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A DISCONTINUOUS PROBLEM INVOLVING THE P-LAPLACIAN OPERATOR AND CRITICAL EXPONENT IN $\mathbb{R}^N$

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Abstract. Using convex analysis, we establish the existence of at least two nonnegative solutions for the quasilinear problem

$$-\Delta_p u = H(u-a)u^{p^*-1} + \lambda h(x) \quad \text{in } \mathbb{R}^N$$

where $\Delta_p u$ is the p-Laplacian operator, H is the Heaviside function, p^* is the Sobolev critical exponent, and h is a positive function.

1. Introduction

The interest in the study of nonlinear partial differential equations with discontinuous nonlinearities has increased because many free boundary problems arising in mathematical physics may be stated in this from. Among these problems, we have the obstacle problem, the seepage surface problem, and the Elenbaas equation; see for example [9, 10, 11].

Among the typical examples, we have chosen the model for the heat conductivity in electrical media. This model has a discontinuity in its constitutive laws. In fact, considering a domain $\Omega \subset \mathbb{R}^3$ (which in particular could be taken as the whole space $\mathbb{R}^3$ [4]) with electrical media, the thermal and electrical conductivity are denoted by $K(x,t)$ and $\sigma(x,t)$, respectively. Here x is in Ω and t represents the temperature. Since we are considering an electrical media, the function σ may have discontinuities in t, and the distribution of the temperature is unknown. The differential equation describing this distribution is

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(K(x,u(x))\frac{\partial u(x)}{\partial x_i}\big) = \sigma(x,u(x)). \tag{1.1}$$

Note that this equation is related to a free boundary problem in which the jump surface of the electrical conductivity is unknown. We describe this surface as being the set

$$\Gamma_\alpha(u) = \{x \in \Omega,\ u(x) = \alpha,\ \sigma \text{ is discontinuous at } \alpha\}. \tag{1.2}$$

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When the thermal conductivity K is constant and the electrical conductivity σ has a single jump and a critical growth, the model becomes

$$-\Delta u = H(u-a)u^{2^*-1} + \lambda h(x) \quad \text{in } \Omega\,. \tag{1.3}$$

Here H is the Heaviside function (i.e. $H(t) = 0$ if $t \le 0$ and $H(t) = 1$ if $t > 0$), $2^* \equiv 2N/(N-2)$ is the well known Sobolev critical exponent for $N > 2$, λ is a positive parameter, and h is a measurable function defined in Ω.

Note that in this model the jump surface of the solution (1.2) is represented by the set

$$\Gamma_a(u) = \{x \in \mathbb{R}^N,\ u(x) = a\}. \tag{1.4}$$

Related to problem (1.3) for the special case of $a = 0$, i.e., without jump discontinuities, we cite the works of Tarantello [17] when p=2, and Alves [2], Cao, Li & Zhou [8] and Gonçalves & Alves [13] for the case $p \ge 2$. In the case $a > 0$, we cite the work of Alves, Bertone & Gonçalves [3].

In this paper we employ variational techniques to study existence and multiplicity of nonnegative solutions of a family of elliptic equations of type (1.3) in the whole space $\mathbb{R}^N$. More precisely, we shall study the quasilinear problem

$$-\Delta_p u = H(u-a)u^{p^*-1} + \lambda h \text{ in } \mathbb{R}^N, \tag{1.5}$$

where here p^* is the critical Sobolev exponent defined by $\frac{pN}{N-p}$ with $N > p$. We consider $a > 0$ and $\lambda > 0$ real parameters, $h : \mathbb{R}^N \to (0,\infty)$ a positive measurable function with

$$h \in L^\theta(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad \frac{1}{\theta} + \frac{1}{p^*} = 1. \tag{1.6}$$

As a solution of (1.5) we understand a function $u \in \mathcal{D}^{1,p}$ verifying

$$-\Delta_p u(x) - \lambda h(x) \in \widehat{f}(u(x)) \quad \text{a.e in } \mathbb{R}^N, \tag{1.7}$$

where $\widehat{f}$ is the multi-valued function

$$\widehat{f}(s) = \begin{cases} \{f(s)\}, & \text{if } s \neq a \\ [f(a--), f(a+)], & \text{if } s = a, \end{cases}$$

with $f(t) = H(t-a)t^{p^*-1}$, $f(t+0) = \lim_{\delta\to 0^+} f(t+\delta)$, $f(t-0) = \lim_{\delta\to 0^+} f(t-\delta)$.

