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ON UNIQUENESS AND EXISTENCE OF ENTROPY SOLUTIONS OF WEAKLY COUPLED SYSTEMS OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS

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Abstract. We prove existence and uniqueness of entropy solutions for the Cauchy problem of weakly coupled systems of nonlinear degenerate parabolic equations. We prove existence of an entropy solution by demonstrating that the Engquist-Osher finite difference scheme is convergent and that any limit function satisfies the entropy condition. The convergence proof is based on deriving a series of a priori estimates and using a general L^p compactness criterion. The uniqueness proof is an adaption of Kružkov's "doubling of variables" proof. We also present a numerical example motivated by biodegradation in porous media.

1. Introduction

In this paper we will prove existence and uniqueness of entropy solutions for weakly coupled systems of nonlinear (strongly) degenerate parabolic equations the form

$$u_t^\kappa + \operatorname{div} F^\kappa(u^\kappa) = \Delta A^\kappa(u^\kappa) + g^\kappa(U), \quad (x,t) \in \Pi_T, \quad \kappa = 1, \dots, K. \tag{1.1}$$

Here $U = (u^1, \dots, u^K)$, $F^\kappa(u^\kappa) = (F_1^\kappa(u^\kappa), \dots, F_d^\kappa(u^\kappa))$, $\Pi_T = \mathbb{R}^d \times [0,T]$, for some T positive. The system (1.1) can more compactly be written as

$$U_t + \operatorname{div} F(U) = \Delta A(U) + G(U), \tag{1.2}$$

when we introduce

$$F_i(U) = \left(F_i^1(u^1), \dots, F_i^K(u^K)\right), \quad A(U) = \left(A^1(u^1), \dots, A^K(u^K)\right),$$
$$G(U) = \left(g^1(U), \dots, g^K(U)\right).$$

We will consider the Cauchy problem for the weakly coupled system (1.1); i.e., we require that

$$U\big|_{t=0} = U_0 \in L^1(\mathbb{R}^d; \mathbb{R}^K) \cap L^\infty(\mathbb{R}^d; \mathbb{R}^K). \tag{1.3}$$

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We assume that the nonlinear (convection and diffusion) flux functions satisfy the general conditions

$$\begin{aligned} F^\kappa &\in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d), \quad F^\kappa(0)=0, \\ A^\kappa &\in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \quad A^\kappa \text{ is nondecreasing with } A^\kappa(0)=0, \end{aligned} \tag{1.4}$$

where $\kappa = 1,\dots,K$. In addition, we assume that

$$G \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^K;\mathbb{R}^K), \quad G(0)=0. \tag{1.5}$$

This class of nonlinear partial differential equations includes several important equations as special cases. When g^κ vanishes identically for all κ, the equation (1.1) becomes K scalar partial differential equations. In particular, the single conservation law

$$u_t + \operatorname{div} f(u) = 0,$$

is a 'simple' special case of (1.1). The regularized conservation law

$$u_t + \operatorname{div} f(u) = \Delta u$$

is another equation within the class analyzed here. Included is also the heat equation

$$u_t = \Delta u, \tag{1.6}$$

the porous medium equation

$$u_t = \Delta u^m, \quad m \ge 1,$$

the two-phase reservoir flow equation

$$u_t + \Big(\frac{u^2}{u^2+(1-u)^2}\Big)_x = A(u)_{xx}, \quad A(u) = \int^u v(1-v)\,dv,$$

as well as the nonlinear strongly degenerate convection-diffusion equation arising the recent theory of sedimentation-consolidation processes (see [7]):

$$u_t + \nabla\cdot f(u) = \Delta D(u), \quad D' \ge 0.$$

Weakly coupled systems arise in relaxation regularizations of conservation laws, where one studies a linear system of equations of the type

$$\begin{aligned} u_t - \sqrt{a}u_x &= -g(u,v), \\ v_t + \sqrt{a}v_x &= g(u,v). \end{aligned}$$

If u is a scalar, then a is a positive number. Furthermore, weakly coupled systems also arise in mathematical models of biodegradation, see Cirpka et al. [13], and the numerical example in Section 5.

Due to the nonlinearity, the mixed hyperbolic-parabolic problem (1.1)–(1.3) will in general possess shock wave solutions, a feature that *may* reflect the physical phenomenon of breaking of waves. This is well-known in the context of conservation laws. Consequently, due to this loss of regularity, it is necessary to work with weak solutions.

A function u^κ is called a weak solution if $u^\kappa \in L^1\cap L^\infty \cap C(0,T;L^1)$, $\nabla A^\kappa(u^\kappa) \in L^2$, u^κ satisfies (1.1) in the sense of distributions, and $u^\kappa(t) \to u_0^\kappa$ in L^1 as $t \downarrow 0$.

However, weak solutions are in general not uniquely determined by their data. We will here consider weak solutions that satisfy a so-called Kružkov type entropy condition (such solutions are called entropy solutions):

$$\begin{aligned} &|u^\kappa - k|_t + \operatorname{div}\big[\operatorname{sign}(u^\kappa - k)\big(F^\kappa(u^\kappa) - F^\kappa(k)\big)\big] - \Delta\,|A^\kappa(u^\kappa) - A^\kappa(k)| \\ &\le \operatorname{sign}(u^\kappa - k) g^\kappa(U) \quad \text{in } \mathcal{D}'(\Pi_T) \quad \text{for all } k \in \mathbb{R}. \end{aligned} \tag{1.7}$$

For a precise statement of the definition of an entropy solution, see Section 2. For pure hyperbolic equations, the entropy condition (2.1) was introduced by Kružkov [33] and Vol'pert [42]. For degenerate parabolic equations, it must be attributed to Vol'pert and Hudjaev [43]. The well-posedness of the entropy solution framework for weakly coupled system of degenerate parabolic equations is the content of the following theorem, which is the main contribution of this paper.

Theorem 1.1. *Assume that the conditions in* (1.4) *and* (1.5) *hold. Then there exists a unique entropy solution of the Cauchy problem* (1.1), (1.3).

We remark that existence and uniqueness of entropy solutions for weakly coupled system of first-order hyperbolic equations have been proved by Natalini and Hanouzet [35] and Rohde [39].

The existence assertion of Theorem 1.1 follows from the results in Section 4. As was done by Evje and Karlsen [20] and Karlsen and Risebro in [29] for scalar equations, existence of an entropy solution is here proved by establishing convergence of suitable finite difference approximations. We mention that for the existence proof one can replace the difference approximations used in this paper by proper adoptions of the numerical approximations studied in [19, 24, 4] or the vanishing viscosity method [43]. For a partial overview of numerical methods for entropy solutions of nonlinear degenerate parabolic equations, we refer to [18].

We now continue with more details about the convergence proof. Let $h > 0$ and $\Delta t > 0$ denote the spatial and temporal discretization parameters, respectively. We then let $u_i^{\kappa,n}$ denote the finite difference approximation of $u^\kappa(ih, n\Delta t)$. In the one-dimensional case, the explicit finite difference scheme takes the form

$$\frac{u_i^{\kappa,n+1} - u_i^{\kappa,n}}{\Delta t} + D_-\Big(F^{\kappa,\mathrm{EO}}\left(u_i^{\kappa,n}, u_{i+1}^{\kappa,n}\right) - D_+ A^\kappa\left(u_i^{\kappa,n}\right)\Big) = g^\kappa\left(U_i^n\right), \quad \kappa = 1, \dots, K, \tag{1.8}$$

where D_- and D_+ are the usual backward and forward difference operators, respectively. In (1.8), $F^{\kappa,\mathrm{EO}}$ denotes the Engquist–Osher numerical flux function [17] defined by

$$F^{\kappa,\mathrm{EO}}(u,v) = \frac{1}{2}\big(F^\kappa(u) + F^\kappa(v)\big) - \int_u^v \Big|\frac{dF^\kappa}{dr}(r)\Big|\, dr.$$

We refer to Section 4 for precise statements in the multi-dimensional case.

The convergence proof is based on deriving uniform L^∞, L^1, and BV bounds on the approximate solution u_h, where $u_h = u_h(x,t)$ denotes a piecewise constant interpolation of $\{u_i^n\}_{i,n}$. These bounds are readily obtained by exploiting that the difference operator on the left-hand side of the equality sign in (1.8) is L^1 contractive, so that the standard estimates from hyperbolic conservation laws apply. Equipped with the BV bound, we use the difference scheme itself and Kružkov's interpolation lemma [32] to show that $u_{\Delta t}$ is uniformly L^1 continuous in time. Kolmogorov's compactness criterion then immediately gives L^1_{loc} convergence (along a subsequence) of $\{u_h\}_{h>0}$ to a function $u \in L^1 \cap L^\infty \cap C([0,T]; L^1)$ such that $u(t) \to$

u_0 in L^1 as $t \downarrow 0$. To ensure that the limit u is the (unique) entropy solution, we first prove that the difference scheme satisfies a so-called discrete entropy inequality and hence it follows, by standard arguments that the entropy condition (1.7) holds true for the limit u. Finally, by an energy type argument we obtain that $A(u_h)$ converges (along a subsequence) to $A(u)$ in L^2_{loc} and $\nabla A(u) \in L^2$. For other papers dealing with numerical schemes for weakly coupled systems, see [40, 37] (and the references therein).

The uniqueness assertion of Theorem 1.1 follows from the results in Section 3. The uniqueness proof is an adaption of the celebrated "doubling of variables" proof due to Kružkov [33] for first-order equations along with an extension to second-order equations by Carrillo [9]. To put the second-order case in a proper perspective, let us illustrate Kružkov's "doubling" device on the simple heat equation (1.6).

At stake is the integral inequality (from which uniqueness readily follows[1])

$$|v-u|_t - \Delta\,|v-u| \le 0 \quad \text{in } \mathcal{D}'(\Pi_T), \tag{1.9}$$

where $v = v(x,t), u = u(x,t)$ are two entropy solutions of the heat equation (1.6).

Following Kružkov [33] closely, we use the entropy inequalities for $v = v(x,t)$ and $u = u(y,s)$ to derive

$$\int \Big(|v-u|\,(\partial_t + \partial_s)\,\phi + |v-u|\,(\Delta_x + \Delta_y)\,\phi \Big)\,dt\,dx\,ds\,dy \ge 0, \tag{1.10}$$

where $\phi = \phi(x,t,y,s)$ is a test function on $\Pi_T \times \Pi_T$. Following the guidelines in [33] once more, a clever choice of test function is

$$\phi(x,t,y,s) = \psi\big(\frac{x+y}{2}, \frac{t+s}{2}\big)\omega_\rho\big(\frac{x-y}{2}, \frac{t-s}{2}\big),$$

where ψ is again a test function and ω_ρ is an approximate delta function with smoothing radius $\rho > 0$.

With this choice, we have

$$(\partial_t + \partial_s)\,\phi = \left[(\partial_t + \partial_s)\,\psi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\right]\omega_\rho\left(\tfrac{x-y}{2}, \tfrac{t-s}{2}\right),$$

so that the singular (as $\rho \downarrow 0$) term cancels out. However, with the second-order operator $\Delta_x + \Delta_y$ we run into problems since there only holds that (see Section 3)

$$(\Delta_x + 2\nabla_x\cdot\nabla_y + \Delta_y)\,\phi = \left[(\Delta_x + 2\nabla_x\cdot\nabla_y + \Delta_y)\,\psi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\right]\omega_\rho\left(\tfrac{x-y}{2}, \tfrac{t-s}{2}\right). \tag{1.11}$$

With $\psi = \psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$ and $\omega_\rho = \omega_\rho\left(\frac{x-y}{2}, \frac{t-s}{2}\right)$, (1.10) then takes the form

$$\int \Big(|v-u|\,(\partial_t + \partial_s)\,\psi + |v-u|\,(\Delta_x + 2\nabla_x\cdot\nabla_y + \Delta_y)\,\psi \Big)\omega_\rho\,dt\,dx\,ds\,dy \ge \mathrm{RHS}, \tag{1.12}$$

where the singular (as $\rho \downarrow 0$) right-hand side is given by

$$\mathrm{RHS} = 2\int |v-u|\,\nabla_x\cdot\nabla_y\phi\,dt\,dx\,ds\,dy = -2\int \nabla_x v\cdot\nabla_y u\,\mathrm{sign}\,'(v-u)\,\phi\,dt\,dx\,ds\,dy.$$

The question emerges how to get rid of the RHS term. At this stage, one should recall that the "entropy dissipation" term has been thrown away in the course of deriving the entropy inequality (1.7). At least formally, the classical derivation of

[1] Formally, one takes a test function that is constant in space and equals the characteristic function on the time interval $[0,t]$. This yields $\|u(t) - v(t)\| \le \|u_0 - v_0\|$.

the entropy condition (see, e.g., Vol'pert and Hudjaev [43]) would actually produce a right-hand side of (1.10) of the form

$$\int \Big(\big|\nabla_x v\big|^2 + \big|\nabla_y u\big|^2 \Big) \operatorname{sign}'(v-u)\phi \, dt\, dx\, ds\, dy. \tag{1.13}$$

We see now that if this term is added to RHS, the result is

$$\widetilde{\mathrm{RHS}} = \int \big|\nabla_x v - \nabla_y u\big|^2 \operatorname{sign}'(v-u)\phi \, dt\, dx\, ds\, dy,$$

The advantage with this term is that $\widetilde{\mathrm{RHS}}$ has a definite sign and can therefore be thrown away. The above argument can be made rigorous by working with a "smooth" approximation of $\operatorname{sign}(\cdot)$, see Section 3. Although the proof of uniqueness for general second-order equations (and weakly coupled systems of such) is more technical, the basic ideas are still those illustrated here on the heat equation.

To finish the story, we follow again [33] when making the change of variables $z = (x-y)/2$, $\tau = (t-s)/2$ and $\tilde{x} = (x+y)/2$, $\tilde{t} = (t+s)/2$, which turns (1.12) into the elegant form

$$\int \Big(|v-u|\psi_{\tilde{t}} + |v-u|\,\Delta_{\tilde{x}}\psi \Big) \omega_\rho(z,\tau)\, d\tilde{t}\, d\tilde{x}\, d\tau\, dz \geq 0. \tag{1.14}$$

Sending $\rho \downarrow 0$ in (1.14), we get (1.9) since ψ was an arbitrary test function.

As we have seen, at least from point of view of carrying out the Kružkov proof for second-order equations, there seems to be a term missing in the entropy condition (for the heat equation the form of this term is hinted in (1.13)). Here a major breakthrough was found recently by Carrillo [9], who exploited the assumption $\nabla A(u) \in L^2$ to "test" the governing equation against $\operatorname{sign}(A(u)-A(c))$, a trick that eventually produced the "entropy dissipation" term needed for the Kružkov proof to work. In our context, Carrillo's trick is carried out in the proof of Lemma 3.1 herein. In [9], scalar equations with $f = f(u), A = A(u)$ were studied. Adopting the ideas in [9], uniqueness results for more general scalar equations with x, t dependent coefficients were proved recently by Karlsen and Risebro [30] and Karlsen and Ohlberger [28]. In this paper, we follow rather closely the presentation in [30]. To make the paper self-contained, we have chosen to give rather detailed proofs, although parts of the proofs are similar to those in [9, 30].

It is worthwhile pointing out that different from [9], we work here with all the derivatives on the test functions ([9] keeps one derivative on the diffusion function) and we exploit fully identity (1.11). We feel that this slightly simplifies the uniqueness proof. There is also a similarity here with the uniqueness proof for *viscosity* solutions of degenerate second-order equations [27].

For some other related papers dealing with the Kružkov's "doubling" device in the context of second-order (scalar) equations of the type studied herein, see (the list is certainly incomplete) [3, 8, 38, 11, 41, 14, 5, 6, 36, 34, 24, 23, 28, 21, 12].

Before ending this discussion about uniqueness, we would like to draw special attention to the paper by Chen and DiBenedetto [11] (see also Chen and Perthame [12]) cited in the above list, which roughly speaking includes the "entropy dissipation" term into their very definition of an entropy solution. This is thus another way of circumventing the problem with extending Kružkov's uniqueness proof to second-order equations. However, from the point of view of establishing existence (i.e., convergence of approximate solutions), this method is less satisfactory since

it is a more involved process to pass to the limit of approximate solutions in an entropy inequality that includes the "entropy dissipation" term than in the standard one (1.7).

