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MULTIPLE SOLUTIONS FOR NONRESONANCE IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

MOUFFAK BENCHOHRA & ABDELGHANI OUAHAB

Abstract. In this paper we investigate the existence of multiple solutions for first and second order impulsive functional differential equations with boundary conditions. Our main tool is the Leggett and Williams fixed point theorem.

1. Introduction

This paper is concerned with the existence of three nonnegative solutions for initial value problems for first and second order impulsive functional differential equations with boundary conditions. Initially we consider the first order impulsive functional differential equation,

$$y'(t) - \lambda y(t) = f(t, y_t), \quad \text{a.e. } t \in [0,T],\ t \neq t_k,\ k = 1, \dots, m, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0],\ y(0) = y(T), \tag{1.3}$$

where $\lambda > 0$, $f : [0,T] \times D \to \mathbb{R}^+$, $I_k \in C(\mathbb{R}, \mathbb{R}^+)$, $0 < r < \infty$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively. $D = \{\Psi : [-r, 0] \to \mathbb{R}^+; \Psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\Psi(\bar{t})$ and $\Psi(\bar{t}^+)$ exist and $\Psi(\bar{t}^-) = \Psi(\bar{t})\}$, $\phi \in D$. For any function y defined on $[-r, T]$ and any $t \in J$, we denote by y_t the element of D defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, to the present time t.

Later, we study the second order impulsive functional differential equations with boundary conditions and fixed moments of the form

$$y''(t) - \lambda y(t) = f(t, y_t), \quad \text{a.e. } t \in [0,T],\ t \neq t_k,\ k = 1, \dots, m, \tag{1.4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.5}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.6}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad y(0) - y(T) = \mu_0, \quad y'(0) - y'(T) = \mu_1, \tag{1.7}$$

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where F, I_k, and ϕ are as in problem (1.1)-(1.3), $\overline{I}_k \in C(\mathbb{R}, \mathbb{R}^+)$ $k = 1, \dots, m$, and $\mu_0, \mu_1 \in \mathbb{R}$.

Note that when $\mu_0 = \mu_1 = 0$ we have periodic boundary conditions. Differential equations with impulses are a basic tool to study evolution processes that are subjected to abrupt changes in their state. Such equations arise naturally from a wide variety of applications, such as space-craft control, inspection processes in operations research, drug administration, and threshold theory in biology. There has been a significant development in impulse theory, especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [7], Lakshmikantham *et al.* [15], and Samoilenko and Perestyuk [17] and the papers of Benchohra *et al* [8], Franco *et al* [11] and the references cited therein.

The existence of multiple solutions for boundary value problems for impulsive differential equations was studied by Guo and Liu [13] and Agarwal and O'Regan [2]. Notice that when the impulses are absent (i.e. $I_k = 0$, $k = 1, \dots, m$) the existence of three solutions and multiple solutions for ordinary differential equations was studied in [1, 3, 4, 5, 6, 14].

The main theorems of this note extend some existence results in the above literature to the impulsive case. Our approach here is based on the Leggett and Williams fixed point theorem in cones [16].

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used for the rest of this paper.

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

$L^1(J, \mathbb{R})$ denotes the Banach space of measurable functions $y : J \to \mathbb{R}$ which are Lebesgue integrable, with

$$\|y\|_{L^1} = \int_0^T |y(t)| dt.$$

Let (a, b) be an open interval and $AC^i(a, b), \mathbb{R})$ be the space of i-times differentiable functions $y : (a, b) \to \mathbb{R}$, whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.