We recall that the solutions of (1.5) are exactly the critical points of the functional $I_{\lambda,a} : \mathcal{D}^{1,p} \to \mathbb{R}$ given by

$$I_{\lambda,a}(u) = \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} F(u)dx - \lambda \int_{\mathbb{R}^N} h(x)udx. \tag{1.8}$$

where $F(u) = \int_0^u f(t)dt$ and $\mathcal{D}^{1,p}$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|\phi\|^p = \int_{\mathbb{R}^N} |\nabla\phi(x)|^p dx.$$

The set $\Gamma_a(u)$ has a relevant role when its Lebesgue measure is zero, since the solutions would satisfy (1.5) in the "strong" sense, i.e.,

$$-\Delta_p u(x) = H(u(x)-a)u(x)^{p^*-1} + \lambda h(u(x)), \tag{1.9}$$

almost everywhere (a.e. for short) in $\mathbb{R}^N$.

Another important remark is that we are considering only *nontrivial* solutions which means that the functions $u \not\equiv 0$ and verify $\operatorname{meas}\{x \in \mathbb{R}^N,\ u(x) > a\} > 0$. We observe that there exists a function w_λ which satisfies

$$-\Delta_p u = \lambda h(x),\ u(x) > 0 \text{ in } \mathbb{R}^N \tag{1.10}$$

and $|w_\lambda|_\infty \leq a$, then it is a solution of (1.5) when λ is small. Furthermore, we will denote by $w = w_\lambda$ the unique solution of (1.10).

Our main result is the following.

Theorem 1.1. *Assume that h satisfies* (1.6)*. Then, there exists $\lambda_* > 0$ and $a_* > 0$ such that if $\lambda \in (0, \lambda_*)$ and $a \in (0, a_*)$, problem* (1.5) *has two nonnegative solutions u_i, $i = 1, 2$ with the following properties:*

(i) $\Delta_p u_i \in L^\theta(\mathbb{R}^N)$;
(ii) $\operatorname{meas}\{x \in \mathbb{R}^N, u_i(x) > a\} > 0$, $i = 1, 2$;
(iii) $\operatorname{meas} \Gamma_a(u_i) = 0$;
(iv) $I_{\lambda,a}(u_2) < 0 < I_{\lambda,a}(u_1)$.

The proof of theorem (1.1) relies on some results of Convex Analysis since the functional $I_{\lambda,a}$ is locally Lipschitz. To get critical points for $I_{\lambda,a}$, we use a version of the Mountain Pass for locally Lipschitz functional and the Ekeland Variational Principle. However, the arguments involved are not standard ones: First of all because we are working with the p-Laplacian operator, which is not linear, the growth of the nonlinear part is critical, and the domain is the whole space $\mathbb{R}^N$. The second reason is that the arguments used when $a = 0$ (the classical case) cannot be used immediately in our context and because of that a new estimates appear, for instance, to prove that the energy functional verifies the Palais-Smale condition at some levels.

To finish this introduction, we would like to say that the our main result complete the results obtained in [1], [2] and [3], in the following sense, in [1] and [2] was considered the case $a = 0$ and in [3] was considered the situation where the operator is the Laplacian and the Heaviside function is multiplying the term involving the function h.

2. Basic results from convex analysis

Throughout this paper X is a Banach space, $\Phi \in \operatorname{Lip}_{\text{loc}}(X, \mathbb{R})$ means that the functional is *Locally Lipschitzian* on X. The *generalized directional derivative of* Φ *in* $u \in X$ is the function denoted by $\Phi^0(u; \cdot)$ and defined by the formula

$$\Phi^0(u; v) = \limsup_{h \to 0,\, \lambda \downarrow 0} \frac{\Phi(u + h + \lambda v) - \Phi(u + h)}{\lambda}.$$