2. Definitions and Preliminaries

Recall that a function $\eta\colon \mathbb{R}\to\mathbb{R}$ is called an *entropy function* if it is convex and C^2. For $\kappa=1,\dots,K$, a vector-valued function $q^\kappa=(q_1^\kappa,\dots,q_d^\kappa)\colon \mathbb{R}\to\mathbb{R}^d$ is called an *entropy flux* if it satisfies the compatibility conditions

$$\frac{dq^\kappa}{du}(u)=\eta'(u)\frac{dF^\kappa}{du}(u).$$

For $\kappa=1,\dots,K$, a function $r^\kappa\colon \mathbb{R}\to\mathbb{R}$ is called a *diffusion entropy flux* if it satisfies the compatibility conditions

$$\frac{dr^\kappa}{du}(u)=\eta'(u)\frac{dA^\kappa}{du}(u).$$

For $k\in\mathbb{R}$, the function $\eta(u)=|u-k|$ is called a *Kružkov entropy function*. The associated functions

$$q^\kappa(u)=\operatorname{sign}(u-k)\big(F^\kappa(u)-F^\kappa(k)\big),\quad r^\kappa(u)=|A^\kappa(u)-A^\kappa(k)|$$

are called the *Kružkov entropy fluxes*. Observe that $r^\kappa(u)=\operatorname{sign}(u-k)(A^\kappa(u)-A^\kappa(k))$. We can now state the following definition of an entropy solution.

Definition 2.1 (Entropy Solution)**.** A vector-valued function $U=(u^1,\dots,u^K)\colon \Pi_T\to\mathbb{R}^K$ is called an *entropy solution* of the Cauchy problem (1.1),(1.3) if for all $\kappa=1,\dots,K$:

(1) $u^\kappa\in L^1(\Pi_T)\cap L^\infty(\Pi_T)\cap C([0,T];L^1(\mathbb{R}^d))$.
(2) $A^\kappa(u^\kappa)\in L^2\big([0,T];H^1(\mathbb{R}^d)\big)$.
(3) For all entropy functions $\eta\colon\mathbb{R}\to\mathbb{R}$ and corresponding entropy fluxes q^κ, r^κ,

$$\eta(u^\kappa)_t+\operatorname{div} q^\kappa(u^\kappa)-\Delta r^\kappa(u^\kappa)\le\eta'(u^\kappa)g^\kappa(U)\quad\text{in }\mathcal{D}'(\Pi_T);$$

that is, for any non-negative test function $\phi(x,t)\in C_0^\infty(\Pi_T)$

$$\iint_{\Pi_T}\Big(\eta(u^\kappa)\phi_t+q^\kappa(u^\kappa)\cdot\nabla\phi+r^\kappa(u^\kappa)\Delta\phi\Big)\,dt\,dx\ge-\iint_{\Pi_T}\eta'(u^\kappa)g^\kappa(U)\phi\,dt\,dx. \tag{2.1}$$

(4) For any ball $\mathcal{B}_r=\big\{x\in\mathbb{R}^d\mid|x|\le r\big\}$,

$$\int_{\mathcal{B}_r}|u^\kappa(x,t)-u_0^\kappa(x)|\,dx\to0\quad\text{essentially as } t\downarrow0+. \tag{2.2}$$

We recall that it is equivalent to require that (2.1) holds for the Kružkov entropies: for any $k\in\mathbb{R}$ and any non-negative test function $\phi(x,t)\in C_0^\infty(\Pi_T)$,

$$\begin{aligned}\iint_{\Pi_T}\Big(&|u^\kappa-k|\phi_t+\operatorname{sign}(u^\kappa-k)\big[F^\kappa(u^\kappa)-F^\kappa(k)\big]\cdot\nabla\phi\\&+|A^\kappa(u^\kappa)-A^\kappa(k)|\,\Delta\phi\Big)\,dt\,dx\\\ge&-\iint_{\Pi_T}\operatorname{sign}(u^\kappa-k)g^\kappa(U)\phi\,dt\,dx.\end{aligned}\tag{2.3}$$

It is well-known that (2.3) in particular implies that U is a weak solution, that is,

$$\iint_{\Pi_T} \Big(u^\kappa \phi_t + F^\kappa(u^\kappa) \cdot \nabla\phi + A^\kappa(u^\kappa)\Delta\phi \Big)\, dt\, dx = -\iint_{\Pi_T} g^\kappa(U)\phi\, dt\, dx, \tag{2.4}$$

for $\kappa = 1, \dots, K$ and any $\phi \in C_0^\infty(\Pi_T)$. We shall need the following five technical lemmas to prove existence of an entropy solution.

Lemma 2.2 (Crandall and Tartar [16]). *Let (Ω, μ) be a measure space and let $D \subset L^1(\Omega)$. Assume that if u and v are in D, then also $u \vee v$ is in D. Let $\mathcal{T}$ be a map $D \to D$ such that*

$$\int_\Omega \mathcal{T}(u)\, d\mu = \int_\Omega u\, d\mu, \quad u \in D.$$

Then the following statements, valid for all u and v in D, are equivalent:

(1) *If $u \le v$, then $\mathcal{T}(u) \le \mathcal{T}(v)$.*
(2) $\int_\Omega \big((\mathcal{T}(u) - \mathcal{T}(v)) \vee 0 \big)\, d\mu \le \int_\Omega \big((u - v) \vee 0 \big)\, d\mu$.
(3) $\int_\Omega |\mathcal{T}(u) - \mathcal{T}(v)|\, d\mu \le \int_\Omega |u - v|\, d\mu$.

Let $u\colon \Pi_T \to \mathbb{R}$ be a function such that $u(\cdot, t) \in L^1(\mathbb{R}^d)$ for all $t \in (0, T)$. By a modulus of continuity, we mean a nondecreasing continuous function $\nu\colon [0, \infty) \to [0, \infty)$ such that $\nu(0) = 0$. We say that u has ν as a spatial modulus of continuity if

$$\sup_{|y| \le r} \int_{\mathbb{R}^d} |u(x + y, t) - u(x, t)|\, dx \le \nu(r; u), \tag{2.5}$$

(where ν may depend on t). We also say that u has ω as a temporal modulus of continuity if there is a modulus of continuity $\omega(\cdot; u)$ such that for each $\tau \in (0, T)$,

$$\sup_{0 \le s \le \tau} \int_{\mathbb{R}^d} |u(x, t + s) - u(x, t)|\, dx \le \omega(\tau; u), \quad t \in (0, T - \tau). \tag{2.6}$$

For proofs of Lemmas 2.3–2.5 we refer to [26].

Lemma 2.3 (L^1 compactness lemma). *Let $\{u_h\}_{h>0}$ be a sequence of functions defined on Π_T and assume that we have that:*

(1) *There exists a constant $C > 0$, independent of h, such that*

$$\|u_h(\cdot, t)\|_{L^1(\mathbb{R}^d)} \le C, \quad \|u_h(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \le C, \quad t \in (0, T);$$

(2) *There exists a spatial modulus of continuity ν, independent of h, such that*

$$\|u_h(\cdot + y, t) - u_h(\cdot, t)\|_{L^1(\mathbb{R}^d)} \le \nu(|y|; u_h), \quad \in \mathbb{R}^d,\ t \in (0, T);$$

(3) *There exists a temporal modulus of continuity ω, independent of h,*

$$\|u_h(\cdot, t + \tau) - u_h(\cdot, t)\|_{L^1(\mathbb{R}^d)} \le \omega(\tau; u_h), \quad \tau \in (0, T) \text{ and } t \in (0, T - \tau).$$

Then $\{u_h\}_{h>0}$ is compact in the strong topology of $L^1_{\mathrm{loc}}(\Pi_T)$. Moreover, any limit point of $\{u_h\}_{h>0}$ belongs to $L^1(\Pi_T) \cap L^\infty(\Pi_T) \cap C([0, T]; L^1(\mathbb{R}^d))$.

Lemma 2.4 (L^2 compactness lemma). *Let $\{u_h\}_{h>0}$ be a sequence of functions defined on Π_T and assume that we have that:*

(1) *There exists a constant $C_1 > 0$, independent of h, such that*

$$\|u_h\|_{L^2(\Pi_T)} \le C_1;$$

(2) *There exists a constant $C_2 > 0$, independent of h, such that*

$$\|u_h(\cdot + y, \cdot) - u_h(\cdot, \cdot)\|_{L^2(\Pi_T)} \le C_2|y|, \quad y \in \mathbb{R}^d;$$

(3) *There exists a constant $C_3 > 0$, independent of h, such that*

$$\|u_h(\cdot, \cdot + \tau) - u_h(\cdot, \cdot)\|_{L^2(\mathbb{R}^d \times (0,T-\tau))} \le C_3\sqrt{\tau}, \quad \tau \in (0, T).$$

Then $\{u_h\}_{h>0}$ is compact in the strong topology of $L^2_{\mathrm{loc}}(\Pi_T)$. Moreover, any limit point of $\{u_h\}_{h>0}$ belongs to $L^2([0,T]; H^1(\mathbb{R}^d))$.

Lemma 2.5 (Kružkov [32])**.** *Let $u(x,t)$ be a bounded measurable function defined on Π_T. For $t \in (0,T)$ assume that u possesses a spatial modulus of continuity*

$$\int_{\mathbb{R}^d} |u(x+\varepsilon, t) - u(x,t)|\, dx \le \nu(|\varepsilon|\,; u), \tag{2.7}$$

where ν does not depend on t. Suppose that for any $\phi \in C_0^\infty(\mathbb{R}^d)$ and any t_1,t_2 $\in (0,T)$,

$$\Big| \int_{\mathbb{R}^d} (u(x,t_2) - u(x,t_1))\, \phi(x)\, dx \Big| \le \mathrm{Const}_T \Big(\sum_{|\alpha| \le m} c_\alpha \|D^\alpha \phi\|_{L^\infty(\mathbb{R}^d)} \Big) |t_2 - t_1|\,, \tag{2.8}$$

where α denotes a multi-index, and c_α are constants not depending on ϕ or t. Then for any $t_1, t_2 \in (0,T)$ and all $\varepsilon > 0$

$$\int_{\mathbb{R}^d} |u(x,t_2) - u(x,t_1)|\, dx \le C\Big(|t_2 - t_1| \sum_{|\alpha| \le m} \frac{c_\alpha}{\varepsilon^{|\alpha|}} + \nu(u;\varepsilon) \Big). \tag{2.9}$$

3. Uniqueness of Entropy Solution

In this section, we prove uniqueness of the entropy solution. Let

$$M^\kappa := \|u^\kappa\|_{L^\infty(\Pi_T)}\,, \quad l^\kappa = A^\kappa(-M^\kappa), \quad L^\kappa = A^\kappa(M^\kappa),$$

and define the function $\big(A^\kappa\big)^{-1}\colon [l^\kappa, L^\kappa] \to \mathbb{R}$ by

$$\big(A^\kappa\big)^{-1}(r) := \min\Big\{ \xi \in [-M^\kappa, M^\kappa] \mid A^\kappa(\xi) = r \Big\}.$$

Notice that this is a lower semicontinuous function and denote by E^κ the set

$$E^\kappa = \big\{ r \in [l^\kappa, L^\kappa] : \big(A^\kappa\big)^{-1} \text{ is discontinuous at } r \big\}.$$

Furthermore, for $\varepsilon > 0$,

$$\mathrm{sign}_\varepsilon(\xi) = \begin{cases} -1, & \xi \le -\varepsilon, \\ \xi/\varepsilon, & -\varepsilon < \xi < \varepsilon, \\ 1 & \xi \ge \varepsilon. \end{cases}$$

To be able to carry out Kružkov's uniqueness proof in our second order context, we need the following version of an important lemma of Carrillo [9].

Lemma 3.1 (Entropy Dissipation Term). *Let u^κ be the κth component of an entropy weak solution of (1.1), (1.3). Then, for any non-negative $\phi \in C_0^\infty(\Pi_T)$ and $k \in \mathbb{R}$ such that $A^\kappa(k) \notin E^\kappa$,*

$$\begin{aligned}
&\iint_{\Pi_T} \Big(|u^\kappa - k| \partial_t \phi + \operatorname{sign}(u^\kappa - k) \big[F^\kappa(u^\kappa) - F^\kappa(k) \big] \cdot \nabla \phi \\
&\quad + |A^\kappa(u^\kappa) - A^\kappa(k)| \, \Delta \phi \Big) \, dt \, dx \\
&= \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \big| \nabla A^\kappa(u^\kappa) \big|^2 \operatorname{sign}_\varepsilon'(A^\kappa(u^\kappa) - A^\kappa(k)) \phi \, dt \, dx \\
&\quad - \iint_{\Pi_T} \operatorname{sign}(u^\kappa - k) g^\kappa(U) \phi \, dt \, dx \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \iint_{|A^\kappa(u^\kappa) - A^\kappa(k)| < \varepsilon} \big| \nabla A^\kappa(u^\kappa) \big|^2 \phi \, dt \, dx - \iint_{\Pi_T} \operatorname{sign}(u^\kappa - k) g^\kappa(U) \phi \, dt \, dx.
\end{aligned} \tag{3.1}$$

Proof. In what follows, we define $u(t) = u_0$ for $t < 0$ and $u(t) = 0$ for $t > T$. Throughout this proof, one should keep in mind that

$$\nabla |A^\kappa(u^\kappa) - A^\kappa(k)| = \operatorname{sign}(u^\kappa - k) \nabla A^\kappa(u^\kappa) \text{ a.e. on } \Pi_T. \tag{3.2}$$

An entropy solution is also a weak solution, and an integration by parts in the weak formulation yields

$$\iint_{\Pi_T} \Big(u^\kappa \phi_t + \big[F^\kappa(u^\kappa) - \nabla A^\kappa(u^\kappa) \big] \cdot \nabla \phi \Big) \, dt \, dx = - \iint_{\Pi_T} g^\kappa(U) \phi \, dt \, dx, \tag{3.3}$$

for any $\phi \in C_0^\infty(\Pi_T)$. In view of (1.4), (1.5), and Definition 2.1, there exists a constant such that

$$\begin{aligned}
&\Big| \iint_{\Pi_T} \Big(\big[F^\kappa(u^\kappa) - \nabla A^\kappa(u^\kappa) \big] \cdot \nabla \phi + g^\kappa(U) \phi \Big) \, dt \, dx \Big| \\
&\le \operatorname{Const} \big(\|u^\kappa\|_{L^2(\Pi_T)} + \|\nabla A^\kappa(u^\kappa)\|_{L^2(\Pi_T)} \big) \|\phi\|_{L^2([0,T]; H^1(\mathbb{R}^d))}.
\end{aligned}$$

This bound implies that (3.3) holds for all $\phi \in H^1(\Pi_T)$ with $\phi|_{t=0,T} = 0$.

