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RADIAL SOLUTIONS OF SINGULAR NONLINEAR BIHARMONIC EQUATIONS AND APPLICATIONS TO CONFORMAL GEOMETRY

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Abstract. Positive entire solutions of the singular biharmonic equation $\Delta^2 u + u^{-q} = 0$ in $\mathbb{R}^n$ with $q > 1$ and $n \geq 3$ are considered. We prove that there are infinitely many radial entire solutions with different growth rates close to quadratic. If $u(0)$ is kept fixed we show that a unique minimal entire solution exists, which separates the entire solutions from those with compact support. For the special case $n = 3$ and $q = 7$ the function $U(r) = \sqrt{1/\sqrt{15} + r^2}$ is the minimal entire solution if $u(0) = 15^{-1/4}$ is kept fixed.

1. Introduction

We consider positive C^4-solutions of the equation

$$\Delta^2 u + u^{-q} = 0 \quad \text{in } D \subset \mathbb{R}^n. \tag{1.1}$$

A solution is called *entire* if it exists in all of $\mathbb{R}^n$. In a recent paper [5] Choi and Xu studied (1.1) in $\mathbb{R}^3$. They proved that under the restriction of exact linear growth at infinity, i.e., $\lim_{x\to\infty} u(x)/|x| = \alpha > 0$ and $q = 7$, problem (1.1) admits (up to translation) only one kind of entire solution given by

$$U(x) = \alpha\sqrt{1/\sqrt{15\alpha^8} + |x|^2}. \tag{1.2}$$

Moreover Choi and Xu prove that (1.1) has no linear growth solution if $4 < q < 7$. Thus, (1.1) in dimension $n = 3$ and with $q = 7$ is a very distinguished case. Indeed, notice that for this choice of parameters (1.1) amounts to

$$\Delta^2 u = (n-4)u^{\frac{n+4}{n-4}}.$$

This equation has explicit geometric relevance as explained in Section 2.

The remarkable result of Choi and Xu led to our present investigation, where we analyze the global structure of the set of all radial solutions of (1.1). In our main result of Theorem 3.1, Section 3, we prove that (1.1) possesses infinitely many entire solutions with almost quadratic, superlinear growth rates. Moreover, one entire solution is distinguished from the others: the minimal entire solution, which coincides with the solution $U(x) = \alpha\sqrt{1/\sqrt{15\alpha^8} + |x|^2}$ found by Choi and Xu.

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The existence of infinitely many entire solutions with different growth rates for the conformally invariant equation $\Delta^2 u + u^{-7} = 0$ in $\mathbb{R}^3$ is in striking contrast to the conformally invariant equation $\Delta^2 u = u^{\frac{n+4}{n-4}}$ in $\mathbb{R}^n$ with $n \geq 5$ and the second order equation $-\Delta u = u^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n$ with $n \geq 3$. In both cases there exists a unique one-parameter family of positive entire solutions, cf. Juncheng Wei and Xingwang Xu [9], and Wenxiong Chen and Congming Li [4].

2. Geometric relevance

2.1. **Yamabe's problem.** Let $g = (g_{ij})$ be the standard Euclidean metric on $\mathbb{R}^n$, $n \geq 3$ with $g_{ij} = \delta_{ij}$. Let $\bar{g} = u^{4/(n-2)} g$ be a second metric derived from g by the positive conformal factor $u : \mathbb{R}^n \to \mathbb{R}$. Then u satisfies Yamabe's equation

$$-\Delta u = \frac{n-2}{4(n-1)} R_{\bar{g}} u^{(n+2)/(n-2)}, \tag{2.1}$$

where $R_{\bar{g}}$ is the scalar curvature of $\bar{g}$, cf. Aubin [1]. If one looks for constant scalar curvature $R_{\bar{g}} = \pm n(n-1)$ then (2.1) has the following explicit solutions

$$U(r) = \big(\frac{2a}{a^2 \pm r^2}\big)^{(n-2)/2} \quad \text{with } r = |x - x_0|,\ a > 0.$$

The corresponding metric is

$$\bar{g}_{ij} = \big(\frac{2a}{a^2 \pm r^2}\big)^2 \delta_{ij}.$$

In case of "+" one finds that $(\mathbb{R}^n, \bar{g})$ is isometrically isomorphic to a sphere $\mathbb{S}^n_a$ of radius a equipped with standard Euclidian metric scaled by $1/a^2$. Moreover, Wenxiong Chen and Congming Li showed in [4] that (2.1) has no other positive solutions. In case of "−", the solution U blows up on $\partial B_a(x_0)$ and one finds that $(B_a(x_0), \bar{g})$ is isometrically isomorphic to the hyperbolic space

$$\mathbb{H}^n_a = \{(y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} : y_1^2 + \dots + y_n^2 - y_{n+1}^2 = -a^2\}$$

with standard Lorentz-Minkowski metric $g(v, w) = \frac{1}{a^2}(v_1 w_1 + \dots v_n w_n - v_{n+1} w_{n+1})$. The explicit form of these solutions and their uniqueness on balls $B_a(0)$ was proved by Loewner, Nirenberg [6].