Let $(E, \|\cdot\|)$ be a Banach space and C be a cone in E. By concave nonnegative continuous functional ψ on C we mean a continuous mapping $\psi :\ C \to [0, \infty)$ with

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y) \quad \text{for all } x, y \in C \ \lambda \in [0, 1].$$

To define solutions of (1.1)-(1.3) we shall consider the space

$$\begin{aligned} PC = \big\{ & y : [0, T] \to \mathbb{R} : y_k \in C(J_k, \mathbb{R}),\ k = 0, \dots, m \text{ and there exist } y(t_k^-) \\ & \text{and } y(t_k^+) \text{ with } y(t_k^-) = y(t_k), k = 1, \dots, m \big\} \end{aligned}$$

which is a Banach space with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_{J_k}, k = 0, \dots, m\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 0, \dots, m$. Set $\Omega = \{y : [-r, T] \to \mathbb{R} : y \in D \cap PC\}$. Then Ω is a Banach space with norm

$$\|y\|_\Omega = \sup\{|y(t)| : t \in [-r, T]\}.$$

Definition 2.1. A map $f : J \times D \to \mathbb{R}^+$ is said to be L^1-Carathéodory if

(i) $t \mapsto f(t, u)$ is measurable for each $y \in D$;
(ii) $u \mapsto f(t, u)$ is continuous for almost all $t \in J$;
(iii) For each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}_+)$ such that $|f(t, u)| \leq h_q(t)$ for all $\|u\|_D \leq q$ and for almost all $t \in J$.

Our consideration is based on the following fixed point theorem given by Leggett and Williams in 1979 [16](see also Guo and Lakshmikantham [12]).

Theorem 2.2. *Let E be a Banach space, $C \subset E$ a cone of E and $R > 0$ a constant. Let $C_R = \{y \in C : \|y\| < R\}$. Suppose a concave nonnegative continuous functional ψ exists on the cone C with $\psi(y) \leq \|y\|$ for $y \in \overline{C}_R$, and let $N : \overline{C}_R \to \overline{C}_R$ be a continuous compact map. Assume there are numbers r, L and K with $0 < r < L < K \leq R$ such that*

(A1) *$\{y \in C(\psi, L, K) : \psi(y) > L\} \neq \emptyset$ and $\psi(N(y)) > L$ for all $y \in C(\psi, L, K)$;*
(A2) *$\|N(y)\| < r$ for all $y \in \overline{C}_r$;*
(A3) *$\psi(N(y)) > L$ for all $y \in C(\psi, L, R)$ with $\|N(y)\| > K$, where $C_K = \{y \in C : \|y\| \leq K\}$ and*

$$C(\psi, L, K) = \{y \in C : \psi(y) \geq L \text{ and } \|y\| \leq K\}.$$

Then N has at least three fixed points y_1, y_2, y_3 in $\overline{C}_R$. Furthermore, we have

$$y_1 \in C_r, \quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\} \quad y_3 \in \overline{C}_R - \{C(\psi, L, R) \cup \overline{C}_r\}.$$

3. First Order Impulsive FDEs

Let us start by defining what we mean by a solution of problem (1.1)–(1.3).

Definition 3.1. A function $y \in \Omega \cap \cup_{k=0}^{m} AC((t_k, t_{k+1}), \mathbb{R})$ is said to be a solution of (1.1)–(1.3) if y satisfies $y'(t) - \lambda y(t) = f(t, y_t)$ a.e. on $J \backslash \{t_1, \dots, t_m\}$, and $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$, and $y(0) = y(T)$.

For the next theorem we need the following assumptions:

(H1) There exist constants c_k such that $|I_k(x)| \leq c_k$, $k = 1, \dots, m$ for each $x \in \mathbb{R}$
(H2) There exist a function $g : [0, \infty) \to [0, \infty)$ continuous and non-decreasing, a function $p \in L^1(J, \mathbb{R}_+)$, $r > 0$, and a constant $0 < M \leq 1$ such that

$$M\, p(t) g(\|u\|) \leq |H(t, s) f(t, u)| \leq p(t) g(\|u\|)$$

for each $(t, s, u) \in J \times J \times D$, and

$$\frac{1}{1 - e^{-\lambda T}} \sum_{k=1}^{m} c_k + g(r) \int_0^T p(t)dt < r;$$

(H3) There exist $L > r$ and an interval $[a, b] \subset (0, T)$ such that

$$\min_{t \in [a,b]} \Big(\sum_{k=1}^{m} H(t, t_k) I_k(y(t_k)) \Big) \geq M \sum_{k=1}^{m} c_k,$$
$$M \Big(\sum_{k=1}^{m} c_k + g(L) \int_0^T p(s)ds \Big) > L;$$