For $\varepsilon > 0$ and $\phi \in C_0^\infty(\Pi_T)$, introduce the functions

$$\mathcal{A}_\varepsilon^\kappa(z; k) = \int_k^z \operatorname{sign}_\varepsilon(A^\kappa(\xi) - A^\kappa(k)) \, d\xi, \quad \psi_\varepsilon^\kappa(u^\kappa) = \operatorname{sign}_\varepsilon(A^\kappa(u^\kappa) - A^\kappa(k)) \phi.$$

We claim that

$$\iint_{\Pi_T} \Big(\mathcal{A}_\varepsilon^\kappa(u^\kappa; k) \phi_t + \big[F^\kappa(u^\kappa) - \nabla A^\kappa(u^\kappa) \big] \cdot \nabla \psi^\varepsilon \Big) \, dt \, dx = - \iint_{\Pi_T} g^\kappa(U) \psi_\varepsilon^\kappa \, dt \, dx. \tag{3.4}$$

To show (3.4), for $\varepsilon > 0$, we introduce the time-regularized test function

$$\psi_\varepsilon^{\kappa, \Delta t}(x, t) = \frac{1}{\Delta t} \int_t^{t + \Delta t} \psi_\varepsilon^\kappa(x, s) \, ds.$$

Observe that $\psi_\varepsilon^\kappa \in L^2([0,T]; H^1(\mathbb{R}))$ and $\psi_\varepsilon^{\kappa, \Delta t} \in H^1(\Pi_T)$, i.e., $\psi_\varepsilon^{\kappa, \Delta t}$ is indeed an admissible test function in the weak formulation of (1.1). Consequently, (3.3) reads

$$\iint_{\Pi_T} \Big(u^\kappa \big(\psi_\varepsilon^{\kappa, \Delta t} \big)_t + \big[F^\kappa(u^\kappa) - \nabla A^\kappa(u^\kappa) \big] \cdot \nabla \psi_\varepsilon^{\kappa, \Delta t} \Big) \, dt \, dx$$

$$= -\iint_{\Pi_T} g^\kappa(U)\psi_\varepsilon^{\kappa,\Delta t}\,dt\,dx.$$

Furthermore

$$\begin{aligned}\iint_{\Pi_T} u^\kappa \left(\psi_\varepsilon^{\kappa,\Delta t}\right)_t\,dt\,dx &= \iint_{\Pi_T} u^\kappa \frac{\psi^\varepsilon(x,t+\Delta t)-\psi_\varepsilon^\kappa(x,t)}{\Delta t}\,dt\,dx\\ &= -\iint_{\Pi_T} \frac{u^\kappa(x,t)-u^\kappa(x,t-\Delta t)}{\Delta t}\psi_\varepsilon^\kappa\,dt\,dx.\end{aligned}$$

Since $\mathcal{A}_\varepsilon^\kappa$ is a convex function, we have

$$\mathcal{A}_\varepsilon^\kappa(z_2;k)-\mathcal{A}_\varepsilon^\kappa(z_1;k)\geq \big(z_2-z_1\big)\,\mathrm{sign}_\varepsilon(A^\kappa(z_1)-A^\kappa(k)),\quad z_1,z_2\in\mathbb{R}. \tag{3.5}$$

In view of (3.5) and the definition of ψ_ε^κ, we have, for a.e. $(x,t)\in\Pi_T$,

$$-\big(u^\kappa(x,t)-u^\kappa(x,t-\Delta t)\big)\psi_\varepsilon^\kappa \leq -\big(\mathcal{A}_\varepsilon^\kappa(u^\kappa(x,t);k)-\mathcal{A}_\varepsilon^\kappa(u^\kappa(x,t-\Delta t);k)\big)\phi.$$

Using this inequality we get

$$\begin{aligned}\iint_{\Pi_T} u^\kappa \left(\psi_\varepsilon^{\kappa,\Delta t}\right)_t\,dtdx &\leq -\iint_{\Pi_T} \frac{\mathcal{A}_\varepsilon^\kappa(u^\kappa(x,t);k)-\mathcal{A}_\varepsilon^\kappa(u^\kappa(x,t-\Delta t);k)}{\Delta t}\phi\,dt\,dx\\ &= \iint_{\Pi_T} \mathcal{A}_\varepsilon^\kappa(u^\kappa;k)\frac{\phi(x,t+\Delta t)-\phi(x,t)}{\Delta t}\,dt\,dx\\ &\to \iint_{\Pi_T} \mathcal{A}_\varepsilon^\kappa(u^\kappa;k)\phi_t\,dt\,dx \quad \text{as } \Delta t\downarrow 0.\end{aligned}$$

Keeping in mind that $\psi^{\varepsilon,\Delta t}\to\psi^\varepsilon$ in $L^2([0,T];H^1(\mathbb{R}))$ as $\Delta t\downarrow 0$, it hence follows that

$$\iint_{\Pi_T}\Big(\mathcal{A}_\varepsilon^\kappa(u^\kappa;k)\phi_t+\big[F^\kappa(u^\kappa)-\nabla A^\kappa(u^\kappa)\big]\cdot\nabla\psi_\varepsilon^\kappa\Big)\,dt\,dx \geq -\iint_{\Pi_T} g^\kappa(U)\psi^\varepsilon\,dt\,dx$$

which is one half of (3.4). To prove the opposite inequality, one proceeds exactly as before using the time-regularized test function $\psi_\varepsilon^{\kappa,\Delta t}(x,t)=\frac{1}{\Delta t}\int_{t-\Delta t}^t\psi^\varepsilon(x,s)\,ds$ and the inequality

$$-\big(u^\kappa(x,t+\Delta t)-u^\kappa(x,t)\big)\psi_\varepsilon^\kappa \geq -\big(\mathcal{A}_\varepsilon^\kappa(u^\kappa(x,t+\Delta t);k)-\mathcal{A}_\varepsilon^\kappa(u^\kappa(x,t);k)\big)\phi.$$

This concludes the proof of our claim (3.4).

Let ϕ,k be as stated in the lemma. Then one can easily check that, as $\varepsilon\downarrow 0$,

$$\mathcal{A}_\varepsilon^\kappa(u^\kappa;k)\to|u^\kappa-k| \quad \text{a.e. in } \Pi_T.$$

Moreover, we have $|\mathcal{A}_\varepsilon^\kappa(u^\kappa;k)|\leq|u^\kappa-k|$, so by Lebesgue's dominated convergence theorem

$$\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T}\mathcal{A}_\varepsilon^\kappa(u^\kappa;k)\partial_t\phi\,dt\,dx=\iint_{\Pi_T}|u^\kappa-k|\partial_t\phi\,dt\,dx.$$

Furthermore,

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T}\Big(F^\kappa(u^\kappa)-F^\kappa(k)-\nabla A^\kappa(u^\kappa)\Big)\cdot\nabla\psi_\varepsilon^\kappa\,dt\,dx\\ &=\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T}\Big(F^\kappa(u^\kappa)-F^\kappa(k)-\nabla A^\kappa(u^\kappa)\Big)\cdot\nabla\,\mathrm{sign}_\varepsilon(A^\kappa(u^\kappa)-A^\kappa(k))\phi\,dt\,dx\\ &\quad+\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T}\mathrm{sign}_\varepsilon(A^\kappa(u^\kappa)-A^\kappa(k))\Big(F^\kappa(u^\kappa)-F^\kappa(k)-\nabla A^\kappa(u^\kappa)\Big)\cdot\nabla\phi\,dt\,dx\end{aligned}$$

$$= \underbrace{\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T} \operatorname{sign}'_\varepsilon(A^\kappa(u^\kappa)-A^\kappa(k))\Big(F^\kappa(u^\kappa)-F^\kappa(k)\Big)\cdot\nabla A^\kappa(u^\kappa)\phi\,dt\,dx}_{I_1}$$
$$- \lim_{\varepsilon\downarrow 0}\iint_{\Pi_T} \big|\nabla A^\kappa(u^\kappa)\big|^2 \operatorname{sign}'_\varepsilon(A^\kappa(u^\kappa)-A^\kappa(k))\phi\,dt\,dx$$
$$+ \underbrace{\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T} \operatorname{sign}_\varepsilon(A^\kappa(u^\kappa)-A^\kappa(k))\Big(F^\kappa(u^\kappa)-F^\kappa(k)-\nabla A^\kappa(u^\kappa)\Big)\cdot\nabla\phi\,dt\,dx}_{I_2}.$$

Regarding I_1 we have

$$\begin{aligned}
|I_1| &= \Big|\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T}\operatorname{div}\Big(\int_{A^\kappa(k)}^{A^\kappa(u^\kappa)}\operatorname{sign}'_\varepsilon(r-A^\kappa(k))\\
&\qquad\times\Big(F^\kappa\big(\big(A^\kappa\big)^{-1}(r)\big)-F^\kappa\big((A^\kappa)^{-1}(A^\kappa(k))\big)\Big)\,dr\Big)\phi\,dt\,dx\Big|\\
&= \Big|\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T}\Big(\int_{A^\kappa(k)}^{A^\kappa(u^\kappa)}\operatorname{sign}'_\varepsilon(r-A^\kappa(k))\\
&\qquad\times\Big(F^\kappa\big(\big(A^\kappa\big)^{-1}(r)\big)-F^\kappa\big((A^\kappa)^{-1}(A^\kappa(k))\big)\Big)\,dr\Big)\cdot\nabla\phi\,dt\,dx\Big|\\
&\le C\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{|r-A^\kappa(k)|<\varepsilon}\Big|\big(A^\kappa\big)^{-1}(r)-\big(A^\kappa\big)^{-1}(A^\kappa(k))\Big|\,dr=0,
\end{aligned}$$

since $A^\kappa(k)\notin E^\kappa$. Here C is some constant that depends on the Lipschitz constant of F^κ and ϕ. Furthermore, we have

$$\begin{aligned}
I_2 &= \lim_{\varepsilon\downarrow 0}\iint_{\Pi_T}\operatorname{sign}_\varepsilon(A^\kappa(u^\kappa)-A^\kappa(k))\big(F^\kappa(u^\kappa)-F^\kappa(k)-\nabla A^\kappa(u^\kappa)\big)\cdot\nabla\phi\,dt\,dx\\
&= \iint_{\Pi_T}\operatorname{sign}(u^\kappa-k)\big(F^\kappa(u^\kappa)-F^\kappa(k)-\nabla A^\kappa(u^\kappa)\big)\cdot\nabla\phi\,dt\,dx.
\end{aligned}$$

In addition

$$\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T}\operatorname{sign}_\varepsilon(A^\kappa(u^\kappa)-A^\kappa(k))g^\kappa(U)\phi\,dt\,dx=\iint_{\Pi_T}\operatorname{sign}(u^\kappa-k)g^\kappa(U)\phi\,dt\,dx.$$

Consequently, sending $\varepsilon\downarrow 0$ in (3.4) and then doing an integration by parts (keeping (3.2) in mind), we obtain the desired equality (3.1). □

Remark 3.2. In the proof of Lemma 3.1, we have in effect proved the following "weak" chain rule (see, e.g., [2, 9, 38]):

$$-\int_0^T\big\langle\partial_t u,\operatorname{sign}_\varepsilon(A(u)-A(k))\phi\big\rangle\,dt=\iint_{\Pi_T}\Big(\int_k^u\operatorname{sign}_\varepsilon(A(\xi)-A(k))\,d\xi\Big)\partial_t\phi\,dt\,dx,$$

for every non-negative function $\phi\in C_0^\infty$ with $\phi|_{t=0}=\phi|_{t=T}=0$.

We are now ready to prove the following theorem:

Theorem 3.3 (Uniqueness)**.** *Assume that* (1.4) *and* (1.5) *hold. Let V,U be two entropy weak solutions of* (1.1), (1.3) *with initial data V_0,U_0, respectively. Then for a.e. $t\in[0,T]$,*

$$\int_{\mathbb{R}^d}\big|V(x,t)-U(x,t)\big|\,dx\le\sqrt{K}\,\exp\big(K\|G\|_{\mathrm{Lip}}\,t\big)\int_{\mathbb{R}^d}\big|V_0(x)-U_0(x)\big|\,dx.\qquad(3.6)$$

In particular, there exists at most one entropy weak solution of the Cauchy problem (1.1), (1.3).

Proof. Let $\phi \in C^\infty(\Pi_T \times \Pi_T)$, $\phi \geq 0$, $\phi = \phi(x,t,y,s)$, and

$$V = V(x,t) = (v^1(x,t), \dots, v^K(x,t)), \quad U = U(y,s) = (u^1(y,s), \dots, v^K(y,s)).$$

Let us introduce the "hyperbolic" sets

$$\mathcal{E}_v^\kappa = \Big\{(x,t) \in \Pi_T : A^\kappa(v^\kappa(x,t)) \in E^\kappa\Big\},$$
$$\mathcal{E}_u^\kappa = \Big\{(y,s) \in \Pi_T : A^\kappa(u^\kappa(y,s)) \in E^\kappa\Big\}.$$

For later use, observe that $\operatorname{sign}(v^\kappa - u^\kappa) = \operatorname{sign}(A^\kappa(v^\kappa) - A^\kappa(u^\kappa))$ a.e. in $\big[(\Pi_T \setminus \mathcal{E}_u^\kappa) \times \Pi_T\big] \cup \big[\Pi_T \times (\Pi_T \setminus \mathcal{E}_v^\kappa)\big]$. Also that $\nabla_x A^\kappa(v^\kappa) = 0$ a.e. in $\mathcal{E}_v^\kappa$ and $\nabla_y A^\kappa(u^\kappa) = 0$ a.e. in $\mathcal{E}_u^\kappa$. From the entropy condition (2.3) for $v^\kappa = v^\kappa(x,t)$ with $k = u^\kappa(y,s)$, we have

$$\begin{aligned} &- \iint_{\Pi_T} \Big(|v^\kappa - u^\kappa| \phi_t + \operatorname{sign}(v^\kappa - u^\kappa) \big[F^\kappa(v^\kappa) - F^\kappa(u^\kappa)\big] \cdot \nabla_x \phi \\ &\quad + |A^\kappa(v^\kappa) - A^\kappa(u^\kappa)| \, \Delta_x \phi \Big) \, dt \, dx \\ &\leq \iint_{\Pi_T} \operatorname{sign}(v^\kappa - u^\kappa) g^\kappa(V) \phi \, dt \, dx. \end{aligned} \tag{3.7}$$

Applying Lemma 3.1 with k replaced by u^κ, we have for all $(y,s) \notin \mathcal{E}_u^\kappa$

$$\begin{aligned} &- \iint_{\Pi_T} \Big(|v^\kappa - u^\kappa| \phi_t + \operatorname{sign}(v^\kappa - u^\kappa) \big[F^\kappa(v^\kappa) - F^\kappa(u^\kappa)\big] \cdot \nabla_x \phi \\ &\quad + |A^\kappa(v^\kappa) - A^\kappa(u^\kappa)| \, \Delta_x \phi \Big) \, dt \, dx \\ &= - \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \big|\nabla_x A^\kappa(v^\kappa)\big|^2 \operatorname{sign}'_\varepsilon(A^\kappa(v^\kappa) - A^\kappa(u^\kappa)) \phi \, dt \, dx \\ &\quad + \iint_{\Pi_T} \operatorname{sign}(v^\kappa - u^\kappa) g^\kappa(V) \phi \, dt \, dx. \end{aligned} \tag{3.8}$$

Integrating over the additional variables (y,s) in (3.7) and (3.8) as well as using Lebesgue's dominated convergence theorem, we find

$$\begin{aligned} &- \iiiint_{\Pi_T \times \Pi_T} \Big(|v^\kappa - u^\kappa| \phi_t + \operatorname{sign}(v^\kappa - u^\kappa) \big[F^\kappa(v^\kappa) - F^\kappa(u^\kappa)\big] \cdot \nabla_x \phi \\ &\quad + |A^\kappa(v^\kappa) - A^\kappa(u^\kappa)| \, \Delta_x \phi \Big) \, dt \, dx \, ds \, dy \\ &= - \iiiint_{\mathcal{E}_u^\kappa \times \Pi_T} \Big(|v^\kappa - u^\kappa| \phi_t + \operatorname{sign}(v^\kappa - u^\kappa) \big[F^\kappa(v^\kappa) - F^\kappa(u^\kappa)\big] \cdot \nabla_x \phi \\ &\quad + |A^\kappa(v^\kappa) - A^\kappa(u^\kappa)| \, \Delta_x \phi \Big) \, dt \, dx \, ds \, dy \\ &\quad - \iiiint_{(\Pi_T \setminus \mathcal{E}_u^\kappa) \times \Pi_T} \Big(|v^\kappa - u^\kappa| \phi_t + \operatorname{sign}(v^\kappa - u^\kappa) \big[F^\kappa(v^\kappa) - F^\kappa(u^\kappa)\big] \cdot \nabla_x \phi \\ &\quad + |A^\kappa(v^\kappa) - A^\kappa(u^\kappa)| \, \Delta_x \phi \Big) \, dt \, dx \, ds \, dy \\ &\leq \iint_{\Pi_T \setminus \mathcal{E}_u^\kappa} \Big(- \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \big|\nabla_x A^\kappa(v^\kappa)\big|^2 \operatorname{sign}'_\varepsilon(A^\kappa(v^\kappa) - A^\kappa(u^\kappa)) \phi \, dt \, dx \Big) \, ds \, dy \end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&+\iiiint_{\Pi_T\times\Pi_T}\operatorname{sign}(v^\kappa-u^\kappa)g^\kappa(V)\phi\,dt\,dx\,ds\,dy\\
&=-\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}_u^\kappa)\times\Pi_T}\left|\nabla_x A^\kappa(v^\kappa)\right|^2\operatorname{sign}_\varepsilon'(A^\kappa(v^\kappa)-A^\kappa(u^\kappa))\phi\,dt\,dx\,ds\,dy\\
&\quad+\iiiint_{\Pi_T\times\Pi_T}\operatorname{sign}(v^\kappa-u^\kappa)g^\kappa(V)\phi\,dt\,dx\,ds\,dy\\
&=-\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}_u^\kappa)\times(\Pi_T\setminus\mathcal{E}_v^\kappa)}\left|\nabla_x A^\kappa(v^\kappa)\right|^2\operatorname{sign}_\varepsilon'(A^\kappa(v^\kappa)-A^\kappa(u^\kappa))\phi\,dt\,dx\,ds\,dy\\
&\quad+\iiiint_{\Pi_T\times\Pi_T}\operatorname{sign}(v^\kappa-u^\kappa)g^\kappa(V)\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