2.2. **A fourth order analog of Yamabe's equation.** For $n \neq 4$ let $\bar{g}$ be given as $\bar{g}_{ij} = u^{4/(n-4)} \delta_{ij}$. Then the conformal factor $u : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\Delta^2 u = \frac{n-4}{2} Q_{\bar{g}} u^{\frac{n+4}{n-4}}, \tag{2.2}$$

where

$$Q_{\bar{g}} = \frac{-1}{2n-2} \Delta R_{\bar{g}} + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_{\bar{g}}^2 - \frac{2}{(n-2)^2} |Ric_{\bar{g}}|^2$$

and $R_{\bar{g}}$, $Ric_{\bar{g}}$ are scalar curvature and Ricci curvature of $\bar{g}$, respectively. Generalizations of (2.2) to the case where g, $\bar{g}$ are conformally related Riemannian metrics on a Riemannian manifold lead to more complicated fourth order equations involving the Paneitz operator instead of Δ^2, cf. Chang [2] and Chang, Yang [3].

The quantity $Q_{\bar g}$ is a curvature term with $Q_{\bar g} \equiv 0$ in dimension $n = 2$. If we assume $Q_{\bar g} \equiv \frac{1}{8} n(n^2 - 4)$ then via a scaling (1.1) and (2.2) are equivalent. In this case (2.2) has the following explicit solutions

$$U(r) = \big(\frac{2a}{a^2 \pm r^2}\big)^{(n-4)/2} \quad \text{with } r = |x - x_0|,\ a > 0$$

producing the same metrics as before, i.e, the metrics representing $\mathbb{S}^n$ and $\mathbb{H}^n$. For the case of "+" and $n \geq 5$, uniqueness of the above family was shown by Juncheng Wei and Xingwang Xu [9]. In the case $n = 3$ uniqueness fails, as it follows from our main result Theorem 3.1. For the case of "−" uniqueness for (2.2) on a ball is open to the best of our knowledge.

2.3. **The uniqueness result of Choi and Xu.** The uniqueness result of Choi and Xu in dimension $n = 3$ has the following geometric meaning: *if u is asymptotically linear as $|x| \to \infty$ then $(\mathbb{R}^3, \bar g)$ is isometrically isomorphic to a standard sphere $\mathbb{S}^3$.* The requirement of u being asymptotically linear means that the metric $\bar g = u^{-4}\delta_{ij}$ on $\mathbb{R}^3$ can be pulled back via inverse stereographic projection to a metric on $\mathbb{S}^3$. However our main result shows that many other radial solutions u of (2.2) exist – in striking contrast to (2.1) for $n \geq 3$ and (2.2) for $n \geq 5$. These metrics cannot be realized as metrics on $\mathbb{S}^3$ but only on $\mathbb{S}^3 \setminus \{P\}$, i.e., on the sphere with one point removed.

A second Theorem of Choi and Xu states the following: if u is an arbitrary entire solution of (1.1) such that the scalar curvature $R_{\bar g}$ of the metric $\bar g_{ij} = u^{4/(n-4)}\delta_{ij}$ is everywhere non-negative then u must be of the form (1.2). Geometrically this means: *if u induces a metric $\bar g$ with everywhere non-negative scalar curvature then $(\mathbb{R}^3, \bar g)$ is isometrically isomorphic to a standard sphere $\mathbb{S}^3$.*

This shows also, that the special solution $U(r)$ of type (1.2) is distinguished from the infinitely many other solutions $u(r)$ found in Theorem 3.1, since they induce metrics on $\mathbb{R}^3$ with sign-changing scalar curvature.

3. Radial solutions of $\Delta^2 u = -u^{-q}$ for $n \geq 3$

We restrict our analysis to radial solutions of (1.1). For this class of solutions the biharmonic operator simplifies to $(\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr})^2$. Therefore we investigate the initial value problem

$$\Big(\frac{d^4}{dr^4} + \frac{2(n-1)}{r}\frac{d^3}{dr^3} + \frac{(n-1)(n-3)}{r^2}\big(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\big)\Big)u + u^{-q} = 0, \tag{3.1}$$

$$u(0) = 1, u'(0) = 0, u''(0) = \delta, u'''(0) = 0. \tag{3.2}$$

In contrast to *entire* solutions, which exist on $(0, \infty)$, we say that a solution u has *compact support* if u is positive on some interval $(0, R)$ and $u(R) = 0$, since then the solution stops to exist. In the following we will say that a real-valued function $f(s), s \in \mathbb{R}$ is *increasing* if $s_1 < s_2$ implies $f(s_1) \leq f(s_2)$. It is *strictly increasing* if $s_1 < s_2$ implies $f(s_1) < f(s_2)$. Similarly we use the word *decreasing* and *strictly decreasing*. We have the following results:

Theorem 3.1. *All solutions of (3.1)-(3.2) are strictly ordered with respect to δ. For $n \geq 3$ and $q > 1$ the following types of solutions are known: there exists a value $\delta_0 > 0$ such that*

(a) *for $-\infty < \delta < \delta_0$ every solution has compact support,*
(b) *for $\delta \geq \delta_0$ every solution is entire,*

(c) *the entire solution* u_0 *with* $u_0''(0) = \delta_0$ *is a separatrix, i.e.,*

$$\begin{aligned} u_0 &= \sup\{u : \ u \text{ is a compact support solution}\} \\ &= \inf\{u : \ u \text{ is an entire solution}\}, \end{aligned}$$

(d) *if* $R(\delta)$ *is the first zero of the solution* u *with* $u''(0) = \delta$, $-\infty < \delta < \delta_0$ *then* $R(\delta)$ *is a continuous, strictly monotone function with* $R(\delta) \to \infty$ *as* $\delta \to \delta_0$ *and* $R(\delta) \to 0$ *as* $\delta \to -\infty$,

(e) *for* $\epsilon > 0$ *sufficiently small there exist solutions which grow faster than* $r^{2-\epsilon}$,

(f) *no solution grows faster than* r^2.

Our proof depends on the construction of suitable sub-, supersolutions and the use of the comparison principle. This technique depends on the fact that (3.1) can be rewritten as a second-order system

$$(r^{n-1}u')' = r^{n-1}U, \quad (r^{n-1}U')' + r^{n-1}u^{-q} = 0. \tag{3.3}$$

Notice that $U(0) = nu''(0)$, $U'(0) = \frac{n+1}{2}u'''(0)$. The next lemma is a comparison result between upper and lower solutions of (3.3). In spirit it follows from corresponding comparison results of Walter [8] for quasimonotone systems.

Lemma 3.2 (Comparison Principle)**.** *Let* (v, V) *and* (w, W) *be two pairs of* C^2*-functions on the interval* $[0, R)$ *with* $v, w > 0$ *on* $[0, R)$ *and with*

$$\begin{gathered} (r^{n-1}v')' = r^{n-1}V, \quad (r^{n-1}w')' = r^{n-1}W, \\ (r^{n-1}V')' + r^{n-1}v^{-q} \le 0, \quad (r^{n-1}W')' + r^{n-1}w^{-q} \ge 0 \end{gathered}$$

on $(0, R)$. *Then the following holds:*

(a) *(Weak comparison) If* $v(0) \le w(0)$, $v'(0) = w'(0) = 0$ *and* $V(0) \le W(0)$, $V'(0) = W'(0) = 0$ *then* $v \le w$, $v' \le w'$, $V \le W$ *and* $V' \le W'$ *on* $[0, R)$.

(b) *(Strong comparison) If for some* $\rho > 0$ *we have* $v < w$ *on the interval* $(0, \rho)$ *then* $v < w$, $v' < w'$, $V < W$, $V' < W'$ *on* $(0, R)$. *A simple way to achieve* $v < w$ *initially is to have* **one** *strict inequality in the initial conditions.*

Proof. We begin with proving part (b). Suppose $v < w$ initially on a small interval $(0, \rho)$. By the second differential inequality we find $(r^{n-1}V')' < (r^{n-1}W')'$ on $(0, \rho)$. By a double integration we get $V < W$ on $(0, \rho)$. Now let $(0, c)$ be the largest interval on which $V < W$, and suppose for contradiction that $c < R$. Using $V < W$ and the first differential inequality we get $(r^{n-1}v')' < (r^{n-1}w')'$ on $(0, c)$ and by a double integration $v' < w'$ and $v < w$ on $(0, c)$. Inserting this into the second differential inequality and integrating twice we get $V' < W'$ on $(0, c)$ and hence $V < W$ on the semi-closed interval $(0, c]$. This contradicts the assumption that $(0, c)$ is the largest interval for which $V < W$. Therefore we have strict inequalities between v, v', V, V' and w, w', W, W' on the entire interval $(0, R)$.