(H4) There exist $R, K, 0 < M_1 < M,\ M_1^{-1}L < K \leq R$ such that

$$\min_{t\in[a,b]}\Big(\sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)) + \int_0^T H(t,s)f(s,y_s)ds\Big)$$
$$\geq M_1 M_*\Big(\sum_{k=1}^{m} c_k + g(R)\int_0^T p(s)ds\Big)$$

where $M_* = \sup_{(t,s)\in[0,T]\times[0,T]} |H(t,s)|$ and

$$\frac{1}{1-e^{-\lambda T}}\sum_{k=1}^{m} c_k + g(R)\int_0^T p(t)dt < R\,.$$

Theorem 3.2. *Assume (H1)–(H4) are satisfied. Then problem (1.1)–(1.3) has at least three solutions.*

Proof. We transform the problem into a fixed point problem. Consider the operator, $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0];\\ \int_0^T H(t,s)f(s,y_s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k^-)), & \text{if } t\in[0,T], \end{cases}$$

where

$$H(t,s) = (e^{-\lambda T}-1)^{-1}\begin{cases} e^{-\lambda(T+s-t)}, & 0\leq s\leq t\leq T,\\ e^{-\lambda(s-t)}, & 0\leq t< s\leq T. \end{cases}$$

Remark 3.3. It is easy to show that the fixed points of N are solutions to the problem (1.1)-(1.3); see [9].

We shall show that N satisfies the assumptions of Theorem 2.2. This will be done in several steps.

Step 1: N is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned} &|N(y_n(t)) - N(y(t))|\\ &\leq \int_0^T |H(t,s)||f(s,y_{ns}) - f(s,y_s)|ds + \sum_{k=1}^m |H(t,t_k)||I_k(y_n(t_k)) - I_k(y(t_k))|\\ &\leq \frac{1}{1-e^{-\lambda T}}\int_0^T |f(s,y_{ns}) - f(s,y_s)|ds + \frac{1}{1-e^{-\lambda T}}\sum_{k=1}^m |I_k(y_n(t_k)) - I_k(y(t_k))|. \end{aligned}$$

Since the functions H, I_k, $k = 1,\dots,m$ are continuous and f is an L^1-Carathéodory,

$$\begin{aligned} &\|N(y_n) - N(y)\|_\Omega\\ &\leq \frac{1}{1-e^{-\lambda T}}\|f(.,y_n) - f(.,y)\|_{L^1} + \frac{1}{1-e^{-\lambda T}}\sum_{k=1}^m |I_k(y_n(t_k)) - I_k(y(t_k))| \end{aligned}$$

which approaches zero as $n \to \infty$.

Step 2: N maps bounded sets into bounded sets in Ω. Indeed, it is sufficient to show that for any $q > 0$ there exists a positive constant ℓ such that for each $y \in B_q = \{y \in \Omega : \|y\|_\Omega \leq q\}$ one has $\|N(y)\|_\Omega \leq \ell$. Let $y \in B_q$. Then for $t \in [0,T]$ we have

$$N(y)(t) = \int_0^T H(t,s)f(s,y_s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)).$$

By (H2) we have for each $t \in [0, T]$

$$\begin{aligned}
|N(y)(t)| &\leq \int_0^T |H(t,s)||f(s,y_s)|d\,s + \sum_{k=1}^m |H(t,t_k)||I_k(y(t_k))| \\
&\leq \int_0^T |H(t,s)|h_q(s)d\,s + \sum_{k=1}^m |H(t,t_k)|c_k.
\end{aligned}$$