Similarly, we derive the inequality

$$
\begin{aligned}
&-\iiiint_{\Pi_T\times\Pi_T}\Big(|u^\kappa-v^\kappa|\phi_s+\operatorname{sign}(u^\kappa-v^\kappa)\big[F^\kappa(u^\kappa)-F^\kappa(v^\kappa)\big]\cdot\nabla_y\phi\\
&\quad+|A^\kappa(u^\kappa)-A^\kappa(v^\kappa)|\,\Delta_y\phi\Big)\,dt\,dx\,ds\,dy \qquad (3.10)\\
&\le-\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}_u^\kappa)\times(\Pi_T\setminus\mathcal{E}_v^\kappa)}\left|\nabla_y A^\kappa(u^\kappa)\right|^2\operatorname{sign}_\varepsilon'(A^\kappa(u^\kappa)-A^\kappa(v^\kappa))\phi\,dt\,dx\,ds\,dy.\\
&\quad+\iiiint_{\Pi_T\times\Pi_T}\operatorname{sign}(u^\kappa-v^\kappa)g^\kappa(U)\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

By adding (3.9) and (3.10), we get

$$
\begin{aligned}
&-\iiiint_{\Pi_T\times\Pi_T}\Big(|v^\kappa-u^\kappa|\,(\partial_t+\partial_s)\,\phi+\operatorname{sign}(v^\kappa-u^\kappa) \qquad (3.11)\\
&\times\big[F^\kappa(v^\kappa)-F^\kappa(u^\kappa)\big]\cdot(\nabla_x+\nabla_y)\,\phi+|A^\kappa(v^\kappa)-A^\kappa(u^\kappa)|\,(\Delta_x+\Delta_y)\,\phi\Big)\,dt\,dx\,ds\,dy\\
&\le-\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}_u^\kappa)\times(\Pi_T\setminus\mathcal{E}_v^\kappa)}\Big(\left|\nabla_x A^\kappa(v^\kappa)\right|^2+\left|\nabla_y A^\kappa(u^\kappa)\right|^2\Big)\operatorname{sign}_\varepsilon'(A^\kappa(v^\kappa)\\
&-A(u^\kappa))\phi\,dt\,dx\,ds\,dy+\iiiint_{\Pi_T\times\Pi_T}\operatorname{sign}(v^\kappa-u^\kappa)\big(g^\kappa(V)-g^\kappa(U)\big)\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

We now use (3.11) to prove that for any non-negative test function $\phi(x,t)\in C_0^\infty(\Pi_T)$,

$$
\begin{aligned}
&-\iint_{\Pi_T}\Big(|v^\kappa-u^\kappa|\phi_t+\operatorname{sign}(v^\kappa-u^\kappa)\big[F^\kappa(v^\kappa)-F^\kappa(u^\kappa)\big]\cdot\nabla\phi\\
&\quad+\big|A^\kappa(v^\kappa)-A^\kappa(u^\kappa)\big|\Delta\phi\Big)\,dt\,dx \qquad (3.12)\\
&\le\iint_{\Pi_T}\operatorname{sign}(v^\kappa-u^\kappa)\,(g^\kappa(V)-g^\kappa(U))\,\phi\,dt\,dx,
\end{aligned}
$$

where $v^\kappa=v^\kappa(x,t)$, $u^\kappa=u^\kappa(x,t)$, $\kappa=1,\dots,K$.

Following Kružkov [33], we introduce a non-negative function $\delta\in C_0^\infty$, satisfying $\delta(\sigma)=\delta(-\sigma)$, $\delta(\sigma)=0$ for $|\sigma|\ge 1$, and $\int_{\mathbb{R}}\delta(\sigma)\,d\sigma=1$. For $\rho>0$ and $t\in\mathbb{R}$, let $\delta_\rho(t)=\frac{1}{\rho}\delta\big(\frac{t}{\rho}\big)$. For $\rho>0$ and $x\in\mathbb{R}^d$, let $\omega_\rho(x)=\frac{1}{\rho}\delta\big(\frac{x_1}{\rho}\big)\dots\frac{1}{\rho}\delta\big(\frac{x_d}{\rho}\big)$. We take $\phi=\phi(x,t,y,s)\in C_0^\infty(\Pi_T\times\Pi_T)$ to be

$$
\phi(x,t,y,s)=\psi\big(\frac{x+y}{2},\frac{t+s}{2}\big)\omega_\rho\big(\frac{x-y}{2}\big)\delta_\rho\big(\frac{t-s}{2}\big), \qquad (3.13)
$$

where $\psi = \psi(x,t) \in C_0^\infty(\Pi_T)$ is another non-negative test function. Observe that

$$(\partial_t + \partial_s)\,\delta_\rho\big(\frac{t-s}{2}\big) = 0, \quad \nabla_{x+y}\omega_\rho\big(\frac{x-y}{2}\big) = 0, \quad \Delta_{xy}\omega_\rho\big(\frac{x-y}{2}\big) = 0,$$

where we have introduced the operators

$$\nabla_{x+y} := \nabla_x + \nabla_y, \quad \Delta_{xy} := \Delta_x + 2\nabla_x \cdot \nabla_y + \Delta_y.$$

After tedious but straightforward computations, we find that

$$\begin{aligned}
(\partial_t + \partial_s)\phi(x,t,y,s) &= \Big[(\partial_t + \partial_s)\psi\big(\frac{x+y}{2}, \frac{t+s}{2}\big)\Big]\omega_\rho\big(\frac{x-y}{2}\big)\delta_\rho\big(\frac{t-s}{2}\big),\\
\nabla_{x+y}\phi(x,t,y,s) &= \Big[\nabla_{x+y}\psi\big(\frac{x+y}{2}, \frac{t+s}{2}\big)\Big]\omega_\rho\big(\frac{x-y}{2}\big)\delta_\rho\big(\frac{t-s}{2}\big),\\
\Delta_{xy}\phi(x,t,y,s) &= \Big[\Delta_{xy}\psi\big(\frac{x+y}{2}, \frac{t+s}{2}\big)\Big]\omega_\rho\big(\frac{x-y}{2}\big)\delta_\rho\big(\frac{t-s}{2}\big).
\end{aligned} \tag{3.14}$$

Inserting (3.13) into (3.11) and then using (3.14), we get

$$\begin{aligned}
&-\iiiint_{\Pi_T\times\Pi_T} \Big(\overline{I}^\kappa_{\rm time}(x,t,y,s) + \overline{I}^\kappa_{\rm conv}(x,t,y,s) + \overline{I}^\kappa_{\rm diff}(x,t,y,s)\Big)\\
&\quad\times \omega_\rho\big(\frac{x-y}{2}\big)\delta_\rho\big(\frac{t-s}{2}\big)\,dt\,dx\,ds\,dy\\
&+\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}^\kappa_u)\times(\Pi_T\setminus\mathcal{E}^\kappa_v)} \Big(\big|\nabla_x A^\kappa(v^\kappa)\big|^2 + \big|\nabla_y A^\kappa(u^\kappa)\big|^2\Big)\,\mathrm{sign}'_\varepsilon(A^\kappa(v^\kappa)\\
&- A^\kappa(u^\kappa))\phi\,dt\,dx\,ds\,dy\\
&+\iiiint_{\Pi_T\times\Pi_T} \overline{I}^\kappa_{\rm xy}(x,t,y,s)\,dt\,dx\,ds\,dy\\
&\le \iiiint_{\Pi_T\times\Pi_T} \overline{I}^\kappa_{\rm sour}(x,t,y,s)\omega_\rho\big(\frac{x-y}{2}\big)\delta_\rho\big(\frac{t-s}{2}\big)\,dt\,dx\,ds\,dy,
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
\overline{I}^\kappa_{\rm diff}(x,t,y,s) &= \big|A^\kappa(v^\kappa(x,t)) - A^\kappa(u^\kappa(y,s))\big|\,\Delta_{xy}\psi\big(\frac{x+y}{2}, \frac{t+s}{2}\big),\\
\overline{I}^\kappa_{\rm xy}(x,t,y,s) &= 2\,\big|A^\kappa(v^\kappa(x,t)) - A^\kappa(u^\kappa(y,s))\big|\,\nabla_x\cdot\nabla_y\phi(t,x,y,s),\\
\overline{I}^\kappa_{\rm time}(x,t,y,s) &= |v^\kappa(x,t) - u^\kappa(y,s)|\big(\partial_t + \partial_s\big)\psi\big(\frac{x+y}{2}, \frac{t+s}{2}\big),\\
\overline{I}^\kappa_{\rm conv}(x,t,y,s) &= \mathrm{sign}(v^\kappa(x,t) - u^\kappa(y,s))\Big[F^\kappa(v^\kappa(x,t)) - F^\kappa(u^\kappa(y,s))\Big]\\
&\qquad\cdot\nabla_{x+y}\psi\big(\frac{x+y}{2}, \frac{t+s}{2}\big),\\
\overline{I}^\kappa_{\rm sour}(x,t,y,s) &= \mathrm{sign}(v^\kappa(x,t) - u^\kappa(y,s))\Big[g^\kappa(V(x,t)) - g^\kappa(U(y,s))\Big]\\
&\qquad\times\psi\big(\frac{x+y}{2}, \frac{t+s}{2}\big).
\end{aligned}$$

Observe that repeated integration by parts gives

$$\begin{aligned}
&-\iiiint_{\Pi_T\times\Pi_T} \overline{I}^\kappa_{\rm xy}(x,t,y,s)\,dt\,dx\,ds\,dy\\
&= -\lim_{\varepsilon\downarrow 0}\iiiint_{\Pi_T\times\Pi_T} 2\Big(\int_{u^\kappa}^{v^\kappa} \mathrm{sign}_\varepsilon(A^\kappa(\xi) - A^\kappa(v^\kappa))\,d\xi\Big)\nabla_x\cdot\nabla_y\phi\,dt\,dx\,ds\,dy
\end{aligned}$$

$$= \lim_{\varepsilon\downarrow 0} \iiiint_{\Pi_T\times\Pi_T} 2\nabla_x A^\kappa(v^\kappa)\cdot\nabla_y A^\kappa(u^\kappa)\,\mathrm{sign}'_\varepsilon(A^\kappa(v^\kappa)-A^\kappa(u^\kappa))\phi\,dt\,dx\,ds\,dy$$
$$= \lim_{\varepsilon\downarrow 0} \iiiint_{(\Pi_T\setminus\mathcal{E}_u^\kappa)\times(\Pi_T\setminus\mathcal{E}_v^\kappa)} 2\nabla_x A^\kappa(v^\kappa)\cdot\nabla_y A^\kappa(u^\kappa)\,\mathrm{sign}'_\varepsilon(A^\kappa(v^\kappa)$$
$$- A^\kappa(u^\kappa))\phi\,dt\,dx\,ds\,dy.$$

Now since

$$\left|\nabla_x A^\kappa(v^\kappa)\right|^2 - 2\nabla_x A^\kappa(v^\kappa)\cdot\nabla_y A^\kappa(u^\kappa) + \left|\nabla_y A^\kappa(u^\kappa)\right|^2$$
$$= \left|\nabla_x A^\kappa(v^\kappa) - \nabla_y A^\kappa(u^\kappa)\right|^2 \geq 0,$$

it follows from (3.15) that

$$\begin{aligned} &-\iiiint_{\Pi_T\times\Pi_T}\Big(\overline{I}^\kappa_{\mathrm{time}}(x,t,y,s)+\overline{I}^\kappa_{\mathrm{conv}}(x,t,y,s)+\overline{I}^\kappa_{\mathrm{diff}}(x,t,y,s)\Big)\\ &\times\omega_\rho\big(\frac{x-y}{2}\big)\delta_\rho\big(\frac{t-s}{2}\big)\,dt\,dx\,ds\,dy \qquad (3.16)\\ &\leq \iiiint_{\Pi_T\times\Pi_T}\overline{I}^\kappa_{\mathrm{sour}}(x,t,y,s)\omega_\rho\left(\frac{x-y}{2}\right)\delta_\rho\big(\frac{t-s}{2}\big)\,dt\,dx\,ds\,dy. \end{aligned}$$

Let us introduce the change of variables

$$\tilde{x}=\frac{x+y}{2},\quad \tilde{t}=\frac{t+s}{2},\quad z=\frac{x-y}{2},\quad \tau=\frac{t-s}{2},$$

which maps $\Pi_T\times\Pi_T$ into

$$\Omega=\mathbb{R}^d\times\mathbb{R}^d\times\left\{(\tilde{t},\tau): 0\leq\tilde{t}+\tau\leq T,\ 0\leq\tilde{t}-\tau\leq T\right\}.$$

As usual with this change of variables, see, e.g., [33],

$$(\partial_t+\partial_s)\,\psi\big(\frac{x+y}{2},\frac{t+s}{2}\big)=\psi_{\tilde{t}}(\tilde{x},\tilde{t}),\quad \nabla_{x+y}\phi(x,t,y,s)=\nabla_{\tilde{x}}\psi(\tilde{x},\tilde{t}).$$

But in addition it has the wonderful property of completely diagonalizing the operator Δ_{xy}:

$$\Delta_{xy}\psi\big(\frac{x+y}{2},\frac{t+s}{2}\big)=\Delta_{\tilde{x}}\psi(\tilde{x},\tilde{t}).$$

Keeping in mind that $x=\tilde{x}+z$, $y=\tilde{x}-z$, $t=\tilde{t}+\tau$, $s=\tilde{t}-\tau$. We may now write (3.16) as

$$\begin{aligned} &-\iiiint_\Omega\Big(I^\kappa_{\mathrm{time}}(\tilde{x},\tilde{t},z,\tau)+I^\kappa_{\mathrm{conv}}(\tilde{x},\tilde{t},z,\tau)-I^\kappa_{\mathrm{diff}}(\tilde{x},\tilde{t},z,\tau)\Big)\omega_\rho(z)\delta_\rho(\tau)\,d\tilde{t}\,d\tilde{x}\,d\tau\,dz\\ &\leq\iiiint_\Omega I^\kappa_{\mathrm{sour}}(\tilde{x},\tilde{t},z,\tau)\omega_\rho(z)\delta_\rho(\tau)\,d\tilde{t}\,d\tilde{x}\,d\tau\,dz, \qquad (3.17)\end{aligned}$$

where

$$\begin{aligned} I^\kappa_{\mathrm{diff}}(\tilde{x},\tilde{t},z,\tau)&=\left|A^\kappa(v^\kappa(\tilde{x}+z,\tilde{t}+\tau))-A^\kappa(u^\kappa(\tilde{x}-z,\tilde{t}-\tau))\right|\Delta_{\tilde{x}}\psi(\tilde{x},\tilde{t}),\\ I^\kappa_{\mathrm{time}}(\tilde{x},\tilde{t},z,\tau)&=|v^\kappa(\tilde{x}+z,\tilde{t}+\tau)-u^\kappa(\tilde{x}-z,\tilde{t}-\tau)|\psi_{\tilde{t}}(\tilde{x},\tilde{t}),\\ I^\kappa_{\mathrm{conv}}(\tilde{x},\tilde{t},z,\tau)&=\mathrm{sign}(v^\kappa(\tilde{x}+z,\tilde{t}+\tau)-u^\kappa(\tilde{x}-z,\tilde{t}-\tau))\\ &\quad\times\Big[F^\kappa(v^\kappa(\tilde{x}+z,\tilde{t}+\tau))-F^\kappa(u^\kappa(\tilde{x}-z,\tilde{t}-\tau))\Big]\cdot\nabla_{\tilde{x}}\psi(\tilde{x},\tilde{t}),\\ I^\kappa_{\mathrm{sour}}(\tilde{x},\tilde{t},z,\tau)&=\mathrm{sign}(v^\kappa(\tilde{x}+z,\tilde{t}+\tau)-u^\kappa(\tilde{x}-z,\tilde{t}-\tau))\\ &\quad\times\Big[g^\kappa(V(\tilde{x}+z,\tilde{t}+\tau))-g^\kappa(U(\tilde{x}-z,\tilde{t}-\tau))\Big]\psi(\tilde{x},\tilde{t}).\end{aligned}$$

After the work of Kružkov [33], it is a routine exercise to use Lebesgue's differentiation theorem to pass to the limit in (3.17) as $\rho \downarrow 0$ to obtain (3.12) (with ψ rather than ϕ).