Part (a) follows from (b) in the following way. Let $f := (r^{n-1}V')' + v^{-q}$ be the defect in the differential inequality for V, and let (v_ϵ, V_ϵ) be the solution of the initial value problem

$$\begin{gathered} (r^{n-1}v_\epsilon')' = r^{n-1}V_\epsilon, \quad v_\epsilon(0) = v(0) - \epsilon, v_\epsilon'(0) = 0, \\ (r^{n-1}V_\epsilon')' + r^{n-1}v_\epsilon^{-q} = f \le 0, \quad V_\epsilon(0) = V(0), V_\epsilon'(0) = 0. \end{gathered}$$

Note that for $\epsilon = 0$ the uniqueness of the initial value problem gives $(v_0, V_0) = (v, V)$. Moreover, for $\epsilon > 0$ the pairs (v_ϵ, V_ϵ) and (w, W) satisfy the hypotheses of part (b), and we can deduce that $(v_\epsilon, v'_\epsilon, V_\epsilon, V'_\epsilon) < (w, w', W, W')$ on $(0, R)$. Letting ϵ tend to 0, the strict inequality becomes a weak inequality. This proves the lemma. $\square$

Lemma 3.2 will be applied to problem (3.1)-(3.2) in the following way. Suppose two positive C^4-functions $v(r), w(r)$ are given with

$$\Delta^2 v + v^{-q} \le 0, \quad v(0) \le 1, \quad v'(0) = 0, \quad v''(0) \le \delta, \quad v'''(0) = 0$$
$$\Delta^2 w + w^{-q} \ge 0, \quad w(0) \ge 1, \quad w'(0) = 0, \quad w''(0) \ge \delta, \quad w'''(0) = 0$$

then v, w are called a sub,- supersolutions relative to the initial value problem

$$\Delta^2 u + u^{-q} = 0, \quad u(0) = 1, u'(0) = 0, \quad u''(0) = \delta, \quad u'''(0) = 0.$$

Lemma 3.2 applied to $(u, u'' + \frac{n-1}{r} u')$ with either $(v, v'' + \frac{n-1}{r} v')$ or $(w, w'' + \frac{n-1}{r} w')$ yields the conclusion that $v \le u \le w$, $v' \le u' \le w'$ on their common interval of existence. Moreover, strict inequality holds as soon as one strict inequality holds in the initial conditions for the function or its second derivative.

Lemma 3.3. *There exists a value $\tilde\delta > 0$ such that for all $\delta \le \tilde\delta$ the solution of* (3.1)-(3.2) *has compact support.*

Proof. Consider the function $w(r) = \epsilon r^2(A - r^2) + 1$ for $\epsilon, A > 0$, which is positive on $(0, \sqrt{A})$. Then $\Delta^2 w = -8\epsilon n(n+2)$. In order to have w as a compact-support supersolution we need $\Delta^2 w + w^{-q} \ge 0$, i.e.

$$-8\epsilon n(n+2) + \big(\epsilon r^2(A - r^2) + 1\big)^{-q} \ge 0 \quad \text{on } (0, \sqrt{A}).$$

The maximum of w over $(0, \sqrt{A})$ is obtained at $\sqrt{A/2}$. Therefore the above equation is satisfied provided

$$\epsilon \frac{A^2}{4} + 1 \le \big(8\epsilon n(n+2)\big)^{-1/q}.$$

For a given value of ϵ such that $0 < \epsilon < 1/(8n(n-2))$ the largest possible value of of A is given by

$$A(\epsilon) := \frac{2}{\sqrt{\epsilon}} \sqrt{(8\epsilon n(n+2))^{-1/q} - 1}$$

and clearly $A(\epsilon) \to \infty$ for $\epsilon \to 0$ and $A(\epsilon) \to 0$ as $\epsilon \to 1/(8n(n+2))$. Furthermore $w(0) = 1, w'(0) = w'''(0) = 0$, and $w''(0) = 2\epsilon A(\epsilon)$ has the properties that $w''(0) \to 0$ as $\epsilon \to 0$ and as $\epsilon \to 1/(8n(n+2))$. The second derivative of w at 0 is maximal for $\epsilon = (1 - 1/q)^q \, 1/(8n(n+2))$ and the value of $w''(0)$ is $\tilde\delta := 4\sqrt{(1 - 1/q)^q / (8n(n+2)(q-1))}$. As a result we have that every solution of (3.1)-(3.2) with $\delta \le \tilde\delta$ stays below $w(r)$, and hence it has compact support. $\square$

Lemma 3.4. *No entire solution of* (3.1)-(3.2) *grows faster than r^2.*

Proof. Let $w(r) = Ar^2 + 1$. Then $\Delta^2 w = 0$ and hence w is a supersolution. Given an arbitrary entire solution u of (3.1)-(3.2) we can choose $2A > u''(0)$. Hence $u < w$ by the comparison principle of Lemma 3.2. $\square$

Lemma 3.5. *There exists $\epsilon_0 = \epsilon_0(n) > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ there exists a value $\bar\delta = \bar\delta(\epsilon)$ with the property that every solution of* (3.1)-(3.2) *with $\delta \ge \bar\delta$ is entire and grows faster than $r^{2-\epsilon}$.*