Then for each $h \in N(B_q)$ we have

$$\|N(y)\|_\Omega \leq \frac{1}{1-e^{-\lambda T}}\Big(\int_0^T h_q(s)d\,s + \sum_{k=1}^m c_k\Big) := \ell.$$

Step 3: N maps bounded set into equicontinuous sets of Ω. Let $\tau_1, \tau_2 \in [0, T]$, $\tau_1 < \tau_2$ and B_q be a bounded set of Ω as in Step 2. Let $y \in B_q$ and $t \in [0, T]$ we have

$$N(y)(t) = \int_0^T H(t,s)f(s,y_s)d\,s + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)).$$

Then

$$\begin{aligned}
&|N(y)(\tau_2) - N(y)(\tau_1)| \\
&\leq \int_0^T |H(\tau_2,s) - H(\tau_1,s)|h_q(s)ds + \sum_{k=1}^m |H(\tau_2,t_k) - H(\tau_1,t_k)|c_k.
\end{aligned}$$

As $\tau_2 \to \tau_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem we can conclude that $N : \Omega \to \Omega$ is completely continuous.

Let $C = \{y \in \Omega : y(t) \geq 0 \text{ for } t \in [-r, T]\}$ be a cone in Ω. Since f, H, I_k, $k = 1, \dots, m$ are positive functions, then $N(C) \subset C$ and $N : \overline{C}_R \to \overline{C}_R$ is compact. By (H1), (H2), (H4) we can show that if $y \in \overline{C}_R$ then $N(y) \subset \overline{C}_R$. Let $\psi : C \to [0, \infty)$ defined by $\psi(y) = \min_{t\in[a,b]} y(t)$. It is clear that ψ is a nonnegative concave continuous functional and $\psi(y) \leq \|y\|_\Omega$ for $y \in \overline{C}_R$. Now it remains to show that the hypotheses of Theorem 2.2 are satisfied. First notice that condition (A2) of Theorem 2.2 holds since for $y \in \overline{C}_r$, and from (H1) and (H2) we have

$$\begin{aligned}
|N(y)(t)| &\leq \int_0^T |H(t,s)||f(s,y_s)|ds + \sum_{k=1}^m |H(t,t_k)|I_k(y(t_k))| \\
&\leq \int_0^T g(\|y_s\|)p(s)ds + \sum_{k=1}^m |H(t,t_k)|I_k(y(t_k))| \\
&\leq g(r)\|p\|_{L^1} + \frac{1}{1-e^{-\lambda T}}\sum_{k=1}^m c_k < r.
\end{aligned}$$

Let $K \geq L$ and $y(t) = \frac{L+K}{2}$ for $t \in [-r, T]$. By the definition of $C(\psi, L, K), y$ belongs to $C(\psi, L, K)$. Then $y \in \{y \in C(\psi, L, K) : \psi(y) > L\}$. Also if $y \in C(\psi, L, K)$ then

$$\psi(N(y)) = \min_{t\in[a,b]}\Big(\sum_{k=1}^m H(t,t_k)I_k(y(t_k)) + \int_0^T H(t,s)f(s,y_s)ds\Big).$$

Then from (H3) we have

$$\begin{aligned}\psi(N(y)) &= \min_{t\in[a,b]}\left(\sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)) + \int_0^T H(t,s)f(s,y_s)ds\right)\\ &\geq \Big(M\sum_{k=1}^{m} c_k + M\int_0^T g(\|y_s\|)p(s)ds\Big)\\ &\geq M\Big(\sum_{k=1}^{m} c_k + g(L)\int_0^T p(s)ds\Big) > L.\end{aligned}$$

So the conditions (A1) and (A2) of Theorem 2.2 are satisfied.