Equipped with (3.12), we can now conclude the proof of the theorem. Pick two (arbitrary but fixed) Lebesgue points $t_1, t_2 \in (0,T)$ of $\|v^\kappa(\cdot,t) - u^\kappa(\cdot,t)\|_{L^1(\mathbb{R}^d)}$, $\kappa = 1, \dots, K$. For any $\nu \in \big(0, \min(t_1, T-t_2)\big)$, let

$$\chi_\nu(t) = H_\nu(t-t_1) - H_\nu(t-t_2), \quad H_\nu(t) = \int_{-\infty}^{t} \delta_\nu(\xi)\, d\xi.$$

Notice that $\chi_\nu'(t) = \delta_\nu(t-t_1) - \delta_\nu(t-t_2)$. Pick a function $\psi \in C_0^\infty(\mathbb{R}^d)$ such that

$$\psi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2, \end{cases}$$

and $0 \le \psi \le 1$ when $1 < |x| < 2$. Let $\varphi_r(x) = \psi\big(\frac{x}{r}\big)$, for $r \ge 1$. We then take the test function ϕ in (3.12) to be of the form

$$\phi(x,t,y,s) = \chi_\nu(t)\varphi_r(x). \tag{3.18}$$

Since $v^\kappa, u^\kappa \in L^1(\Pi_T)$, we obviously have that

$$\iint_{\Pi_T} \Big(\operatorname{sign}(v^\kappa - u^\kappa)\big[F^\kappa(v^\kappa) - F^\kappa(u^\kappa)\big] \cdot \nabla\varphi_r + |A^\kappa(v^\kappa) - A^\kappa(u^\kappa)|\, \Delta\varphi_r \Big)\, dt\, dx$$

approaches zero as $r \uparrow \infty$. Consequently, sending $r \uparrow \infty$ in (3.12) yields

$$\begin{aligned} &- \iint_{\Pi_T} |v^\kappa(x,t) - u^\kappa(x,t)|\, \chi_\nu'(t)\, dt\, dx \\ &\le |g^\kappa|_{\mathrm{Lip}} \sum_{\ell=1}^{K} \iint_{\Pi_T} \big|v^\ell(x,t) - u^\ell(x,t)\big|\, \chi_\nu(t)\, dt\, dx. \end{aligned} \tag{3.19}$$

Summing (3.19) over κ we find

$$\begin{aligned} &- \sum_{\kappa=1}^{K} \iint_{\Pi_T} |v^\kappa(x,t) - u^\kappa(x,t)|\, \chi_\nu'(t)\, dt\, dx \\ &\le C \sum_{\kappa=1}^{K} \iint_{\Pi_T} |v^\kappa(x,t) - u^\kappa(x,t)|\, \chi_\nu(t)\, dt\, dx, \end{aligned} \tag{3.20}$$

where $C := K \max_\kappa \big(|g^\kappa|_{\mathrm{Lip}} \big)$. Sending $\nu \downarrow 0$ in (3.20), we get

$$\begin{aligned} &\sum_{\kappa=1}^{K} \int_{\mathbb{R}^d} |v^\kappa(x,t_2) - u^\kappa(x,t_2)|\, dx \\ &\le \sum_{\kappa=1}^{K} \int_{\mathbb{R}^d} |v^\kappa(x,t_1) - u^\kappa(x,t_1)|\, dx + C \int_{t_1}^{t_2} \Big(\sum_{\kappa=1}^{K} \int_{\mathbb{R}^d} |v^\kappa(x,t) - u^\kappa(x,t)|\, dx \Big)\, dt. \end{aligned}$$

An application of Gronwall's inequality now gives

$$\begin{aligned} &\sum_{\kappa=1}^{K} \int_{\mathbb{R}^d} |v^\kappa(x,t_2) - u^\kappa(x,t_2)|\, dx \\ &\le \exp\big(C(t_2 - t_1)\big) \sum_{\kappa=1}^{K} \int_{\mathbb{R}^d} |v^\kappa(x,t_1) - u^\kappa(x,t_1)|\, dx. \end{aligned}$$

$$\xrightarrow{t_1\downarrow 0} \exp\big(Ct_2\big)\sum_{\kappa=1}^{K}\int_{\mathbb{R}^d}|v^\kappa(x,0)-u^\kappa(x,0)|\,dx.$$

By using the inequality $\sum_\kappa |v^\kappa - u^\kappa| \le \sqrt{K}\|V-U\|$ and since t_2 is an arbitrary Lebesgue point, the theorem is proved. □

Remark 3.4. We note that the proof of Theorem 3.3 is slightly different from the corresponding proof in [9]. We here work with the term

$$|A^\kappa(v^\kappa) - A^\kappa(u^\kappa)|\,(\Delta_x\phi + \Delta_y\phi)\,,$$

and exploits fully the identity

$$\Delta_x\Phi(x-y) + 2\nabla_x\cdot\nabla_y\Phi(x-y) + \Delta_y\Phi(x-y) = 0, \tag{3.21}$$

which holds for any (smooth) function $\Phi\colon \mathbb{R}\to\mathbb{R}$. In [9], the author works instead with the term

$$\operatorname{sign}(v^\kappa - u^\kappa)\Big(\nabla_x A^\kappa(v^\kappa)\cdot\nabla_x\phi - \nabla_y A^\kappa(u^\kappa)\cdot\nabla_y\phi\Big),$$

and exploits eventually the usual "Kružkov identity"

$$\nabla_x\Phi(x-y) + \nabla_y\Phi(x-y) = 0.$$

The interested reader is hereby invited to have a look at Ishii's paper [27] to see how the identity (3.21) (implicitly) plays a central role in the uniqueness proof for *viscosity* solutions of degenerate second-order partial differential equations.

Remark 3.5. Following [30] and [28], one can prove that Theorem 3.3 holds for more general systems of the type

$$u_t^\kappa + \operatorname{div} F^\kappa(x,t,u^\kappa) = \Delta_x\big(K^\kappa(x,t)A^\kappa(u^\kappa)\big) + g^\kappa(x,t,U), \quad \kappa = 1,\dots,K,$$

where $F^\kappa, K^\kappa, A^\kappa, g^\kappa$ satisfy the same assumptions as in [30, 28]. In particular, K^κ is a diagonal matrix that needs to be bounded away from zero a.e.

4. Existence of Entropy Solution

In this section, we prove existence of an entropy solution by establishing convergence of certain finite difference approximations. To this end, we shall assume that u_0 belongs to $L^1(\mathbb{R}^d)\cap BV(\mathbb{R}^d)$ and has compact support, the latter implies that all subsequent sums over I are finite. Furthermore, we shall assume that F and A are C^1. These additional assumptions on u_0, F, A will be removed towards the end of this section (see the proof of Theorem 4.9).

Let $I=(i_1,\dots,i_d)\in\mathbb{Z}^d$ be a multi-index and let $e_i\in\mathbb{Z}^d$ the multi-index with with zeros everywhere except for a 1 at the ith place. Selecting a mesh size $h>0$, a time step $\Delta t>0$, and integer N such that $N\Delta t = T$, the value of our finite difference approximation of u^κ at the point $(x_I,t_n)=(hI,n\Delta t)$, with $I\in\mathbb{Z}^d$ and $n=0,\dots,N$, will be denoted by $u_I^{\kappa,n}$ for $\kappa=1,\dots,K$. Sometimes we write U_I^n for the vector $\big(u_I^{1,n},\dots,u_I^{K,n}\big)$. To simplify the notation, we introduce the (backward and forward) finite difference operators

$$D_{i,-}u_I^{\kappa,n} = \frac{1}{h}\big(u_I^{\kappa,n} - u_{I-e_i}^{\kappa,n}\big), \quad D_{i,+}u_I^{\kappa,n} = \frac{1}{h}\big(u_{I+e_i}^{\kappa,n} - u_I^{\kappa,n}\big), \quad i=1,\dots,d.$$

As already mentioned in the introduction, we shall analyze the Engquist–Osher (generalized upwind) scheme. For a scalar flux function $F_i^\kappa(u)$, the associated Engquist–Osher numerical flux function [17] can be written as

$$F_i^{\kappa,\mathrm{EO}}(u,v) = \frac{1}{2}\big(F_i^\kappa(u) + F_i^\kappa(v)\big) - \frac{1}{2}\int_u^v \big|\frac{dF_i^\kappa}{dr}(r)\big|\,dr, \tag{4.1}$$

which is Lipschitz (actually C^1) in both variables with (common) Lipschitz constant $|F_i^\kappa|_{\mathrm{Lip}}$. We may write

$$F_i^{\kappa,\mathrm{EO}}(u,v) = F_i^{\kappa,+}(u) + F_i^{\kappa,-}(v), \tag{4.2}$$

where (recall that $F_i^\kappa(0)=0$)

$$F_i^{\kappa,+}(u) = \int_0^u \Big(\frac{dF_i^\kappa}{dr}(r)\vee 0\Big)\,dr, \quad F_i^{\kappa,-}(v) = \int_0^v \Big(\frac{dF_i^\kappa}{dr}(r)\wedge 0\Big)\,dr.$$

Remark 4.1. *For a monotone flux function F_i^κ, the Engquist-Osher flux reduces to the upwind flux, i.e.,*

$$F_i^{\kappa,\mathrm{EO}}(u,v) = F_i^\kappa(u) \quad \text{if } \frac{dF_i^\kappa}{dr}\ge 0, \quad F_i^{\kappa,\mathrm{EO}}(u,v) = F_i^\kappa(v) \quad \text{if } \frac{dF_i^\kappa}{dr}<0.$$

The Engquist-Osher finite difference scheme now takes the form

$$\frac{u_I^{\kappa,n+1}-u_I^{\kappa,n}}{\Delta t} + \sum_{i=1}^d D_{i,-}\Big(F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n}\right) - D_{i,+}A^\kappa\left(u_I^{\kappa,n}\right)\Big) = g^\kappa\left(U_I^n\right), \tag{4.3}$$

for $\kappa=1,\dots,K$. Letting $\lambda=\frac{\Delta t}{h}$ and $\mu=\frac{\Delta t}{h^2}$, we assume hereafter that the following CFL condition holds:

$$\mathrm{CFL}^\kappa := \lambda\sum_{i=1}^d \big\|\frac{dF_i^\kappa}{du}\big\|_\infty + 2\mu d\big\|\frac{dA^\kappa}{du}\big\|_\infty \le 1, \quad \kappa=1,\dots,K. \tag{4.4}$$

For later use, we note that we may write the finite difference scheme (4.3) as

$$\begin{aligned} &\frac{\overline{u}_I^{\kappa,n+1}-u_I^{\kappa,n}}{\Delta t} + \sum_{i=1}^d D_{i,-}\Big(F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n}\right) - D_{i,+}A^\kappa\left(u_I^{\kappa,n}\right)\Big) = 0,\\ &u_I^{\kappa,n+1} = \overline{u}_I^{\kappa,n+1} + \Delta t g^\kappa\left(U_I^n\right), \quad \kappa=1,\dots,K. \end{aligned} \tag{4.5}$$

Sometimes we will write $\overline{U}_I^n$ for the vector $\big(\overline{u}_I^{1,n},\dots,\overline{u}_I^{K,n}\big)$. The approximate solution $U_h=(u_h^1,\dots,u_h^K)$ is then defined as

$$U_h(x,t) = U_I^n, \quad \text{for } (x,t)\in\chi_I\times[t_n,t_{n+1}), \tag{4.6}$$

where χ_I denotes the set

$$\chi_I = \{x\in\mathbb{R}^d : h(i_j-1/2)\le x_j < h(i_j+1/2),\ j=1,\dots,d\}, \quad I=(i_1,\dots,i_d). \tag{4.7}$$

We initialize the scheme by setting

$$U_I^0 = \frac{1}{|\chi_I|}\int_{\chi_I} U_0(x)\,dx = h^{-d}\int_{\chi_I} U_0(x)\,dx. \tag{4.8}$$

Our first lemma provides uniform L^1, L^∞, BV estimates for U_h.

Lemma 4.2. *There exists a constant C, independent of h, such that $t\in(0,T)$*

$$\|U_h(\cdot,t)\|_{L^1(\mathbb{R}^d)}\le C, \quad \|U_h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}\le C, \quad |U_h(\cdot,t)|_{BV(\mathbb{R}^d)}\le C.$$

Proof. First note that we can write $\overline{u}_I^{\kappa,n+1} = \mathcal{S}^\kappa(u^{\kappa,n}; I)$, where $\mathcal{S}^\kappa\colon L^1(\mathbb{Z}^d) \to L^1(\mathbb{Z}^d)$ maps the sequence $u^{\kappa,n} = \{u_I^{\kappa,n}\}_I$ according to the formula

$$\mathcal{S}^\kappa(u^{\kappa,n}; I) = u_I^{\kappa,n} - \Delta t \sum_{i=1}^d D_{i,-}\Big(F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n}\right) - D_{i,+}A^\kappa\left(u_I^{\kappa,n}\right)\Big).$$

An easy exercise will reveal that the CFL condition (4.4) implies that $\mathcal{S}^\kappa(\cdot)$ is a monotone function of all its arguments. Since the difference approximation has compact support, we get $\sum_I \mathcal{S}^\kappa(u^{\kappa,n}; I) = \sum_I u_I^{\kappa,n}$. Since $\mathcal{S}^\kappa$ is monotone and obviously commutes with spatial translations, it follows from Lemma 2.2 that

$$\left\|\overline{u}^{\kappa,n+1}\right\|_{L^1(\mathbb{Z}^d)} \le \left\|u^{\kappa,n}\right\|_{L^1(\mathbb{Z}^d)}. \tag{4.9}$$

For a grid function $u = \{u_I\}_I$, we recall that the L^p norms are defined as

$$\|u\|_{L^p(\mathbb{Z}^d)}^p = \sum_I |u_I|^p, \quad p < \infty, \quad \|u\|_{L^\infty(\mathbb{Z}^d)} = \sup_I |u_I|.$$

Furthermore, using a standard argument, the inequality (4.9) is also valid with $L^1(\mathbb{Z}^d)$ replaced by $L^\infty(\mathbb{Z}^d)$. For completeness we repeat the argument here in the case $d = 1$. Rewriting (4.5) and using (4.2) we find

$$\begin{aligned}
\overline{u}_I^{\kappa,n+1} &= u_I^{\kappa,n} - \lambda\big(F^{\kappa,+}(u_I^{\kappa,n}) - F^{\kappa,+}(u_{I-1}^{\kappa,n}) + F^{\kappa,-}(u_{I+1}^{\kappa,n}) - F^{\kappa,-}(u_I^{\kappa,n})\big) \\
&\quad + \mu\big(A^\kappa(u_{I+1}^{\kappa,n}) + A^\kappa(u_{I-1}^{\kappa,n}) - 2A^\kappa(u_I^{\kappa,n})\big) \\
&= \big(1 - \mu d A_I^\kappa - \mu d A_{I+1}^\kappa - \lambda d F_I^{\kappa,+} + \lambda d F_I^{\kappa,-}\big)u_I^{\kappa,n} \\
&\quad + \big(\mu d A_I^\kappa + \lambda d F_I^{\kappa,+}\big)u_{I-1}^{\kappa,n} + \big(\mu d A_{I+1}^\kappa - \lambda d F_I^{\kappa,-}\big)u_{I+1}^{\kappa,n}.
\end{aligned} \tag{4.10}$$