Proof. For the construction of an entire subsolution we consider $v(r) = (1 + b^2r^2)^{1-\frac{\epsilon}{2}}$. Let $p(r)$ be the function such that

$$\Delta^2 v + p(r)v^{-q} = 0 \quad \text{on } (0,\infty).$$

With the help of MAPLE we compute

$$p(r) = 2b^4\epsilon(1-\epsilon/2)(1+b^2r^2)^{-3-\frac{\epsilon}{2}(1+q)+q}\rho(r)$$

where

$$\rho(r) = \big(b^4r^4(\epsilon^2 - 2n\epsilon + 2\epsilon - 2n + n^2) + b^2r^2(-4\epsilon - 2n\epsilon + 2n^2 - 8) + n^2 + 2n\big)$$

For small $\epsilon < \epsilon_0(n)$ and for $n \geq 3$ we see that

$$c(1+b^2r^2)^2 \leq \rho(r) \leq C(1+b^2r^2)^2 \quad \text{on } (0,\infty)$$

for two constants c, C independent of b and ϵ. Hence

$$p(r) \geq 2cb^4\epsilon(1-\epsilon/2)(1+b^2r^2)^{-1-\frac{\epsilon}{2}(1+q)+q} \geq 1$$

if we ensure that $\epsilon < 2(q-1)/(1+q)$ and if we choose b sufficiently large. For this choice of the parameters ϵ and b we find

$$\Delta^2 v + v^{-q} \leq 0 \quad \text{on } (0,\infty),$$

i.e., v is indeed a subsolution with $v(0) = 1$, $v'(0) = v'''(0) = 0$ and $v''(0) = b^2(2-\epsilon)$. Together with the supersolution $w = Ar^2 + 1$ for large A from Lemma 3.4, they give rise to an entire solution with growth rate larger than $r^{2-\epsilon}$. $\square$

Proof of Theorem 3.1. Part (a): By Lemma 3.3 there is a $\tilde\delta > 0$ such that the solution of (3.1)-(3.2) with $u''(0) = \tilde\delta$ has compact support. Via the comparison principle we see that for $-\infty < \delta < \tilde\delta$ the solutions also have compact support. Therfore, we may define

$$\delta_0 := \sup\big\{\delta: \text{ the solution with } u''(0) = \delta \text{ has compact support.}\big\}$$

By the entire solution found in Lemma 3.5 the value δ_0 is finite and positive, and any solution with $u''(0) > \delta$ must be entire. As we will see in the proof of Part (d), the first zero $R(\delta)$ tends to ∞ as $\delta \to \delta_0$. Therefore the separatrix-solution u with $u''(\delta_0)$ must be entire, too. Hence Part (b) and (c) are established. Part (e) follows from Lemma 3.5, Part (f) from Lemma 3.4.

Part (d): Let $R(\delta)$ be the first zero of the solution u with $u''(0) = \delta$. Notice that $u(r) = -\int_r^{R(\delta)} u'(t)\,dt$ and that u is absolutely continuous on $[0, R(\delta)]$. We want to show that R is a continuous function of δ with $R \to 0$ as $\delta \to -\infty$ and $R \to \infty$ as $\delta \to \delta_0$. By the comparison principle the function $R(\delta)$ is monotone in δ. Moreover, for two solution u_1, u_2 with $u_1''(0) = \delta_1 < \delta_2 = u_2''(0)$ we find by the comparison principle that $(u_2 - u_1)' > 0$, i.e. the gap between the solutions is increasing. Therefore $R(\delta)$ is a strictly monotone function of δ, and hence continuity can only fail if $R(\delta)$ has jump-discontinuities, which are excluded by the continuous dependence of the solution on initial values. Next we assume for contradiction that $R(\delta)$ tends to a finite limit as $\delta \to \delta_0$. Since the solutions u with $u''(0) > \delta_0$ must be entire by the definition of δ_0 we get again a contradiction to the continuous dependence principle. Similarly, $R(\delta)$ cannot approach a positive limit as $\delta \to -\infty$. $\square$

4. Entire solutions of $\Delta^2 u = -u^{-q}$ for $n = 3$

Since radially symmetric functions in $\mathbb{R}^3$ satisfy $\Delta^2 u(r) = u^{(iv)} + 4u'''/r$ we can prove the following variant of the comparison principle of Lemma 3.2:

Lemma 4.1 (Comparison principle for $n = 3$). *Let (v, w) be a pair of C^4-functions on the interval $[0, R)$ with $v, w > 0$ on $[0, R)$ and with*

$$v^{(iv)} + \frac{4}{r}v''' + v^{-q} \le 0, \quad w^{(iv)} + \frac{4}{r}w''' \ge 0 \quad \text{on } (0, R).$$

Then the following holds:

(a) *(Weak comparison) If $v(0) \le w(0)$, $v'(0) = w'(0) = 0$, $v''(0) \le w''(0)$, $v'''(0) = w'''(0) = 0$ then $v \le w$, $v' \le w'$, $v'' \le w''$ and $v''' \le w'''$ on $[0, R)$.*

(b) *(Strong comparison) If for some $\rho > 0$ we have $v < w$ on the interval $(0, \rho)$ then $v < w$, $v' < w'$, $v'' < w''$, $v''' < w'''$ on $(0, R)$. A simple way to achieve $v < w$ initially is to have* **one** *strict inequality in the initial conditions.*