Finally we will be prove that (A3) of Theorem 2.2 holds. Let $y \in C(\psi, L, R)$ with $\|N(y)\|_\Omega > K$ Thus

$$\begin{aligned}\psi(N(y)) &= \min_{t\in[a,b]}\Big(\sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)) + \int_0^T H(t,s)f(s,y_s)ds\Big)\\ &\geq M_1M_*\Big(\sum_{k=1}^{m} c_k + g(R)\int_0^T p(s)ds\Big)\\ &\geq M_1\|N(y)\|_\Omega > M_1K > L.\end{aligned}$$

Thus condition (A3) holds. Then Leggett and Williams fixed point theorem implies that N has at least three fixed points y_1, y_2, y_3 which are solutions to problem (1.1)–(1.3). Furthermore, we have

$$y_1 \in C_r,\quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\},\quad y_3 \in C_R - \{C(\psi, L, R) \cup (C_r)\}.$$

□

4. Second Order Impulsive FDEs

In this section we give an existence result for the boundary-value problem (1.4)–(1.7).

Definition 4.1. A function $y \in \Omega \cap \cup_{k=0}^m AC^1((t_k, t_{k+1}), \mathbb{R})$ is said to be a solution of (1.4)–(1.7) if y satisfies $y''(t) - \lambda y(t) = f(t, y_t)$ a.e. on $J\backslash\{t_1, \dots, t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-))$, $k = 1, \dots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$, $y(0) - y(T) = \mu_0$, $y'(0) - y'(T) = \mu_1$.

We now consider the "linear problem"

$$y''(t) - \lambda y(t) = g(t),\quad t \neq t_k,\ k = 1, \dots, m, \tag{4.1}$$

subjected to the conditions (1.5), (1.6), (1.7), and where $g \in L^1([t_k, t_{k+1}], \mathbb{R})$. Note that (1.5)–(1.7), (4.1) is not really a linear problem since the impulsive functions are not necessarily linear. However, if $I_k, \bar{I}_k$, $k = 1, \dots, m$ are linear, then (1.5)–(1.7), (4.1) is a linear impulsive problem.

We need the following auxiliary result:

Lemma 4.2. *$y \in \Omega \cap \cup_{k=0}^m AC^1((t_k, t_{k+1}), \mathbb{R})$ is a solution of (1.5)–(1.7), (4.1), if and only if $y \in \Omega$ is a solution of the impulsive integral functional equation,*

$$y(t) = \begin{cases} \phi(t), & t \in [-r, 0],\\ \int_0^T M(t,s)h(s)ds + M(t,0)\mu_1 + N(t,0)\mu_0 & \\ + \sum_{k=1}^m [M(t,t_k)I_k(y(t_k)) + N(t,t_k)\bar{I}_k(y(t_k))], & t \in [0, T],\end{cases} \tag{4.2}$$

where

$$M(t,s)=\frac{-1}{2\sqrt{\lambda}(e^{\sqrt{\lambda}T}-1)}\begin{cases}e^{\sqrt{\lambda}(T+s-t)}+e^{\sqrt{\lambda}(t-s)}, & 0\le s\le t\le T,\\ e^{\sqrt{\lambda}(T+t-s)}+e^{\sqrt{\lambda}(s-t)}, & 0\le t<s\le T,\end{cases}$$

and

$$N(t,s)=\frac{\partial}{\partial t}M(t,s)=\frac{1}{2(e^{\sqrt{\lambda}T}-1)}\begin{cases}e^{\sqrt{\lambda}(T+s-t)}-e^{\sqrt{\lambda}(t-s)}, & 0\le s\le t\le T,\\ e^{\sqrt{\lambda}(s-t)}-e^{\sqrt{\lambda}(T+t-s)}, & 0\le t<s\le T.\end{cases}$$

We omit the proof of this lemma since it is similar to the proof of results in [10].