Since $\Phi^0(u; \cdot)$ is continuous and convex it makes sense to consider the subdifferential of $\Phi^0(u; \cdot)$, which is, by definition,

$$\partial \Phi^0(u; z) = \{\mu \in X^* : \langle \mu, v - z \rangle_{X^*, X} \leq \Phi^0(u; v) - \Phi^0(u; z)\ \forall v \in X\}.$$

We define as *generalized gradient* of Φ *in* u the set

$$\partial \Phi(u) = \{\mu \in X^* \mid \langle \mu, v \rangle_{X^*, X} \leq \Phi^0(u; v)\ \forall v \in X\},$$

and we shall denote it by $\partial \Phi(u)$. Since $\Phi^0(u; 0) = 0$ we have

$$\partial \Phi(u) = \partial \Phi^0(u; 0).$$

An important property of the generalized gradient is the following: If $u \in X$ then $\partial\Phi(u)$ is a nonempty convex set and it is w^*-compact. In particular, there exists $\hat{\omega} \in \partial\Phi(u)$ such that $\|\hat{\omega}\|_{X^*} = \min_{\omega\in\partial\Phi(u)} \|\omega\|_{X^*}$.

We say that $\{u_n\}$ verifies *the Palais Smale Condition for the functional* Φ *and the value* c (denoted by $(PS)_c$) if $\{u_n\}$ verifies

$$\Phi(u_n) \to c \quad \text{and} \quad \|\omega_n\| = \min_{\omega\in\partial\Phi(u_n)} \|\omega\|_{X^*} \to 0, \tag{2.1}$$

then it implies that there is a subsequence of u_n which converges in $\mathcal{D}^{1,p}$.

Next we shall enunciate two crucial results that will be used throughout this work. One is the well known Mountain Pass theorem, in a locally Lipchitzian version. The other is a characterization of the elements of the generalized gradient of a determined functional. The proof of these results can be found in [3].

Theorem 2.1. *Let* $\Phi \in Lip_{\rm loc}(X,\mathbb{R})$. *Suppose that* $\Phi(0) = 0$ *and there is* $\eta,\ r_1 > 0$, $e \in X$ *with* $\|e\| > r_1$ *such that*

$$\Phi(u) \geq \eta \textit{ if } \|u\| = r_1,\ \Phi(e) \leq 0. \tag{2.2}$$

If $c \equiv \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \Phi(\gamma(t))$ *and*

$$\Gamma \equiv \{\gamma \in \mathcal{C}([0,1],\ X) \mid \gamma(0) = 0,\ \gamma(1) = e\},$$

then $c > 0$ *and there exists a sequence* $\{u_n\} \subset X$ *satisfying (2.1).*

Proposition 2.2. *Let* $\Phi(u) = \int_{\mathbb{R}^N} F(u)dx$ *be the functional defined in (1.8). Then,* $\Phi \in Lip_{loc}(L^{p^*}(\mathbb{R}^N);\mathbb{R})$, $\partial\Phi(u) \subset (L^{p^*}(\mathbb{R}^N))'$ *and if* $\omega \in \partial\Phi(u)$, *it satisfies*

$$\omega(x) \in \widehat{f}(u(x)),\ \textit{a.e. } x \in \mathbb{R}^N. \tag{2.3}$$

3. Preliminary Results

Hereafter we shall use $L^s,\ s > 1$ to represent the Lebesgue space $L^s(\mathbb{R}^N)$ and $|\cdot|_s$ its usual norm. Besides, if g is a Lebesgue integrable function, we shall write $\int g$ for $\int_{\mathbb{R}^N} gdx$ and S denotes the best Sobolev constant of the imbedding $\mathcal{D}^{1,p} \hookrightarrow L^{p^*}$, that is,

$$S = \min_{u\in\mathcal{D}^{1,p},\, u\neq 0} \frac{\int |\nabla u|^p}{\left(\int |u|^{p^*}\right)^{\frac{p}{p^*}}}$$

Our first Lemma is a version for vectorial functions in $\mathbb{R}^N$ of a result due to Brezis & Lieb (see [6]). Its proof can be found in [1].