Proof. First we notice that the differential inequality for v is equivalent to

$$\begin{aligned} v_1' &= v_2, \quad v_1(0) = 1, \\ v_2' &= v_3, \quad v_2(0) = 0, \\ v_3' &= v_4, \quad v_3(0) = \delta, \\ v_4' &\le -v_4/r - v_1^{-q}, \quad v_4(0) = 0, \end{aligned} \tag{4.1}$$

with a similar system for w. System (4.1) is of quasimonotone type, i.e., the j-th right-hand side, $j = 1, \dots, 4$ is increasing in the variables v_k for all $k \in \{1, \dots, 4\} \setminus \{j\}$. Thus the comparison theorem from Walter [8] for quasimonotone systems of ordinary differential equations applies. $\square$

Let $u : [0, \infty) \to \mathbb{R}$ and let a, b, A, B be positive constants (depending on u). We say that u has

linear growth	if $ar + b \le u(r) \le Ar + B$,
exactly linear growth	if $\lim_{r\to\infty} u(r)/r > 0$,
at least linear growth	if $u(r) \ge ar + b$,
superlinear growth	if $u(r) \ge ar + b$, but u does not have linear growth.

The following theorem gives a more detailed picture of the entire radial solutions in dimension $n = 3$. A stronger version of part (b) including non-radial solutions has been obtained by Choi and Xu [5] as part of their main result. Here we give a different proof. To the best of our knowledge, part (a) is new for $q > 7$.

Theorem 4.2. *For $n = 3$ entire solutions of (3.1)-(3.2) have the following properties:*

(a) *For $q \ge 7$ a unique solution with linear growth exist. It coincides with the separatrix and has exactly linear growth. For $q = 7$ it is given by $\sqrt{1 + r^2/\sqrt{15}}$.*

(b) *For $4 < q < 7$ there is no solution with linear growth.*

(c) *For $4 < q < 7$ the separatrix has superlinear growth.*

(d) *For $1 < q \le 4$ the separatrix has at least linear growth.*

It is unknown whether in (b) and (c) the condition $4 < q < 7$ can be weakened to $0 < q < 7$.

Interpretation. To understand Part (b) of Theorem 4.2 let us say that $u^{\frac{2n}{n-4}} = u^{-6}$ has "critical growth" and set $F(u) = u^{1-q}/(1-q)$. Then $F(u)/u^{-6}$ is decreasing for $1 < q < 7$. In other words u^{-q} for $1 < q < 7$ has "subcritical growth". For "subcritical growth" it is well known that (2.1) and (2.2) with positive exponents have no entire positive solution. Part (b) of Theorem 4.2 is such a subcritical non-existence result for certain solution classes.

It was observed by Choi and Xu that the uniqueness of the explicit solution $\sqrt{1 + r^2/\sqrt{15}}$ can also be obtained by a geometric condition different to the linear growth condition:

Corollary 4.3 (Choi and Xu)**.** *For $n = 3$ and $q = 7$ the uniqueness result of Part (a) of Theorem 4.2 holds in the class of functions u such that the scalar curvature of $\bar{g} = u^{\frac{4}{n-4}}\delta_{ij}$ is everywhere positive.*

Proof. The scalar curvature of $\bar{g}$ is given as (see Section 2.1)

$$R_{\bar{g}} = -4\frac{n-1}{n-2} u^{\frac{n+2}{4-n}} \Delta u^{\frac{n-2}{n-4}} \tag{4.2}$$

which reduces in dimension $n = 3$ and for a radial function u to

$$R_{\bar{g}} = \frac{8}{r^2} u^5 (r^2 u' u^{-2})'. \tag{4.3}$$

Hence, positivity of $R_{\bar{g}}$ implies that $r^2 u' u^{-2}$ is increasing. Integration of the inequality $r^2 u' u^{-2} \geq c_0$ for $r \geq r_0$ implies $1/u(r) - 1/u(s) \geq c_0/r - c_0/s$ for $r_0 < r < s$. Letting $s \to \infty$ we obtain $u(r) \leq r/c_0$ for $r \geq r_0$. Since u is also convex, cf. Lemma 4.4, this implies that u has linear growth, and Theorem 4.2 applies. $\square$

For radially symmetric functions in $\mathbb{R}^3$ we find

$$\Delta^2 u(r) = u^{(iv)} + 4u'''/r = r^{-4}(r^4 u''')'.$$

Lemma 4.4. *For $n = 3$ a solution u of* (3.1)-(3.2) *is entire if and only if it is convex.*

Remark. It is unknown whether this classification of entire solutions also holds for $n \geq 4$.