We are now in a position to state and prove our existence result for problem (1.4)-(1.7). We first list the following hypotheses:

(H5) There exist constants d_k such that $|\overline{I}_k(x)|\le d_k,\ k=1,\dots,m$ for each $x\in\mathbb{R}$

(H6) There exist a function $g^*:[0,\infty)\to[0,\infty)$ continuous and non-decreasing, a function $p\in L^1(J,\mathbb{R}_+)$, $r^*>0$, and $0<M^*\le 1$ such that

$$M^*p(t)g^*(\|u\|)\le|M(t,s)f(t,u)|\le p(t)g^*(\|u\|)$$

for each $(t,s,u)\in J\times J\times D$, and

$$C\sum_{k=1}^m(c_k+d_k)+C_*[|\mu_1|+|\mu_0|]+\sup_{(t,s)\in[0,T]\times[0,T]}|M(t,s)|g^*(r^*)\int_0^T p(s)ds<r^*$$

where

$$C=\max(\sup_{(t,s)\in[0,T]\times[0,T]}|M(t,s)|,\sup_{(t,s)\in[0,T]\times[0,T]}|N(t,s)|),$$
$$C_*=\max(\sup_{t\in[0,T]]}|M(t,0)|,\sup_{t\in[0,T]}|N(t,0)|)$$

(H7) There exist $L^*>r^*$, $0<M^*\le 1$ and an interval $[a,b]\subset(0,T)$ such that

$$\begin{aligned}&\min_{t\in[a,b]}\Big(\sum_{k=1}^m[M(t,t_k)I_k(y(t_k))+N(t,t_k)\overline{I}_k(y(t_k))]+\int_0^T M(t,s)f(s,y_s)ds\Big)\\&\ge M^*\min_{t\in[a,b]}\Big(M(t,0)\mu_1+N(t,0)\mu_0+g^*(L^*)\int_0^T p(s)d\,s+\sum_{k=1}^m[c_k+d_k]\Big)\\&>L^*\end{aligned}$$

(H8) There exist $R^*,K,0<M_1^*\le M^*$ such that $M_1^{*-1}L<K\le R^*$,

$$C\sum_{k=1}^m[c_k+d_k]+C_*(|\mu_0|+|\mu_1|)+\sup_{(t,s)\in[0,T]\times[0,T]}|M(t,s)|g^*(R^*)\int_0^T p(s)ds<R^*$$

and

$$\begin{aligned}&\min_{t\in[a,b]}\Big(M(t,0)\mu_1+N(t,0)\mu_0+\sum_{k=1}^m M(t,t_k)I_k(y(t_k))\\&+N(t,t_k)\overline{I}_k(y(t_k))+\int_0^T M(t,s)f(s,y_s)ds\Big)\\&\ge M_1^*\Big(C_*(|\mu_1|+|\mu_0|)\end{aligned}$$

$$+ \sup_{(t,s)\in[0,T]\times[0,T]} |M(t,s)|g^*(R^*)\int_0^T p(s)ds + C\sum_{k=1}^m [c_k+d_k]\Big).$$

Theorem 4.3. *Suppose that hypotheses* $(H1)$, $(H5)-(H8)$ *are satisfied. Then the problem (1.4)–(1.7) has at least three positive solutions.*

Proof. We transform the problem into a fixed point problem. Consider the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1(y)(t) = \begin{cases} \phi(t), & t\in[-r,0],\\ \int_0^T M(t,s)f(s,y_s)ds + M(t,0)\mu_1 + N(t,0)\mu_0 & \\ +\sum_{k=1}^m [M(t,t_k)I_k(y(t_k)) + N(t,t_k)\overline{I}_k(y(t_k))], & t\in[0,T], \end{cases}$$

As in Theorem 3.2 we can show that N_1 is compact. Now we prove only that the hypotheses of Theorem 2.2 are satisfied.