Here the quantities dA_I^κ, dA_{I+1}^κ, $dF_I^{\kappa,\pm}$ denote derivatives of A^κ and $F^{\kappa,\pm}$, respectively, evaluated at points between $u_I^{\kappa,n}$ and $u_{I\pm1}^{\kappa,n}$ using the mean value theorem. Applying the CFL condition (4.4) we see that

$$\begin{aligned}
\left|\overline{u}_I^{\kappa,n+1}\right| &\le \left(1 - \mu d A_I^\kappa - \mu d A_{I+1}^\kappa - \lambda d F_I^{\kappa,+} + \lambda d F_I^{\kappa,-}\right)\left|u_I^{\kappa,n}\right| \\
&\quad + \left(\mu d A_I^\kappa + \lambda d F_I^{\kappa,+}\right)\left|u_{I-1}^{\kappa,n}\right| + \left(\mu d A_{I+1}^\kappa - \lambda d F_I^{\kappa,-}\right)\left|u_{I+1}^{\kappa,n}\right| \\
&\le \left\|u^{\kappa,n}\right\|_{L^\infty(\mathbb{Z}^d)}.
\end{aligned}$$

Thus we have shown

$$\|\overline{u}^{\kappa,n+1}\|_{L^\infty(\mathbb{Z}^d)} \le \|u^{\kappa,n}\|_{L^\infty(\mathbb{Z}^d)}. \tag{4.11}$$

Using (4.5), (4.9) and summing over $\kappa = 1, \dots, K$, we get

$$\sum_{\kappa=1}^K \left\|u^{\kappa,n+1}\right\|_{L^p(\mathbb{Z}^d)} \le \Big(1 + K \max_{\kappa=1,\dots,K}\big(|g^\kappa|_{\mathrm{Lip}}\big)\Delta t\Big)\sum_{\kappa=1}^K \|u^{\kappa,n}\|_{L^p(\mathbb{Z}^d)}, \quad p = 1, \infty,$$

from which it follows that

$$\begin{aligned}
\sum_{\kappa=1}^K \|u^{\kappa,n}\|_{L^p(\mathbb{Z}^d)} &\le \Big(1 + K \max_{\kappa=1,\dots,K}\big(|g^\kappa|_{\mathrm{Lip}}\big)\Delta t\Big)^n \sum_{\kappa=1}^K \left\|u^{\kappa,0}\right\|_{L^p(\mathbb{Z}^d)}, \\
&\le \exp\Big(\hat{C}t_n\Big)\sum_{\kappa=1}^K \left\|u^{\kappa,0}\right\|_{L^p(\mathbb{Z}^d)}, \quad n = 0, \dots, N, \quad p = 1, \infty,
\end{aligned}$$

for some constant $\hat{C}$ independent h.

Similarly, an application of Lemma 2.2 gives

$$\sum_I \big|\overline{u}_I^{\kappa,n+1} - \overline{u}_{I-e_i}^{\kappa,n+1}\big| \le \sum_I \big|u_I^{\kappa,n} - u_{I-e_i}^{\kappa,n}\big|.$$

Hence, from (4.5) and after summing over $\kappa = 1, \dots, K$, we get

$$\sum_{\kappa=1}^{K}\sum_I \big|u_I^{\kappa,n} - u_{I-e_i}^{\kappa,n}\big| \le \exp\big(\hat{C}t_n\big) \sum_{\kappa=1}^{K}\sum_I \big|u_I^{\kappa,0} - u_{I-e_i}^{\kappa,0}\big|, \quad n = 0, \dots, N.$$

This concludes the proof of the lemma. □

The next lemma shows that $U_h(\cdot,t)$ is L^1 Hölder continuous in time.

Lemma 4.3. *There exists a constant C, independent of h, such that*

$$\|U_h(\cdot, t+\tau) - U_h(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le C\sqrt{\tau}, \quad \tau \in (0,T) \text{ and } t \in (0, T-\tau).$$

Proof. Let $\phi = \phi(x)$ be a $C_0^\infty(\mathbb{R}^d)$ function and set $\phi_I = \phi(x_I)$. From (4.3) we get

$$\begin{aligned}
&\Big|h^d \sum_I \phi_I \big(u_I^{\kappa,n+1} - u_I^{\kappa,n}\big)\Big| \\
&\le \Big(\underbrace{h^d \sum_I \sum_{i=1}^{d} \big|\phi_I D_{i,-} F_i^{\kappa,\mathrm{EO}}(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n})\big|}_{B_1} \\
&\quad + \underbrace{h^d \sum_I \sum_{i=1}^{d} \big|D_{i,+}\phi_I D_{i,+} A^\kappa\left(u_I^{\kappa,n}\right)\big|}_{B_2} + \underbrace{h^d \sum_I \big|\phi_I g^\kappa\left(U_I^n\right)\big|}_{B_3} \Big)\Delta t.
\end{aligned}$$

Equipped with Lemma 4.2, we get the following estimates:

$$|B_1| \le 2 \max_{i,u} \Big|\frac{dF_i^\kappa}{du}(u)\Big| \|\phi\|_{L^\infty(\mathbb{R}^d)} h^d \sum_I \sum_{i=1}^{d} |D_{i,-} u_I^{\kappa,n}| \le C_1 \|\phi\|_{L^\infty(\mathbb{R}^d)},$$

$$|B_2| \le \max_{i,u} \Big|\frac{dA^\kappa}{du}(u)\Big| \max_i \|\phi_{x_i}\|_{L^\infty(\mathbb{R}^d)} h^d \sum_I \sum_{i=1}^{d} |D_{i,+} u_I^{\kappa,n}| \le C_2 \max_i \|\phi_{x_i}\|_{L^\infty(\mathbb{R}^d)},$$

$$|B_3| \le \|\phi\|_{L^\infty(\mathbb{R}^d)} \sum_{\kappa=1}^{K} \big(|g^\kappa|_{\mathrm{Lip}}\, h^d \sum_I |u_I^{\kappa,n}|\big) \le C_3 \|\phi\|_{L^\infty(\mathbb{R}^d)},$$

for some constant C_1, C_2, C_3 that are independent of h. From these estimates it now follows that

$$\big|h^d \sum_I \big(u_I^{\kappa,n+1} - u_I^{\kappa,n}\big)\phi_I\big| \le C_4 \Big(\|\phi\|_{L^\infty(\mathbb{R}^d)} + \max_i \|\phi_{x_i}\|_{L^\infty(\mathbb{R}^d)} \Big)\Delta t, \tag{4.12}$$

for some constant C_4 that is independent of h. Regarding u_h^κ, we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^d} \big(u_h^\kappa(x, t_{n+1}) - u_h^\kappa(x,t_n)\big)\phi(x)\, dx \Big| \\
&\le \Big| h^d \sum_I \big(u_I^{\kappa,n+1} - u_I^{\kappa,n}\big)\phi_I \Big| + \underbrace{\sum_I \big|u_I^{\kappa,n+1} - u_I^{\kappa,n}\big| \int_{\chi_I} |\phi(x) - \phi_I|\, dx}_{I_4},
\end{aligned} \tag{4.13}$$

where χ_I is defined in (4.7). Let us estimate the additional error term I_4. Using the finite difference scheme (4.3) and Lemma 4.2, we can do this as we did for B_1, B_2, B_3:

$$\begin{aligned}
|I_4| &\le \max_i \|\phi_{x_i}\|_{L^\infty(\mathbb{R}^d)} h^{d+1} \sum_I \left|u_I^{\kappa,n+1} - u_I^{\kappa,n}\right| \\
&\le \max_i \|\phi_{x_i}\|_{L^\infty(\mathbb{R}^d)} \Big(h^{d+1} \sum_I \sum_{i=1}^d \left| D_{i,-} F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n}\right)\right| \\
&\quad + h^d \sum_I \sum_{i=1}^d \left|D_{i,+} A^\kappa\left(u_I^{\kappa,n}\right)\right| + h^d \sum_I \sum_{i=1}^d \left|D_{i,+} A^\kappa\left(u_{I-e_i}^{\kappa,n}\right)\right| + h^{d+1} \sum_I |g^\kappa| \Big) \Delta t \\
&\le C_5 \max_i \|\phi_{x_i}\|_{L^\infty(\mathbb{R}^d)} \Delta t,
\end{aligned}$$

for some constant C_5 that is independent of h.

From this estimate as well as (4.13) and (4.12), we get

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^d} \Big(u_h^\kappa(x, t_{n+1}) - u_h^\kappa(x, t_n)\Big) \phi(x)\, dx \Big| \\
&\le C_6 \Big(\|\phi\|_{L^\infty(\mathbb{R}^d)} + \max_i \|\phi_{x_i}\|_{L^\infty(\mathbb{R}^d)} \Big) \Delta t, \quad C_6 := C_4 + C_5.
\end{aligned} \tag{4.14}$$

Let τ, t be as in the lemma. Using (4.14), it is not difficult to show that

$$\Big| \int_{\mathbb{R}^d} \Big(u_h^\kappa(t+\tau) - u_h^\kappa(t)\Big) \phi(x) \Big| \le C_7 \Big(\|\phi\|_{L^\infty(\mathbb{R}^d)} + \max_i \|\phi_{x_i}\|_{L^\infty(\mathbb{R}^d)} \Big) \tau, \tag{4.15}$$

for some constant C_7 that is independent of h. In view of (4.15), the lemma now follows from an application of Lemma 2.5. □

The next lemma provides us with a uniform L^2 space translation estimate for $A^\kappa(U_h)$.

Lemma 4.4. *There exists a constant C, independent of h, such that*

$$\|A^\kappa\left(U_h(\cdot + y, \cdot)\right) - A^\kappa\left(U_h(\cdot, \cdot)\right)\|_{L^2(\Pi_T)} \le C\sqrt{|y|\left(|y| + h\right)}, \quad y \in \mathbb{R}^d. \tag{4.16}$$

Proof. We shall derive a discrete energy estimate. Multiplying (4.3) by $\Delta t\, h^d\, u_I^{\kappa,n}$, summing over n, I, and then doing summation by parts in I, we find that

$$\begin{aligned}
&h^d \sum_{n,I} u_I^{\kappa,n} \left(u_I^{\kappa,n+1} - u_I^{\kappa,n}\right) + \Delta t\, h^d \sum_{n,I} \sum_{i=1}^d u_I^{\kappa,n} D_{i,-} F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n}\right) \\
&\quad + \Delta t\, h^d \sum_{n,I} \sum_{i=1}^d D_{i,+} u_I^{\kappa,n} D_{i,+} A^\kappa\left(u_I^{\kappa,n}\right) - \Delta t\, h^d \sum_{n,I} u_I^{\kappa,n} g^\kappa\left(U_I^n\right) = 0.
\end{aligned} \tag{4.17}$$

Observe that we can write

$$u_I^{\kappa,n}\big(u_I^{\kappa,n+1} - u_I^{\kappa,n}\big) = \frac{1}{2}\Big(\big(u_I^{\kappa,n+1}\big)^2 - \big(u_I^{\kappa,n}\big)^2 - \big(u_I^{\kappa,n+1} - u_I^{\kappa,n}\big)^2 \Big),$$

Assuming (without loss of generality) $\max_u dA^\kappa(u)/du > 0$ and since $dA^\kappa/du \ge 0$, we also have

$$\frac{1}{\max_u \frac{dA^\kappa}{du}(u)} \Big(D_{i,+} A^\kappa\left(u_I^{\kappa,n}\right)\Big)^2 \le D_{i,+} u_I^{\kappa,n} D_{i,+} A^\kappa\left(u_I^{\kappa,n}\right).$$

From these observations, we get from (4.17) that

$$
\begin{aligned}
&\frac{\Delta t\, h^d}{\max_u \frac{dA^\kappa}{du}(u)} \sum_{n,I} \sum_{i=1}^{d} \left(D_{i,+} A^\kappa\left(u_I^{\kappa,n}\right)\right)^2 \\
&\le -\frac{h^d}{2} \sum_{n,I} \left(\left(u_I^{\kappa,n+1}\right)^2 - \left(u_I^{\kappa,n}\right)^2 \right) + \frac{h^d}{2} \sum_{n,I} \left(u_I^{\kappa,n+1} - u_I^{\kappa,n}\right)^2 \\
&\quad - \Delta t\, h^d \sum_{n,I} \sum_{i=1}^{d} u_I^{\kappa,n} D_{i,-} F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n}\right) + \Delta t\, h^d \sum_{n,I} u_I^{\kappa,n} g^\kappa\left(U_I^n\right) \\
&= \frac{h^d}{2} \sum_{I} \left(\left(u_I^{\kappa,0}\right)^2 - \left(u_I^{\kappa,N}\right)^2 \right) + \frac{h^d}{2} \sum_{n,I} \left(u_I^{\kappa,n+1} - u_I^{\kappa,n}\right)^2 \\
&\quad - \Delta t\, h^d \sum_{n,I} \sum_{i=1}^{d} u_I^{\kappa,n} D_{i,-} F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n}\right) + \Delta t\, h^d \sum_{n,I} u_I^{\kappa,n} g^\kappa\left(U_I^n\right) \\
&\le C_1 + \frac{h^d}{2} \sum_{n,I} \left(u_I^{\kappa,n+1} - u_I^{\kappa,n}\right)^2 + 2 \max_{n,I} |u_I^{\kappa,n}| \\
&\quad \times \max_{i,u} \left|\frac{dF_i^\kappa}{du}(u)\right| \Delta t\, h^d \sum_{n,I} \sum_{i=1}^{d} |D_{i,-} u_I^{\kappa,n}| + \sum_{\kappa=1}^{K} \left(|g^\kappa|_{\mathrm{Lip}}\, \Delta t\, h^d \sum_{n,I} |u_I^{\kappa,n}| \right) \\
&\le \frac{h^d}{2} \sum_{n,I} \left(u_I^{\kappa,n+1} - u_I^{\kappa,n}\right)^2 + C_2,
\end{aligned}
\tag{4.18}
$$

for constants C_1, C_2 that are independent of h. To derive the last two inequalities, we used that the finite difference solution is uniformly bounded in the L^1, L^∞, and BV norms.

From (4.3) and the inequality $\left(\sum_{i=1}^r a_i\right)^2 \le c_r \sum_{i=1}^r (a_i)^2$ for any integer $r \ge 1$, we find that

$$
\begin{aligned}
\frac{1}{2}\left(u_I^{\kappa,n+1} - u_I^{\kappa,n}\right)^2 \le C_d \underbrace{\Delta t^2 \sum_{i=1}^{d} \left(D_{i,+} F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n}\right)\right)^2}_{B_1} \\
+ C_d \underbrace{\Delta t^2 \sum_{i=1}^{d} \left(D_{i,+} A^\kappa\left(u_I^{\kappa,n}\right)\right)^2}_{B_2} + C_d \underbrace{\Delta t^2 \left(g^\kappa\left(U_I^n\right)\right)^2}_{B_3},
\end{aligned}
\tag{4.19}
$$

for some constant C_d that is independent of h but it depends on d (the number of spatial dimensions). In view of (the hyperbolic part of) (4.4) and Lemma 4.2, we have that

$$
\begin{aligned}
\left|\frac{h^d}{2} \sum_{n,I} B_1\right| &\le \lambda \max_{i,u} \left|\frac{dF_i^\kappa}{du}(u)\right| \max_{n,I} |u_I^{\kappa,n}|\, \Delta t\, h^d \sum_{n,I} \sum_{i=1}^{d} \left| D_{i,-} F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n}, u_{I+e_i}^{\kappa,n}\right)\right| \\
&\le \max_{n,I} |u_I^{\kappa,n}| \max_{i,u} \left|\frac{dF_i^\kappa}{du}(u)\right| \Delta t\, h^d \sum_{n,I} \sum_{i=1}^{d} |D_{i,-} u_I^{\kappa,n}| \le C_3,
\end{aligned}
$$

for some constant C_3 that is independent of h. Similarly, in view of (the parabolic part of) (4.4) and the L^1, L^∞ bounds in Lemma 4.2, we have that

$$\begin{aligned}\Big|\frac{h^d}{2}\sum_{n,I} B_2\Big| &\le \frac{\Delta t}{h}\max_u \frac{dA^\kappa}{du}(u)\max_{n,I}|u_I^{\kappa,n}|\,\Delta t\,h^d\sum_{n,I}\sum_{i=1}^d |D_{i,+}A^\kappa\,(u_I^{\kappa,n})|\\ &\le 2\mu d\big(\max_u \frac{dA^\kappa}{du}(u)\big)^2\max_{n,I}|u_I^{\kappa,n}|\,\Delta t\,h^d\sum_{n,I}|u_I^{\kappa,n}| \le C_4.\end{aligned}$$

Finally, we have $\big|\frac{h^d}{2}\sum_{n,I} B_3\big| \le C_5$.