Proof of Lemma 4.4. If u is convex then u is entire. Now suppose there exists r_0 such that $u''(r_0) < 0$. Since $(r^4 u''')' \leq 0$ and $u'''(0) = 0$ we find that $u''' \leq 0$, i.e. u'' is decreasing. Therefore u'' stays negative once it is negative at r_0. Thus u cannot be entire. $\square$

Lemma 4.5. *Suppose $n = 3$. Let $u(r)$ be an entire solution of* (3.1)-(3.2) *with linear growth. Then there exists a constant $C > 0$ such that $0 \leq u'(r) \leq C$ if $q > 3$ and moreover:*

$$0 \leq u'' \leq \begin{cases} C(1+r)^{-3} & \text{if } q > 5, \\ C(1+r)^{-3}\log(2+r) & \text{if } q = 5, \\ C(1+r)^{-q+2} & \text{if } 2 < q < 5, \end{cases}$$

$$0 \geq u''' \geq \begin{cases} -C(1+r)^{-4} & \text{if } q > 5, \\ -C(1+r)^{-4}\log(2+r) & \text{if } q = 5, \\ -C(1+r)^{-q+1} & \text{if } 1 < q < 5. \end{cases}$$

Proof. We begin with the estimate for u'''. Since $(r^4u''')' \le 0$ and $u'''(0) = 0$ we see that $u''' \le 0$. Hence
$$0 \ge u'''(r) = \frac{-1}{r^4}\int_0^r s^4 u(s)^{-q}\,ds \ge \frac{-C}{r^4}\int_0^r (as+b)^{4-q}\,ds.$$
The estimate follows by performing the integration. Likewise, since
$$u''(r) - u''(s) = \int_s^r u'''(t)\,dt$$
and since $\int^\infty u'''(t)\,dt$ converges for $q > 2$ we have that $\lim_{r\to\infty} u''(r)$ exists and vanishes due the assumption of linear growth. Moreover $u'' > 0$ by Lemma 4.4. Thus
$$0 \le u''(s) = \int_s^\infty -u'''(t)\,dt,$$
and the estimate for u'' follows by integrating the one for u'''. A final integration leads to $0 \le u'(r) = \int_0^r u''(t)\,dt < \infty$ provided $q > 3$. □

Lemma 4.6. *Suppose $n = 3$. Let $q > 4$ and suppose u is a solution of* (3.1)-(3.2) *with linear growth. Then $\lim_{r\to\infty} u - ru'$ exists.*

Proof. Integration by parts yields
$$\begin{aligned}\frac{1}{2}\int_0^r u^{-q}s^3\,ds &= \frac{-1}{2}\int_0^r (s^4u''')'\frac{1}{s}\,ds\\ &= -\frac{r^3}{2}u'''(r) - \frac{r^2}{2}u''(r) + ru'(r) - u(r) + u(0).\end{aligned}$$
By the assumption $q > 4$ we find that the left-hand side converges as $r \to \infty$. Moreover, by Lemma 4.5, $r^3u''', r^2u'' \to 0$ as $r \to \infty$. Hence
$$\lim_{r\to\infty} u - ru' = u(0) - \frac{1}{2}\int_0^\infty u^{-q}r^3\,dr,$$
as claimed. □

Lemma 4.7. *Suppose $n = 3$. Let $q > 4$ and suppose u is a solution of* (3.1)-(3.2) *with linear growth. Then $u - ru' \ge 0$ on $[0,\infty)$.*

Proof. Let $h = u - ru'$. We derive a differential inequality for h. By direct computation
$$\Delta^2 h = -7u^{(iv)} - ru^{(v)} - \frac{8}{r}u'''.$$
Differentiation of (3.1) and multiplication by r yields
$$ru^{(v)} + 4u^{(iv)} - \frac{4}{r}u''' = rqu^{-1-q}u'.$$
Substituting this into the expression for $\Delta^2 h$ and using (3.1) again we get
$$\begin{aligned}\Delta^2 h &= -rqu^{-1-q}u' + 3u^{-q}\\ &= 3u^{-1-q}h + (3-q)ru^{-1-q}u'\\ &\le 3u^{-1-q}h\end{aligned}\tag{4.4}$$
since $q > 4$ by assumption. Notice that h is decreasing since $h' = -ru'' < 0$. Suppose for contradiction that $h(r) < 0$ for $r \ge r_0$. Then (4.4) implies $\Delta^2 h < 0$ on (r_0,∞), i.e., $h'''r^4$ is decreasing on (r_0,∞). Now we distinguish two cases:

Case (a): h''' is negative somewhere in (r_0, ∞). Then h''' stays negative, say, on (r_1, ∞), i.e. h' is a concave, negative function on (r_1, ∞). This implies that h is unbounded below, which is impossible by Lemma 4.6.