Lemma 3.1. *Let* $\eta_n : \mathbb{R}^N \to \mathbb{R}^K$ $(K \geq 1)$ *with* $\eta_n \in L^p(\mathbb{R}^N) \times \ldots \times L^p(\mathbb{R}^N)$ $(p \geq 2)$, $\eta_n(x) \to 0$ *a.e in* $\mathbb{R}^N$ *and* $A(y) = |y|^{p-2}y$, *for all* $y \in \mathbb{R}^K$. *Then, if* $|\eta_n|_{L^p(\mathbb{R}^N)} \leq C$, *for all* $n \in \mathbb{N}$ *we have*

$$\int_{\mathbb{R}^N} |A(\eta_n + w) - A(\eta_n) - A(w)|^{\frac{p}{p-1}} = o_n(1),$$

for each $w \in L^p(\mathbb{R}^N) \times \ldots \times L^p(\mathbb{R}^N)$ *fixed.*

The next lemma is standard and its proof use similar arguments to those in [3]. It shows that the functional $I_{\lambda,a}$ verifies the mountain pass geometry.

Lemma 3.2. *There is* $\lambda_0 > 0$ *such that for* $\lambda \in (0, \lambda_0)$ *the functional* $I_{\lambda,a}$ *verifies the mountain pass geometry (2.2), for all* $a > 0$.

Using the lemma above, we conclude by Theorem 2.1 that there exists $\{u_n\}$ in $\mathcal{D}^{1,p}$ such that

$$I_{\lambda,a}(u_n) \to c \text{ and } \|w_n\| = \min_{w_n \in \partial I_{\lambda,a}(u_n)} \|w\| \to 0$$

Lemma 3.3. *The functional $I_{\lambda,a}$ satisfies the condition $(PS)_c$, for*

$$c \in (-\infty, \frac{1}{N} S^{\frac{N}{p}} - c_1 \lambda^{\frac{p}{p-1}}),$$

where $c_1 = c_1(N, S, \theta, |h|_\theta)$ is a positive constant that verifies the following inequality

$$\frac{1}{N} t^p - \frac{\lambda}{\theta} |h|_\theta t \geq -c_1 \lambda^{\frac{p-1}{p}}, \quad \textit{for all } t \geq 0.$$

Proof. Suppose u_n satisfies (2.1). One has u_n bounded and there exists $u_0 \in \mathcal{D}^{1,p}$ such that u_n converges weakly in $\mathcal{D}^{1,p}$ and a.e. in $\mathbb{R}^N$ to u_0. Let $v_n = u_n - u_0$ and suppose that $\|v_n\|^p \to l > 0$. Thus,

$$\langle w_n, v_n \rangle = \int |\nabla u_n|^{p-2} \nabla u_n \nabla v_n - \lambda \int h(x) v_n - \langle \rho_n, v_n \rangle$$

where $\rho_n \in \partial \Phi(u_n)$. Using Proposition 2.1, we have

$$0 \leq \rho_n(x) \leq u_n^{p^*-1}(x) \quad \text{a.e in } \mathbb{R}^N$$

and repeating similar arguments explored in [13], it is possible to show the existence of a set $\Gamma \subset \mathbb{R}^N$ empty or finite such that $\{u_n\}$ is strongly convergent in $L^{p^*}(K)$ for all $K \subset (\mathbb{R}^N \setminus \Gamma)$ compact set. The above information imply that, up to subsequence, we can assume

$$\rho_n(x) \to \rho_0(x) \quad \text{a.e in } \mathbb{R}^N.$$

The above properties involving the sequences $\{u_n\}$ and $\{\rho_n\}$ together with the arguments explored in [13] are sufficient to show

$$\nabla u_n(x) \to \nabla u_0(x) \quad \text{a.e in } \mathbb{R}^N.$$

From Lemma 3.1 we have

$$\langle w_n, v_n \rangle = \int |\nabla v_n|^p + \int |\nabla u_0|^{p-2} \nabla u_0 \nabla v_n - \langle \rho_n, v_n \rangle + o_n(1).$$

Hence, $\langle w_n, v_n \rangle = l - \langle \rho_n, v_n \rangle + o_n(1)$, which implies

$$\lim_{n\to\infty} \langle \rho_n, v_n \rangle = l. \tag{3.1}$$

Moreover, by recalling that

$$\langle \rho_n, v_n \rangle \leq \int f(u_n + 0) v_{n+} + \int f(u_n - 0)(-v_{n-}),$$

we get

$$\langle \rho_n, v_n \rangle \leq \int u_n^{p^*-1} v_{n+} = \int_{u_n > u_0} u_n^{p^*-1}(u_n - u_0).$$