Let $C = \{y \in \Omega : y(t) \ge 0 \text{ for } t\in[-r,T]\}$ be a cone in Ω. It is clear that $N_1(C)\subset C$ and $N_1 : \overline{C}_{R^*} \to \overline{C}_{R^*}$ is compact. By (H1), (H5), (H6), (H8) we can show that if $y\in \overline{C}_{R^*}$ then $N_1(y)\in\overline{C}_{R^*}$. Let $\psi : C\to[0,\infty)$ defined by $\psi(y) = \min_{t\in[a,b]} y(t)$. It is clear that ψ is a nonnegative concave continuous functional and $\psi(y)\le \|y\|_\Omega$ for $y\in\overline{C}_{R^*}$. Now it remains to show that the hypotheses of Theorem 2.2 are satisfied. First notice that condition (A2) holds since for $y\in\overline{C}_{r^*}$, we have from (H1), (H5), (H6),

$$\begin{aligned}|N_1(y)(t)| &\le \int_0^T |M(t,s)||f(s,y_s)|d\,s + |M(t,0)||\mu_1| + |N(t,0)||\mu_0|\\ &\quad + \sum_{k=1}^m [|M(t,t_k)|c_k + |N(t,t_k)|d_k]\\ &\le \int_0^T g^*(\|y_s\|)p(s)d\,s + C_*(|\mu_1|+|\mu_0|) + C\sum_{k=1}^m (c_k+d_k)\\ &\le \sup_{t\in[0,T]\times[0,T]} |M(t,s)|g^*(L^*)\int_0^T p(s)ds + C_*(|\mu_1|+|\mu_0|)\\ &\quad + C\sum_{k=1}^m (c_k+d_k) < r^*.\end{aligned}$$

Let $K^*\ge L^*$ and $y(t) = \frac{L^*+K^*}{2}$ for $t\in[-r,T]$. By the definition of $C(\psi,L^*,K^*)$, y is in $C(\psi,L^*,K^*)$. Then $y\in\{y\in C(\psi,L^*,K^*) : \psi(y) > L^*\}$. Also if $y\in C(\psi,L^*,K^*)$ we have

$$\begin{aligned}\psi(N_1(y)) = \min_{t\in[a,b]}\Big(&\int_0^T M(t,s)f(s,y_s)ds + M(t,0)\mu_1 + N(t,0)\mu_0\\ &+ \sum_{k=1}^m [M(t,t_k)I_k(y(t_k)) + N(t,t_k)\overline{I}_k(y(t_k))]\Big).\end{aligned}$$

Then from (H7) we get

$$\begin{aligned}\psi(N_1(y)) &\ge M^*\min_{t\in[a,b]}\Big(M(t,0)\mu_1 + N(t,0)\mu_0 + \sum_{k=1}^m (c_k+d_k) + g^*(L^*)\int_0^T p(s)ds\Big)\\ &> L^*.\end{aligned}$$

So conditions (A1) and (A2) of Theorem 2.2 are satisfied.

Finally to see that (A3) holds let $y \in C(\psi, L^*, R^*)$ with $\|N_1(y)\|_\Omega > K^*$ then from (H8) we have

$$
\begin{aligned}
&\psi(N_1(y))\\
&\geq M_1^*\Big(\sum_{k=1}^{m}(c_k+d_k)+C_*|\mu_1|+C|\mu_0|+\sup_{t\in[0,T]\times[0,T]}|M(t,s)|g^*(R^*)\int_0^T p(s)ds\Big)\\
&\geq M_1^*\Big(C_*|\mu_1|+C|\mu_0|+C^*\sum_{k=1}^{m}(c_k+d_k)+\sup_{t\in[0,T]\times[0,T]}|M(t,s)|g^*(R^*)\int_0^T p(s)ds\Big)\\
&\geq M_1^*\|N_1(y)\|_\Omega > M_1^*K > L^*.
\end{aligned}
$$

Thus condition (A3) for Theorem 2.2 holds. As consequence of Leggett and Williams theorem we deduce that N_1 has at least three fixed points y_1, y_2, y_3 which are solutions to problem (1.4)–(1.7). Furthermore, we have

$$
y_1 \in C_{r^*}, \quad y_2 \in \{y \in C(\psi, L^*, R^*) : \psi(y) > L^*\}, \quad y_3 \in C_{R^*} - \{C(\psi, L^*, R^*) \cup C_{r^*}\}.
$$

□

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