Summing up, from (4.19) and the uniform bounds just obtained for B_1, B_2, B_3, we have

$$\Big|\frac{h^d}{2}\sum_{n,I}\big(u_I^{\kappa,n+1}-u_I^{\kappa,n}\big)^2\Big| \le C_6,$$

for some constant C_6 that is independent of h. Inserting this estimate into (4.17), we finally get

$$\Delta t\,h^d\sum_{n,I}\sum_{i=1}^d\big(D_{i,+}A^\kappa\,(u_I^{\kappa,n})\big)^2 \le C_7, \tag{4.20}$$

for some constant C_7 that is independent of h.

Let us now derive (4.16) from (4.20). To facilitate this we introduce some notation inspired by [22]. Let $\bar\chi_I$ denote the the set

$$\bar\chi_I = \big\{x\in\mathbb{R}^d : h\,(i_j-1/2)\le x\le h\,(i_j+1/2)\,,\ j=1,\dots,d\big\}\,,\quad I=(i_1,\dots,i_d)\,,$$

and for x and y in $\mathbb{R}^d$ let $\sigma(x,y)$ denote the line from x to $x+y$. Then for x and y in $\mathbb{R}^d$ we define

$$\chi_{I+e_i/2}(x,y) = \begin{cases}1 & \text{if } \sigma(x,y)\cap\bar\chi_I\cap(\bar\chi_{I+e_i})\ne\emptyset,\\ 0 & \text{otherwise.}\end{cases} \tag{4.21}$$

Using this notation, we have

$$|A^\kappa\,(u^{\kappa,n}(x+y))-A^\kappa\,(u^{\kappa,n}(x))| \le \sum_I\sum_{i=1}^d \chi_{I+e_i/2}(x,y)\,|D_{i,+}A^\kappa\,(u_I^{\kappa,n})|\,,$$

which by the Cauchy-Schwarz inequality implies

$$\begin{aligned}&\big(A^\kappa\,(u^{\kappa,n}(x+y))-A^\kappa\,(u^{\kappa,n}(x))\big)^2\\ &\le \sum_I\sum_{i=1}^d \chi_{I+e_i/2}(x,y)h^2\sum_I\sum_{i=1}^d \chi_{I+e_i/2}(x,y)\,|D_{i,+}A^\kappa\,(u_I^{\kappa,n})|^2\,.\end{aligned} \tag{4.22}$$

If we let $n_i(y)$ denote the number of edges crossed by y, then we find that

$$n_i(y)\le \mathrm{floor}\big(\frac{|y_i|}{h}\big)+1,$$

where $\mathrm{floor}(a)$ denotes the integer part of a, and y_i the ith component of y. Thus

$$\sum_I\sum_{i=1}^d \chi_{I+e_i/2}(x,y)h \le \sum_{i=1}^d n_i(y)h \le h\sum_{i=1}^d\Big(\big(\frac{|y_i|}{h}\big)+1\Big) \le \sqrt{d}\,|y|+dh.$$

Furthermore, we have the relation

$$\int_{\mathbb{R}^d} \chi_{I+e^i/2}(x,y)\,dx = h^{d-1}\,|y_i|\,.$$

Hence, integrating (4.22) over x, and then over t (which amounts to summing over n), we find that

$$\begin{aligned}
&\iint_{\Pi_T} \left(A^\kappa\left(U_h(x+y,t)\right) - A^\kappa\left(U_h(x,t)\right)\right)^2\,dt\,dx\\
&\le (|y|+h)h\Delta t \sum_{n,I}\sum_{i=1}^{d} \int_{\mathbb{R}^d} \chi_{I+e_i/2}(x,y)\,dx\,\left|D_{i,+}A^\kappa\left(u_I^{\kappa,n}\right)\right|^2.\\
&\le d^{3/2}(|y|+h)\,|y|\,\Delta t h^d \sum_{n,I}\sum_{i=1}^{d} \left|D_{i,+}A^\kappa\left(u_I^{\kappa,n}\right)\right|^2\\
&\le C_8(|y|+h)\,|y|\,,
\end{aligned}$$

by (4.20). This concludes the proof of (4.16). □

Remark 4.5. It is possible to derive Lemma 4.4 without using BV regularity of the approximate solution, see [29] and also [1, 10, 22, 25, 31].

The next lemma provides us with a uniform L^2 time translation estimate for $A(U_h)$.

Lemma 4.6. *There exists a constant C, independent of h, such that*

$$\left\|A^\kappa\left(U_h(\cdot,\cdot+\tau)\right) - A^\kappa\left(U_h(\cdot,\cdot)\right)\right\|_{L^2(\mathbb{R}^d\times(0,T-\tau))} \le C\sqrt{\tau}, \quad \tau\in(0,T). \tag{4.23}$$

Proof. We will use the space estimate (4.20) and the finite difference scheme (4.3) to show that $A^\kappa\left(u_h^\kappa\right)$ is also L^2 continuous in time. For $t\in[t_n,t_{n+1})$, $t+\tau\in[t_{n+m_\alpha},t_{n+m_\alpha+1})$ for some m_α, $\alpha=1,2$, and $m_2=m_1+1$. Furthermore $m_2\Delta t\le\tau+\Delta t$. Using this notation we have that

$$\begin{aligned}
&\iint_{\Pi_{T-\tau}} \Big(A^\kappa\left(u_h^\kappa(x,t+\tau)\right) - A^\kappa\left(u_h^\kappa(x,t)\right)\Big)^2\,dt\,dx\\
&= \sum_{\alpha=1}^{2} c_\alpha \Delta t h^d \sum_{n,I}\Big(A^\kappa\left(u_I^{\kappa,m_\alpha}\right) - A^\kappa\left(u_I^{\kappa,n}\right)\Big)^2,
\end{aligned} \tag{4.24}$$

for some weights $c_1,c_2\in[0,1]$ satisfying $c_1+c_2=1$. Now

$$\begin{aligned}
&\left(A^\kappa\left(u_I^{\kappa,m_\alpha}\right) - A^\kappa\left(u_I^{\kappa,n}\right)\right)^2\\
&\le \max_u \frac{dA^\kappa}{du}(u)\Big(A^\kappa\left(u_I^{\kappa,n+m_\alpha}\right) - A^\kappa\left(u_I^{\kappa,n}\right)\Big)\Big(u_I^{\kappa,m_\alpha} - u_I^{\kappa,n}\Big)\\
&\le C_1\Big(A^\kappa\left(u_I^{\kappa,n+m_\alpha}\right) - A^\kappa\left(u_I^{\kappa,n}\right)\Big)\sum_{m=n}^{n+m_\alpha-1}\Big(u_I^{\kappa,m+1} - u_I^{\kappa,m}\Big)\\
&= C_1\Big[\underbrace{-\Big(A^\kappa\left(u_I^{\kappa,n+m_\alpha}\right) - A^\kappa\left(u_I^{\kappa,n}\right)\Big)\Delta t \sum_{m=n}^{n+m_\alpha-1}\sum_{i=1}^{d} D_{i,-}F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,m}, u_{I+e_i}^{\kappa,m}\right)}_{B_{1,\alpha}(n,I)}
\end{aligned}$$

$$+\underbrace{\Big(A^\kappa\left(u_I^{\kappa,n+m_\alpha}\right)-A^\kappa\left(u_I^{\kappa,n}\right)\Big)\Delta t\sum_{m=n}^{n+m_\alpha-1}\sum_{i=1}^{d}D_{i,-}D_{i,+}A^\kappa\left(u_I^{\kappa,m}\right)}_{B_{2,\alpha}(n,I)}$$

$$+\underbrace{\Big(A^\kappa\left(u_I^{\kappa,n+m_\alpha}\right)-A^\kappa\left(u_I^{\kappa,n}\right)\Big)\Delta t\sum_{m=n}^{n+m_\alpha-1}g^\kappa\left(U_I^m\right)}_{B_{3,\alpha}(n,I)}\Big].$$

Using Lemma 4.2 (as before), we get the uniform bound

$$\Delta t h^d\sum_{n,I}|B_{1,\alpha}(n,I)|+|B_{3,\alpha}(n,I)|\le C_2\tau, \tag{4.25}$$

where C_2 is independent of h and we have used that $(m_2-1)\Delta t\le\tau$. Regarding $B_{2,\alpha}$, we use summation by parts to obtain

$$\begin{aligned}\sum_I B_{2,\alpha}(n,I)&=-\Delta t\sum_{m=n}^{n+m_\alpha-1}\sum_I\sum_{i=1}^{d}D_{i,+}A^\kappa\left(u_I^{\kappa,n+m_\alpha}\right)D_{i,+}A^\kappa\left(u_I^{\kappa,m}\right)\\&\quad+\Delta t\sum_{m=n}^{n+m_\alpha-1}\sum_I\sum_{i=1}^{d}D_{i,+}A^\kappa\left(u_I^{\kappa,n}\right)D_{i,+}A^\kappa\left(u_I^{\kappa,m}\right)\\&\le\frac{\Delta t}{2}\left(m_\alpha-1\right)\sum_I\Big(\sum_{i=1}^{d}\left(D_{i,+}A^\kappa\left(u_I^{\kappa,n+m_\alpha}\right)\right)^2+\left(D_{i,+}A^\kappa\left(u_I^{\kappa,n}\right)\right)^2\Big)\\&\quad+\Delta t\sum_{m=n}^{n+m_\alpha-1}\sum_I\sum_{i=1}^{d}\left(D_{i,+}A^\kappa\left(u_I^{\kappa,m}\right)\right)^2,\end{aligned}$$

where we have used the identity $ab\le\frac{1}{2}\left(a^2+b^2\right)$ for all $a,b\in\mathbb{R}$. Now that we have used the scheme to get rid of all time differences we use (4.20) to conclude that

$$\Delta t h^d\sum_{n,I}|B_{2,\alpha}(n,I)|\le C_3\tau, \tag{4.26}$$

for some constant C_3 independent of Δt. Now (4.24), (4.25) and (4.26) closes the proof of (4.23). □

Remark 4.7. Observe that if we went directly via Lemmas 4.2 and 4.3 (interpolating between L^1 and L^∞), then we would have obtained the (not optimal) estimate

$$\left\|A\left(U_h(\cdot,t+\tau)\right)-A\left(U_h(\cdot,t)\right)\right\|_{L^2(\mathbb{R}^d)}\le C\tau^{1/4},\quad t\in(0,T),$$

for some constant that is independent of h.

We next show that the finite difference scheme satisfies a discrete entropy condition. Let $\eta\colon\mathbb{R}\to\mathbb{R}$ be an entropy function. In this case the associated Engquist–Osher (numerical) entropy flux $q^{\kappa,\mathrm{EO}}(u,v)=\left(q_1^{\kappa,\mathrm{EO}},\dots,q_d^{\kappa,\mathrm{EO}}\right)$ is defined by (see, e.g., Kröner [31, p. 184])

$$q_i^{\kappa,\mathrm{EO}}(u,v)=\int_0^u\eta'(\xi)\Big(\frac{dF_i^\kappa}{du}(\xi)\vee0\Big)\,d\xi+\int_0^v\eta'(\xi)\Big(\frac{dF_i^\kappa}{du}(\xi)\wedge0\Big)\,d\xi,\ i=1,\dots,d. \tag{4.27}$$

The next lemma provides us with a cell entropy inequality for the Engquist–Osher scheme (4.3).

Lemma 4.8. *For any entropy function $\eta\colon \mathbb{R}\to\mathbb{R}$ and corresponding entropy fluxes $q^{\kappa,\mathrm{EO}}, r^\kappa$,*

$$\begin{aligned}&\frac{\eta(u_I^{\kappa,n+1})-\eta\left(u_I^{\kappa,n}\right)}{\Delta t}+\sum_{i=1}^d D_{i,-}q_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n},u_{I+e_i}^{\kappa,n}\right)-\sum_{i=1}^d D_{i,-}D_{i,+}r^\kappa\left(u_I^{\kappa,n}\right)\\&\le\eta'\left(u_I^{\kappa,n+1}\right)g^\kappa\left(U_I^n\right),\quad \kappa=1,\dots,K.\end{aligned}\tag{4.28}$$

Proof. Assume for the moment that the following inequality holds:

$$\frac{\eta(\overline{u}_I^{\kappa,n+1})-\eta\left(u_I^{\kappa,n}\right)}{\Delta t}+\sum_{i=1}^d D_{i,-}q_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n},u_{I+e_i}^{\kappa,n}\right)-\sum_{i=1}^d D_{i,-}D_{i,+}r^\kappa\left(u_I^{\kappa,n}\right)\le 0.\tag{4.29}$$

Then using (4.5) and convexity of the entropy function η, it follows that

$$\eta\big(\overline{u}_I^{\kappa,n+1}\big)\ge\eta\big(u_I^{\kappa,n+1}\big)-\eta'\big(u_I^{\kappa,n+1}\big)\Delta t g^\kappa\big(U_I^n\big).\tag{4.30}$$

Combining (4.29) and (4.30), we get the desired cell entropy inequality (4.28).

It remains to prove (4.29). The proof is based on a monotonicity property ensured by the CFL condition. We refer to Kröner [31] for a similar proof in the context of hyperbolic conservation laws. For $\kappa=1,\dots,K$, define the function $H^\kappa\colon\mathbb{R}^{2d}\to\mathbb{R}$ by

$$\begin{aligned}&H^\kappa\left(u_{I-e_1}^{\kappa,n},u_{I+e_1}^{\kappa,n},\dots,u_{I-e_d}^{\kappa,n},u_{I+e_d}^{\kappa,n}\right)\\&=\eta\left(\overline{u}_I^{\kappa,n+1}\right)-\eta\left(u_I^{\kappa,n}\right)+\lambda\sum_{i=1}^d\Big(q_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n},u_{I+e_i}^{\kappa,n}\right)-q_i^{\kappa,\mathrm{EO}}\left(u_{I-e_i}^{\kappa,n},u_I^{\kappa,n}\right)\Big)\\&\quad+\mu\sum_{i=1}^d\Big(r^\kappa\left(u_{I-e_i}^{\kappa,n}\right)-2r^\kappa\left(u_I^{\kappa,n}\right)+r^\kappa\left(u_{I+e_i}^{\kappa,n}\right)\Big),\end{aligned}$$

where

$$\begin{aligned}\overline{u}_I^{\kappa,n+1}&=u_I^{\kappa,n}-\lambda\sum_{i=1}^d\Big(F_i^{\kappa,\mathrm{EO}}\left(u_I^{\kappa,n},u_{I+e_i}^{\kappa,n}\right)-F_i^{\kappa,\mathrm{EO}}\left(u_{I-e_i}^{\kappa,n},u_I^{\kappa,n}\right)\Big)\\&\quad+\mu\sum_{i=1}^d\Big(A^\kappa\left(u_{I-e_i}^{\kappa,n}\right)-2A^\kappa\left(u_I^{\kappa,n}\right)+A^\kappa\left(u_{I+e_i}^{\kappa,n}\right)\Big).\end{aligned}$$

Observe that $H^\kappa(u_I^{\kappa,n},\dots,u_I^{\kappa,n})=0$. Furthermore, using a first-order Taylor expansion along with the CFL condition (4.4) and convexity of η, it is not hard to check that

$$\begin{aligned}&\partial_\ell H^\kappa\left(u_I^{\kappa,n},\dots,\xi_\ell,\dots,u_I^{\kappa,n}\right)>0,\quad \xi_\ell<u_I^{\kappa,n},\quad \ell=1,\dots,2d.\\&\partial_\ell H^\kappa\left(u_I^{\kappa,n},\dots,\xi_\ell,\dots,u_I^{\kappa,n}\right)<0,\quad \xi_\ell>u_I^{\kappa,n},\quad \ell=1,\dots,2d.\end{aligned}$$

From this we conclude that H is a non-positive function and hence (4.29) follows. □

We now have the necessary tools to prove our main result of this section.