Consequently,

$$\langle \rho_n, v_n \rangle \leq \int u_0^{p^*} + \int |v_n|^{p^*} - \int_{u_n \leq u_0} u_n^{p^*} - \int u_n^{p^*-1} u_0 + \int_{u_n \leq u_0} u_n^{p^*-1} u_0 + o_n(1).$$

Therefore,

$$\langle \rho_n, v_n \rangle \leq \int |v_n|^{p^*} + o_n(1).$$

The last inequality implies that

$$\lim_{n\to\infty}\langle \rho_n, v_n\rangle \le \lim_{n\to\infty}\int |v_n|^{p^*}. \tag{3.2}$$

Now, from (3.1) and (3.2), we obtain that $Sl^{\frac{p}{p^*}} \le l$, which infers

$$S^{\frac{N}{p}} \le l. \tag{3.3}$$

On the other hand, we have

$$I_{\lambda,a}(u_n) + o_n(1) = I_{\lambda,a}(u_n) - \frac{1}{p^*}\langle w_n, u_n\rangle$$

that is,

$$I_{\lambda,a}(u_n) + o_n(1) \ge \frac{1}{N}\|u_n\|^p - \frac{\lambda}{\theta}\int h(x)u_n.$$

Thus,

$$I_{\lambda,a}(u_n) + o_n(1) \ge \frac{1}{N}\|v_n\|^p - \lambda^{\frac{p}{p-1}}c_1 + o_n(1),$$

where $c_1 = c_1(N, S, \theta, |h|_\theta)$ is the constant stated in the Lemma.
From the last inequality and (3.3), we get

$$c \ge \frac{S^{\frac{N}{p}}}{N} - \lambda^{\frac{p}{p-1}}c_1,$$

which contradicts that $c \in (-\infty, \frac{S^{\frac{N}{p}}}{N} - \lambda^{\frac{p}{p-1}}c_1)$. Therefore, we should have $l = 0$ and consequently $u_n \to u_0$ in $\mathcal{D}^{1,p}$. This finished the proof of the lemma. □

Lemma 3.4. *There exists $\lambda_1 > 0$, $a_* > 0$, and $e \in \mathcal{D}^{1,p}$ such that, for $\lambda \in (0, \lambda_1)$ and $a \in (0, a_*)$, we have $e \in B^c_\rho(0)$ with $I_{\lambda,a}(e) < 0$, and*

$$0 < r \le c = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I(\gamma(t)) < \frac{1}{N}S^{\frac{N}{p}} - c_1\lambda^{\frac{p}{p-1}}, \tag{3.4}$$

with $\Gamma = \{\gamma \in C([0,1], \mathcal{D}^{1,p}),\ \gamma(0) = 0,\ \gamma(1) = e\}$.

Proof. Let $\lambda_2 > 0$ such that $\frac{S^{\frac{N}{p}}}{N} - \lambda_2^{\frac{p}{p-1}}c_1 > 0$ $\forall\lambda \in (0, \lambda_2)$. It is known by Talenti in [16] that the family of functions

$$w_\varepsilon(x) = \frac{\Big[N\varepsilon\big(\frac{N-p}{p-1}\big)^{p-1}\Big]^{\frac{N-p}{p^2}}}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} \quad \varepsilon > 0$$

satisfies

$$\|w_\varepsilon\|^p = |w_\varepsilon|^{p^*}_{p^*} = S^{\frac{N}{p}}.$$

Note that, there is $t_0 > 0$ such that for $t \le t_0$, we have

$$I_{\lambda,a}(tw_\varepsilon) \le \frac{1}{N}S^{\frac{N}{p}} - c_1\lambda^{\frac{p}{p-1}} \ \ \forall\lambda \in (0, \lambda_2).$$

