $u \in \mathrm{D}(h_0 + h_1)$ and $(u, v)_{h_0+h_1} = 0$ for all $v \in C_c^\infty(E)$. We will show that $u = 0$. We have

$$0 = h_0(u, v) + h_1(u, v) + (u, v) = (u, \nabla^*\nabla v) + \int \langle V_1 u, v\rangle \, d\mu + (u, v). \tag{3.5}$$

Here we used integration by parts in the first term on the right hand side of the second equality.

By Corollary 3.6 it follows that $V_1 u \in L^1_{\rm loc}(E)$, and from (3.5) we get the following equality of distributional sections of E:

$$\nabla^*\nabla u = (-V_1 - 1)u. \tag{3.6}$$

From (3.6) we have $\nabla^*\nabla u \in L^1_{\rm loc}(E)$. By Lemma 3.1 and by (3.6), we obtain

$$\Delta_M |u| \leq \mathrm{Re}\langle \nabla^*\nabla u, \operatorname{sign} u\rangle = \langle -(V_1 + 1)u, \operatorname{sign} u\rangle \leq -|u|. \tag{3.7}$$

The last inequality in (3.7) follows since $V_1 \geq 0$ (as an operator $E_x \to E_x$). Therefore,

$$(\Delta_M + 1)|u| \leq 0. \tag{3.8}$$

By Lemma 3.2, it follows that $|u| \leq 0$. So $u = 0$, and the proof is complete. □

Lemma 3.10. *$C_c^\infty(E)$ is a core of the quadratic form $h = (h_0 + h_1) + h_2$.*

Since the quadratic form h_2 is (h_0+h_1)-bounded, the lemma follows immediately from Lemma 3.9.

Proof of Theorem 2.3. We will show that $S = H$. By Lemma 3.8 we have $H \subset S$, so it is enough to show that $\mathrm{Dom}(S) \subset \mathrm{Dom}(H)$.

Let $u \in \mathrm{Dom}(S)$. By definition of $\mathrm{Dom}(S)$, we have $u \in \mathrm{D}(h_0) \subset \mathrm{D}(h_2)$ and $u \in \mathrm{D}(h_1)$. Hence $u \in \mathrm{D}(h)$. For all $v \in C_c^\infty(E)$ we have

$$h(u, v) = h_0(u, v) + h_1(u, v) + h_2(u, v) = (u, \nabla^*\nabla v) + \int \langle Vu, v\rangle \, d\mu = (H_V u, v).$$

The last equality holds since $H_V u = Su \in L^2(E)$. By Lemma 3.10 it follows that $C_c^\infty(E)$ is a form core of h. Now from property (iii) of operator H we have $u \in \mathrm{Dom}(H)$ with $Hu = H_V u$. This concludes the proof of the Theorem. □

Proof of Remark 2.1. Let p be as in Remark 2.1. We may assume that $\|V_2\|_{L^p(\mathrm{End}\,E)}$ is arbitrarily small because there exists a sequence $V_2^{(k)} \in L^\infty(\mathrm{End}\,E) \bigcap L^p(\mathrm{End}\,E)$, $k \in \mathbb{Z}_+$, such that

$$\|V_2^{(k)} - V_2\|_{L^p(\mathrm{End}\,E)} \to 0, \quad \text{as } k \to \infty,$$

and $V_2^{(k)}$, $k \in \mathbb{Z}_+$, contributes to h_2 only a bounded form.

For the rest of this article, we will assume that $\|V_2\|_{L^p(\mathrm{End}\,E)}$ is arbitrarily small. By Cauchy-Schwartz inequality and Hölder's inequality we have

$$\Big|\int \langle V_2 u, u\rangle\, d\mu\Big| \le \int |\langle V_2 u, u\rangle|\, d\mu \le \int |V_2||u|^2\, d\mu \le \|V_2\|_{L^p(\mathrm{End}\,E)}\|u\|^2_{L^t(E)}, \tag{3.9}$$

where $|V_2|$ denotes the norm of the operator $V_2(x)\colon E_x \to E_x$ and

$$\frac{1}{p} + \frac{2}{t} = 1. \tag{3.10}$$

With our assumptions on (M, g), E and ∇, the usual Sobolev embedding theorems for $W^{1,2}(\mathbb{R}^n)$ also hold for $W^{1,2}(E)$ (see Sec. A1.1 in [10]).

For $n \ge 3$, we know by hypothesis that $p = n/2$, so from (3.10) we get $1/t = 1/2 - 1/n$. By the Sobolev embedding theorem (see, for example, the first part of Theorem 2.10 in [1]) we have

$$\|u\|_{L^t(E)} \le C(\|\nabla u\|_{L^2(T^*M\otimes E)} + \|u\|_{L^2(E)}), \quad \text{for all } u \in W^{1,2}(E),$$

where $C > 0$ is a positive constant.

For $n = 2$, we know by hypothesis that $p > 1$, so from (3.10) we get $2 < t < \infty$. In this case, it is well-known (see, e.g., the first part of Theorem 2.10 in [1]) that

$$\|u\|_{L^t(E)} \le C(\|\nabla u\|_{L^2(T^*M\otimes E)} + \|u\|_{L^2(E)}), \quad \text{for all } u \in W^{1,2}(E),$$

where $C > 0$ is a positive constant.

For $n = 1$, we know by hypothesis that $p = 1$, so from (3.10) we get $t = \infty$. In this case, it is well-known (see e.g. the second part of Theorem 2.10 in [1]) that

$$\|u\|_{L^\infty(E)} \le C(\|\nabla u\|_{L^2(T^*M\otimes E)} + \|u\|_{L^2(E)}), \quad \text{for all } u \in W^{1,2}(E),$$

where $C > 0$ is a positive constant.

Combining each of the last three inequalities with (3.9), we get (2.6) (with constant $b < 1$ because $\|V_2\|_{L^p(\mathrm{End}\,E)}$ is arbitrarily small). $\square$

References

[1] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer-Verlag, Berlin, 1998.

[2] M. Braverman, O. Milatovic, M. Shubin, *Essential self-adjointness of Schrödinger type operators on manifolds*, Russian Math. Surveys, **57**(4) (2002), 641–692.

[3] H. Brézis, T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl., **58**(9) (1979), 137–151.

[4] H. Hess, R. Schrader, A. Uhlenbrock, *Kato's inequality and the spectral distribution of Laplacians on compact Riemannian manifold*, J. Differential Geom., **15** (1980), 27–37.

[5] T. Kato, *Schrödinger operators with singular potentials,* Israel J. Math., **13** (1972), 135–148.

[6] T. Kato, *A second look at the essential selfadjointness of the Schrödinger operators*, Physical reality and mathematical description, Reidel, Dordrecht, 1974, 193–201.

[7] T. Kato, *On some Schrödinger operators with a singular complex potential*, Ann. Sc. Norm. Sup. Pisa, Ser. IV, Vol. 5 (1978), 105–114.

[8] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1980.

[9] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I, II: Functional analysis. Fourier analysis, self-adjointness*, Academic Press, New York e.a., 1975.

[10] M. A. Shubin, *Spectral theory of elliptic operators on noncompact manifolds,* Astérisque No. 207 (1992), 35–108.
[11] M. Taylor, *Partial Differential Equations II: Qualitative Studies of Linear Equations*, Springer-Verlag, New York e.a., 1996.

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