Theorem 4.9 (Existence)**.** *Assume that* (1.4) *and* (1.5) *hold. There exists an entropy solution of the Cauchy problem* (1.1), (1.3)*. Furthermore, the entropy solution can be constructed as the limit of a sequence of finite difference approximations.*

Proof. Let us first treat the case where u_0 belongs to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ and has compact support. Furthermore, we assume that F, A are C^1. In view of the h uniform estimates in Lemmas 4.2 and 4.3, Lemma 2.3 tells us that the sequence $\{u_h^\kappa\}_{h>0}$ is compact in $L^1_{\mathrm{loc}}(\Pi_T)$. Moreover, any limit point of this sequence satisfies (1) and (4) in Definition 2.1. Using Lemma 4.8 and standard arguments analogous to the ones used to prove the classical Lax–Wendroff theorem, we eventually conclude that any limit point of $\{u_h^\kappa\}_{h>0}$ satisfies the entropy condition (2.1). In view of Lemma 4.16, Lemma 4.6, and since $A^\kappa(u_h^\kappa)$ obviously belongs to $L^2(\Pi_T)$, Lemma 2.4 tells us that the sequence $\{A^\kappa(u_h^\kappa)\}_{h>0}$ is compact in $L^2_{\mathrm{loc}}(\Pi_T)$. Moreover, any limit point of this sequence satisfies (3) in Definition (2.1).

To treat the general case where u_0 only belongs to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we use the L^1 stability result in Theorem 3.3 along with an approximation procedure. This argument is classical and it is thus omitted, see instead Crandall and Majda [15], for example. Similarly, the case that F, A are merely Lipschitz continuous can be treated by approximating F, A with C^1 functions F_ℓ, A_ℓ and noting that all previous estimates are robust with respect to sending $\ell \uparrow \infty$. □

5. A numerical example

As an illustration of the ideas set forth in this paper, we consider a simplified model of biodegradation of a contaminant in a porous medium. Assume that a contaminant (e.g., oil) is injected into a porous medium containing water with dissolved oxygen. The contaminant reacts with oxygen to some third component, which we assume does not influence the model. We also assume that the oxygen is passively advected along with the flow, and that it dissolves equally well in the water and the contaminant. To be precise, we study the following model

$$\begin{aligned} u_t + \mathbf{v} \cdot \nabla (f(u)) &= \varepsilon \Delta u + g(u,c), \\ c_t + \mathbf{v} \cdot \nabla c &= \varepsilon \Delta c + g(u,c). \end{aligned} \tag{5.1}$$

Here, u denotes the concentration of the contaminant, and c the concentration of the oxygen. The velocity field $\mathbf{v}$ is given by

$$\mathbf{v}(x,y) = -\frac{\mathbf{r}_1}{|\mathbf{r}_1|} + \frac{\mathbf{r}_2}{|\mathbf{r}_2|}$$

with $\mathbf{r}_1 = (x - 0.1, y - 0.5)$, $\mathbf{r}_2 = (x - 1, y - 0.5)$. The flux function is

$$f(u) = \frac{u^2}{u^2 + (1-u)^2},$$

and the source term g models the reaction by Monod kinetics via

$$g(u,c) = K \frac{uc}{(0.2+u)(0.2+c)}, \quad K = 3.5. \tag{5.2}$$

Finally, we set $\varepsilon = 0.25$. We consider this model in the rectangle $(x,y) \in [0,1] \times [0,0.5]$. To compute numerical approximations we use a straightforward modification of the Engquist–Osher scheme (4.3), using Neumann boundary conditions. We remark that this model is strongly inspired by a similar model in [37]. In Figure 1

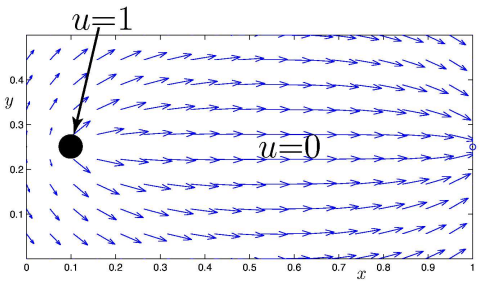

FIGURE 1. The velocity field and the geometric configuration.

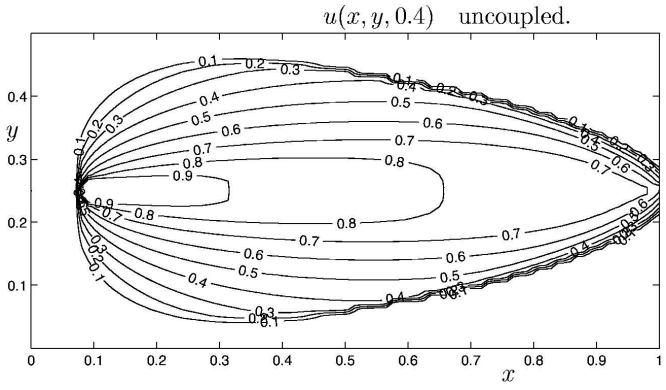

FIGURE 2. Numerical solution of the uncoupled equations, the u-component.

we show the velocity field $\mathbf{v}$ and the setup for our computations. The "inlet" is at the point $(0.1, 0.25)$ and is modeled by setting

$$u(x,y,0) = \begin{cases} 1 & (x,y) \in D, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$D = \left\{ (x,y) : (x-0.1)^2 + (y-0.25)^2 \le 0.025 \right\}.$$

Furthermore, we also set $u(x,y,t) = 1$ for $(x,y) \in D$. The initial "oxygen" saturation is everywhere 1, i.e., $c(x,y,0) = 1$. We used $\Delta x = \Delta y = 1/100$ for our simulation.

In Figure 2 we show the saturation u at $t = 0.4$ if $K = 0$ in (5.2), i.e., we have a scalar conservation law. Compare this with Figure 3 where we show the approximate solution of (5.1) at $t = 0.4$. In Figure 4 we show the corresponding c variable. It is not difficult to see the effect of the coupling of the equations.

REFERENCES

[1] M. Afif and B. Amaziane. Convergence of finite volume schemes for a degenerate convection-diffusion equation arising in two-phase flow in porous media. *Comput. Methods. Appl. Engrg.*, 191:5265–5286, 2002.

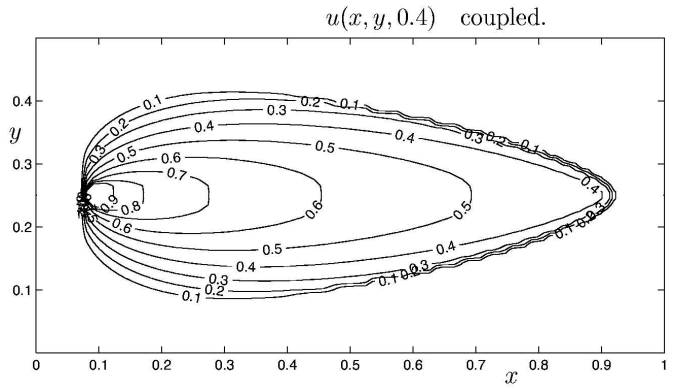

FIGURE 3. Numerical solution of (5.1), $\Delta x = \Delta y = 0.01$.

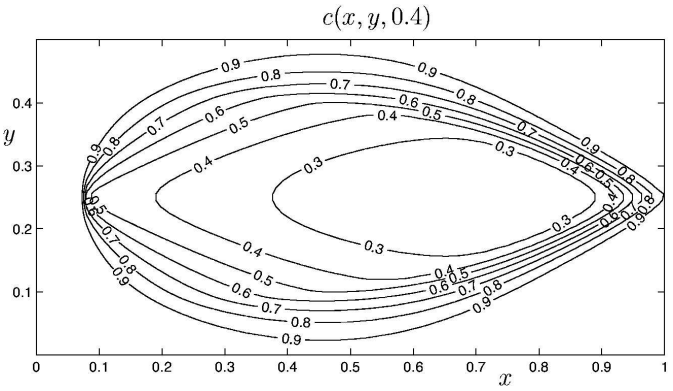

FIGURE 4. Numerical solution of (5.1), the c-component.

[2] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983.

[3] P. Bénilan and H. Touré. Sur l'équation générale $u_t = a(\cdot, u, \phi(\cdot, u)_x)_x + v$ dans L^1. II. Le problème d'évolution. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 12(6):727–761, 1995.

[4] F. Bouchut, F. R. Guarguaglini, and R. Natalini. Diffusive BGK approximations for nonlinear multidimensional parabolic equations. *Indiana Univ. Math. J.*, 49:723–749, 2000.

[5] R. Bürger, S. Evje, and K. H. Karlsen. On strongly degenerate convection-diffusion problems modeling sedimentation-consolidation processes. *J. Math. Anal. Appl.*, 247(2):517–556, 2000.

[6] R. Bürger and K. H. Karlsen. A strongly degenerate convection-diffusion problem modeling centrifugation of flocculated suspensions. In *Int. Series of Numerical Mathematics*, Vol. 140 (2001), 207-216, Birkhuser Verlag.

[7] M. C. Bustos, F. Concha, R. Bürger, and E. M. Tory. *Sedimentation and Thickening: Phenomenological Foundation and Mathematical Theory.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[8] J. Carrillo. On the uniqueness of the solution of the evolution dam problem. *Nonlinear Anal.*, 22(5):573–607, 1994.

[9] J. Carrillo. Entropy solutions for nonlinear degenerate problems. *Arch. Rational Mech. Anal.*, 147(4):269–361, 1999.

[10] S. Champier, T. Gallouët, and R. Herbin. Convergence of an upstream finite volume scheme for a nonlinear hyperbolic equation on a triangular mesh. *Numer. Math.*, 66(2):139–157, 1993.

[11] G.-Q. Chen and E. DiBenedetto. Stability of entropy solutions to the Cauchy problem for a class of hyperbolic-parabolic equations. *SIAM J. Math. Anal.*, 33(4): 751–762, 2001.
[12] G.-Q. Chen and B. Perthame. Well-posedness for anisotropic degenerate parabolic-hyperbolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, to appear.
[13] O. Cirpka, E. Frind, and R. Helmig. Numerical simulation of biodegradation controlled by transverse mixing. *J. Contaminant Hydrology*, 40:159–182, 1999.
[14] B. Cockburn and G. Gripenberg. Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. *J. Differential Equations*, 151(2):231–251, 1999.
[15] M. G. Crandall and A. Majda. Monotone difference approximations for scalar conservation laws. *Math. Comp.*, 34(149):1–21, 1980.
[16] M. G. Crandall and L. Tartar. Some relations between nonexpansive and order preserving mappings. *Proc. Amer. Math. Soc.*, 78(3):385–390, 1980.
[17] B. Engquist and S. Osher. One-sided difference approximations for nonlinear conservation laws. *Math. Comp.*, 36(154):321–351, 1981.
[18] M. S. Espedal and K. H. Karlsen. Numerical solution of reservoir flow models based on large time step operator splitting algorithms. In *Filtration in Porous Media and Industrial Applications (Cetraro, Italy, 1998)*, volume 1734 of *Lecture Notes in Mathematics*, pages 9–77. Springer, Berlin, 2000.
[19] S. Evje and K. H. Karlsen. Viscous splitting approximation of mixed hyperbolic-parabolic convection-diffusion equations. *Numer. Math.*, 83(1):107–137, 1999.
[20] S. Evje and K. H. Karlsen. Monotone difference approximations of BV solutions to degenerate convection-diffusion equations. *SIAM J. Numer. Anal.*, 37(6):1838–1860, 2000.
[21] S. Evje and K. H. Karlsen. An error estimate for viscous approximate solutions of degenerate parabolic equations. *J. Nonlinear Math. Phys.*, 9:262–281, 2002.
[22] R. Eymard, T. Gallouët, M. Ghilani, and R. Herbin. Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by finite volume schemes. *IMA J. Numer. Anal.*, 18(4):563–594, 1998.
[23] R. Eymard, T. Gallouët, and R. Herbin. Error estimate for approximate solutions of a nonlinear convection-diffusion problem. *Advances in Differential Equations*, 7(4):419–440, 2002.
[24] R. Eymard, T. Gallouët, R. Herbin, and A. Michel. Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. *Numer. Math.*, 92(1):41–82, 2002.
[25] R. Eymard, T. Gallouët, D. Hilhorst, and Y. Naït Slimane. Finite volumes and nonlinear diffusion equations. *RAIRO Modél. Math. Anal. Numér.*, 32(6):747–761, 1998.
[26] H. Holden and N. H. Risebro. *Front Tracking for Hyperbolic Conservation Laws*. Springer, New York, 2002.
[27] H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs. *Comm. Pure Appl. Math.*, 42(1):15–45, 1989.
[28] K. H. Karlsen and M. Ohlberger. A note on the uniqueness of entropy solutions of nonlinear degenerate parabolic equations. *J. Math. Anal. Appl.*, 275(1):439-458, 2002.
[29] K. H. Karlsen and N. H. Risebro. Convergence of finite difference schemes for viscous and inviscid conservation laws with rough coefficients. *M2AN Math. Model. Numer. Anal.*, 35(2):239–269, 2001.
[30] K. H. Karlsen and N. H. Risebro. On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. *Discrete Contin. Dynam. Systems - Series A*, 9(5):1081–1104, 2003.
[31] D. Kröner. *Numerical Schemes for Conservation Laws*. John Wiley & Sons Ltd., Chichester, 1997.
[32] S. N. Kružkov. Results concerning the nature of the continuity of solutions of parabolic equations, and some of their applications. *Mat. Notes*, 6:517–523, 1969.
[33] S. N. Kružkov. First order quasi-linear equations in several independent variables. *Math. USSR Sbornik*, 10(2):217–243, 1970.
[34] C. Mascia, A. Porretta, and A. Terracina. Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations. *Arch. Ration. Mech. Anal.*, 163:87–124, 2002.
[35] R. Natalini and B. Hanouzet. Weakly coupled systems of quasilinear hyperbolic equations. *Differential Integral Equations*, 9(6):1279–1292, 1996.
[36] M. Ohlberger. A posteriori error estimates for vertex centered finite volume approximations of convection-diffusion-reaction equations. *M2AN Math. Model. Numer. Anal.*, 35(2):355-387, 2001.

[37] M. Ohlberger and C. Rohde. Adaptive finite volume approximations for weakly coupled convection dominated parabolic systems. *IMA J. Numer. Anal.* 22(2):253–280, 2002.
[38] F. Otto. L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations. *J. Differential Equations*, 131(1):20–38, 1996.
[39] C. Rohde. Entropy solutions for weakly coupled hyperbolic systems in several space dimensions. *Z. Angew. Math. Phys.*, 49(3):470–499, 1998.
[40] C. Rohde. Upwind finite volume schemes for weakly coupled hyperbolic systems of conservation laws in 2D. *Numer. Math.*, 81(1):85–123, 1998.
[41] É. Rouvre and G. Gagneux. Solution forte entropique de lois scalaires hyperboliques-paraboliques dégénérées. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(7):599–602, 1999.
[42] A. I. Vol'pert. The spaces BV and quasi-linear equations. *Math. USSR Sbornik*, 2(2):225–267, 1967.
[43] A. I. Vol'pert and S. I. Hudjaev. Cauchy's problem for degenerate second order quasilinear parabolic equations. *Math. USSR Sbornik*, 7(3):365–387, 1969.

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