Moreover, if $t \ge t_0$

$$\Omega_a = \{t_0 w_\varepsilon > a\} \subset \{tw_\varepsilon > a\},$$

thus,

$$\begin{aligned}I_{\lambda,a}(tw_\varepsilon) &\le \frac{t^p}{p}S^{\frac{N}{p}} - \lambda t\int h(x)w_\varepsilon - \int_{\Omega_a} F(tw_\varepsilon)\\ &= \frac{t^p}{p}S^{\frac{N}{p}} - \lambda t\int h(x)w_\varepsilon - \frac{t^{p^*}}{p^*}\int_{\Omega_a} w_\varepsilon^{p^*} + \frac{a^{p^*}}{p^*}|\Omega_a|.\end{aligned}$$

Therefore, the function

$$P(t) = \frac{t^p}{p}S^{\frac{N}{p}} - \lambda t\int h(x)w_\varepsilon - \frac{t^{p^*}}{p^*}\int_{\Omega_a} w_\varepsilon^{p^*} + \frac{a^{p^*}}{p^*}|\Omega_a|,$$

has a maximum at $\gamma_1 > 0$ and the function $g(t) = \frac{t^p}{p} - \frac{t^{p^*}}{p^*}$ attains its maximum in $t = 1$. As a consequence we get

$$I_{\lambda,a}(tw_\varepsilon) \le \frac{1}{N}S^{\frac{N}{p}} - \lambda t_0\int h(x)w_\varepsilon + \frac{\gamma_1^{p^*}}{p^*}\int_{\Omega_a^c} w_\varepsilon^{p^*} + \frac{a^{p^*}}{p^*}|\Omega_a|.$$

Now, noticing that

$$|\Omega_a| \le \frac{\omega_N K_\varepsilon t_0^{\frac{N}{N-p}}}{a^{\frac{N}{N-p}}},$$

where K_ε is a constant that dependents of ε, one obtains

$$a^{p^*}|\Omega_a| \to 0 \quad \text{as } a \to 0.$$

Then, by taking $\lambda_3 > 0$ such that

$$\lambda t_0\int h(x)w_\varepsilon > \lambda^{\frac{p}{p-1}}c_1,$$

for all $\lambda \in (0,\lambda_3)$, we choose $a^* = a(\lambda_3)$ satisfying

$$-\lambda t_0\int h(x)w_\varepsilon + \frac{\gamma_1^{p^*}}{p^*}\int_{\Omega_a^c} w_\varepsilon^{p^*} + \frac{a^{p^*}}{p^*}|\Omega_a| < -\lambda^{\frac{p}{p-1}}c_1 \quad \forall a \in (0,a^*).$$

Finally, for $a \in (0,a_*)$ and $\lambda \in (0,\lambda_1)$, with $\lambda_1 = \min\{\lambda_2,\lambda_3\}$, we have

$$I_{\lambda,a}(tw_\varepsilon) \le \frac{1}{N}S^{\frac{N}{p}} - c_1\lambda^{\frac{p}{p-1}} \quad \forall\lambda \in (0,\lambda_2) \quad \forall t \ge t_0,$$

and the proof is complete □

4. Proof of Theorem 1.1

4.1. **First solution (Mountain Pass).** Let $\lambda_* = \min\{\lambda_0,\lambda_1\}$, where λ_0 and λ_1 were given by Lemmas 3.2 and 3.4. By Theorem 2.1 there exists a sequence $(PS)_c$, for c defined in (3.4). Therefore we obtain $\rho_n \in \partial\Phi(u_n)$ such that

$$w_n = Q'(u_n) - \Psi'(u_n) - \rho_n, \tag{4.1}$$

where here w_n was defined in (2.1), and

$$Q(u) = \frac{1}{p}\int |\nabla u|^p dx, \quad \Psi(u) = \lambda\int h(x)u(x)dx.$$

Using straightforward arguments, we find that $\{u_n\}$ is bounded in $\mathcal{D}^{1,p}$. Moreover, using the fact that

$$\langle w_n, u_{n-}\rangle = o_n(1)$$

we have $\|u_{n-}\| \to 0$, where u_{n-} is the negative part of u_n. Then there exists a nonnegative $u_1 \in \mathcal{D}^{1,p}$ such that $u_n \rightharpoonup u_1$, $u_n(x) \to u_1(x)$, a.e. $x \in \mathbb{R}^N$. Besides, there exists $\rho_0 \in L^\theta$ such that $\rho_n \rightharpoonup \rho_0$ in L^θ. Now, since $\rho_n \in \partial\Phi(u_n)$, repeating the same arguments explored in the proof of Lemma 3.3 we have

$$\begin{gathered}\rho_n(x) \in \widehat{f}(u_n(x)), \quad \text{a.e. } x \in \mathbb{R}^N,\\ \rho_0(x) \in \widehat{f}(u_1(x)), \quad \text{a.e. } x \in \mathbb{R}^N,\end{gathered}$$

and for $\varphi \in \mathcal{D}^{1,p}$,

$$\int |\nabla u_1|^{p-2}\nabla u_1 \nabla\varphi - \lambda \int h(x)\varphi - \int \rho_0\varphi = 0. \tag{4.2}$$

Proof of i): $\Delta_p u_1 \in L^\theta$. In this subsection, we shall adapt for our problem some arguments that could be found in [5]. From (4.2), we have

$$-\Delta_p u_1 = J_1 + J_2 \quad \text{in } (\mathcal{D}^{1,p})',$$

where $J_1, J_2 : \mathcal{D}^{1,p} \to \mathbb{R}$ are linear functionals:

$$J_1(v) = \lambda \int h(x)v \quad \text{and} \quad J_2(v) = \int \rho_0(x)v.$$

Note that $J_1, J_2 \in (L^{p^*})' \subset (\mathcal{D}^{1,p})'$. Thus, by Riesz's Theorem, $J_1,\ J_2 \in L^\theta$ and so $\Delta_p u_1 \in L^\theta$. Since (4.2) holds, then

$$-\Delta_p u_1 = \lambda h + \rho_0 \quad \text{a.e } \mathbb{R}^N$$

and, from this equality, we get

$$-\Delta_p u_1(x) - \lambda h(x) \in \widehat{f}(u(x)), \quad \text{a.e. } x \in \mathbb{R}^N.$$

This has proved that u_1 is a solution of (1.5).

Proof of ii): $\text{meas}\{x \in \mathbb{R}^N; u_1 > a\} > 0$. Now, we shall prove that u_1 is a nontrivial solution. By Lemmas 3.3 and 3.4 we get $u_n \to u_1$ and $I(u_1) > 0$, so that $u_1 \not\equiv 0$. Suppose, by contradiction, that $u_1 \le a$ in $\mathbb{R}^N$. Then, u_1 would verify

$$\|u_1\|^p = \lambda \int h(x)u_1, \quad \text{in } \mathbb{R}^N,$$

and as a consequence

$$I(u_1) = \frac{-\lambda(p-1)}{p} \int h(x)u_1 < 0.$$

This contradicts the fact that $I(u_1) > 0$.

4.2. **Proof of iii):** $\text{meas}(\Gamma_a(u_1)) = 0$. Assume by contradiction that $\text{meas}(\Gamma_a(u_i)) > 0$. By using the Morrey-Stampacchia's Theorem (see [14] and [15]), we have that $-\Delta_p u(x) = 0$ a.e. $x \in \Gamma_a(u)$. Hence,

$$-\lambda h(x) \in [0, a^{p^*}],$$

which is a contradiction. Thus $\text{meas}(\Gamma_a(u_i)) = 0$, $i = 1, 2$.

4.3. **Second solution (Local Minimization).** To prove the existence of u_2, we observe that, fixed a positive function $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\lim_{t\to 0} I_{\lambda,a}(t\psi) < 0.$$

Consequently

$$\widetilde{c} = \inf_{\overline{B_\rho}} I_{\lambda,a} < 0, \quad \text{for } a \in (0, a_*),$$

and $-\infty < \widetilde{c} < 0$. Now, considering $I_{\lambda,a}|_{\overline{B_\rho}}$, we apply the Ekeland variational principle (see [12]) to obtain $u_\varepsilon \in \overline{B_\rho}$ such that

$$I_{\lambda,a}(u_\varepsilon) < \inf_{\overline{B_\rho}} I_{\lambda,a} + \varepsilon, \tag{4.3}$$

and

$$I_a(u_\varepsilon) < I_a(u) + \varepsilon\|u - u_\varepsilon\|, \quad u \neq u_\varepsilon. \tag{4.4}$$

Let ε be a positive number defined by

$$0 < \varepsilon < \inf_{\partial B_\rho} I_{\lambda,a} - \inf_{\overline{B_\rho}} I_{\lambda,a}.$$

For this choice of ε, one has

$$I_{\lambda,a}(u_\varepsilon) \le \inf_{\overline{B_\rho}} I_{\lambda,a} + \varepsilon < \inf_{\partial B_\rho} I_{\lambda,a},$$

which implies that $u_\varepsilon \in B_\rho$. Let $\gamma > 0$ be small enough that $u_\gamma = u_\varepsilon + \gamma v \in B_\rho$, and $v \in \mathcal{D}^{1,p}$. From (4.4) we get

$$I_{\lambda,a}(u_\varepsilon + \gamma v) - I_{\lambda,a}(u_\varepsilon) + \gamma\varepsilon\|v\| \ge 0.$$

Thus we have

$$-\varepsilon\|v\| \le \limsup_{\gamma \downarrow 0} \frac{I_{\lambda,a}(u_\varepsilon + \gamma v) - I_{\lambda,a}(u_\varepsilon)}{\gamma} \le I^0_{\lambda,a}(u_\varepsilon; v).$$

Now, since the equality below

$$I^0_{\lambda,a}(u; v) = \max_{\mu \in \partial I_{\lambda,a}(u)} \langle \mu, v\rangle, \ u, \ v \in \mathcal{D}^{1,p},$$

holds, it follows that

$$-\varepsilon\|v\| \le I^0_{\lambda,a}(u_\varepsilon, v) = \max_{\omega \in \partial I_{\lambda,a}(u_\varepsilon)} \langle \omega, v\rangle, \quad \text{for all } v \in \mathcal{D}^{1,p}.$$

Interchanging v and $-v$ we obtain

$$-\varepsilon\|v\| \le \max_{\omega \in \partial I_{\lambda,a}(u_\varepsilon)} \langle \omega, -v\rangle = -\min_{\omega \in \partial I_{\lambda,a}(u_\varepsilon)} \langle \omega, v\rangle, \quad v \in \mathcal{D}^{1,p}$$

Therefore,

$$\min_{\omega \in \partial I_{\lambda,a}(u_\varepsilon)} \langle \omega, v\rangle \le \varepsilon\|v\|, \quad v \in \mathcal{D}^{1,p},$$

concluding that

$$\sup_{\|v\|=1} \min_{\omega \in \partial I_{\lambda,a}(u_\varepsilon)} \langle \omega, v\rangle \le \varepsilon.$$

Finally, by Ky Fan's Min-max theorem ([7, Proposition 1.8]), we get

$$\min_{\omega \in \partial I_{\lambda,a}(u_\varepsilon)} \sup_{v \in B_1} \langle \omega, v\rangle \le \varepsilon,$$

which along with (4.3) yields the existence of $u_n \in B_\rho$ such that $I_{\lambda,a}(u_n) \to c$, and $\min_{\omega \in \partial I_{\lambda,a}(u_n)} \|\omega\| \to 0$. Therefore, by Lemma 3.3, there exists $u_2 \in \mathcal{D}^{1,p}$ and a subsequence u_{n_i} of u_n such that $u_{n_i} \to u_2$ in $\mathcal{D}^{1,p}$ and $I_{\lambda,a}(u_2) = c = \inf_{\overline{B_\rho}} I < 0$.

Moreover, we have that $u_2 > a$ in a open $\omega \in \mathbb{R}^N$ because, otherwise, we would have $u(x) \leq a$ in $\mathbb{R}^N$. This implies that u_2 is a solution of (1.10) and by uniqueness we would have $u_2 = u_2(a)$, for all $a \in (0, a^*)$. On the other hand,

$$S|u_2|_{p^*}^p \leq \|u_2\|^p < \lambda \int h(x)u_2(x) \leq \lambda |h|_1 a,$$

which implies that u_2 goes to zero in $\mathcal{D}^{1,p}$ as a goes to zero, hence $ii)$ and $iv)$ hold for u_2. The arguments to proof that u_2 also verifies $i)$ and $iii)$ are the same explored in the section 4.1. This conclude the proof of Theorem 1.1.

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