Case (b): $h''' \geq 0$ in (r_0, ∞). Then h''' is decreasing on (r_0, ∞), i.e., h'' is concave on (r_0, ∞). For large enough r_1 we have either case (b1) $h'' < 0$ on (r_1, ∞) or case (b2) $h'' > 0$ on (r_1, ∞). In case (b1) h is a concave decreasing function contradicting Lemma 4.6. Hence we can assume case (b2), i.e., $h'' > 0$ on (r_1, ∞). Thus h'' is a positive concave function on (r_1, ∞) and hence increasing at ∞. However, by Lemma 4.5, we have $r^2 h''(r) \to 0$ as $r \to \infty$. This is incompatible with the fact that h'' is positive increasing at ∞, and finishes the discussion of case (b). □

Lemma 4.8 (Pohožaev's identity)**.** *Suppose $n \geq 3$. Let u be an entire solution of (3.1)-(3.2). Then the following identity holds*

$$\begin{aligned}
&\int_0^\rho u^{1-q}\Big(\frac{n}{1-q} - \frac{n-4}{2}\Big) r^{n-1}\,dr \\
&= -\frac{\rho^n}{2}(u'')^2 + \frac{\rho^n}{1-q}u^{1-q} + \frac{n}{2}\rho^{n-1}u'u'' + \frac{(n-1)(n-4)}{2}\rho^{n-2}uu'' \\
&\quad + \frac{n-4}{2}\rho^{n-1}uu''' + \rho^n u'u''' - \frac{n-1}{2}\rho^{n-2}(u')^2 - \frac{(n-1)(n-4)}{2}\rho^{n-3}uu'
\end{aligned}$$

for every $\rho > 0$.

Proof. The result follows from a general identity of Pucci, Serrin [7], Proposition 4, for the one-dimensional Lagrangian $\mathcal{F} = (\frac{1}{2}(u'' + \frac{n-1}{r}u')^2 + \frac{1}{1-q}u^{1-q})r^{n-1}$. □

Proof of Theorem 4.2. Part (a): For $q \geq 7$ the function $U(r) = \sqrt{1 + r^2/\sqrt{15}}$ is a subsolution. Therefore every compact support solution stays below $U(r)$, and thus the separatrix must stay below $U(r)$. Hence, the separatrix $S(r)$ has linear growth. Let $t(r)$ be the slope of the tangent of $S(r)$. By convexity, $t(r)$ is increasing, and by the upper bound $U(r)$ we see that $t(r)$ is bounded, i.e. convergent with $t = \lim_{r\to\infty} t(r)$. Hence the separatrix $S(r)$ has exactly linear growth with slope t.

Let u be an entire, linear growth solution with $u(0) = 1$. If $u''(0) < S''(0)$ then by Lemma 4.5 we have $u'''(r) < S'''(r)$, and hence $c := (S-u)''(0) < (S-v)''(r)$, i.e. $u''(r) \leq S''(r) - c$. However, since $S''(r) \to 0$ as $r \to \infty$ by Lemma 4.5, we get that $u''(r) < 0$ for large r. Hence u becomes concave eventually, stays concave, and hence cannot be entire. Now suppose $u''(0) > S''(0)$. Then $u''(r) \geq S''(r) + u''(0) - S''(0) > 0$, i.e. $\liminf_{r\to\infty} u''(r) > 0$. This contradicts Lemma 4.5. Hence, among all solutions with linear growth, $S(r)$ is unique. This uniqueness shows that in case $q = 7$ the separatrix $S(r)$ must have the explicit form $\sqrt{1 + r^2/\sqrt{15}}$.

Part (b): Suppose $4 < q < 7$. Then in the right-hand side of Pohožaev's identity all terms except the last two converge individually to 0 as $\rho \to \infty$. The last two terms are

$$uu' - \rho(u')^2 = u'(u - \rho u') \geq 0$$

by Lemma 4.7. Hence the lim inf of the entire right hand side is ≥ 0. In contrast, the left-hand side is negative. Hence no linear growth solution exists.

Part (c): For $4 < q < 7$ the separatrix cannot have linear growth. By convexity it has a least linear growth, i.e, it has superlinear growth.

Part (d): The reasoning of Part (c) shows that the separatrix has at least linear growth. However, linear growth itself can no longer be excluded. □

5. Compact support solutions of $\Delta^2 u = -u^{-q}$ for $n = 3$

Let u be a function with support $[0, R]$ and let a, A be a positive constants (depending on u). We say that u has

square root growth if $a\sqrt{R-r} \le u(r) \le A\sqrt{R-r}$
exactly square root growth if $\lim_{r\to R} u(r)/\sqrt{R-r} > 0$,
at least square root growth if $a\sqrt{R-r} \le u(r)$.

Theorem 5.1. *For $n = 3$ compact support solutions of (3.1)-(3.2) have the following properties:*

(a) *for $6 < q < 7$ there is no compact support solution with square-root growth at its zero,*
(b) *for $1 < q \le 6$ there is no compact support solution with a least square-root growth at its zero.*
(c) *For $q = 7$ there are compact support solutions with exactly square root growth given by $\alpha\sqrt{1/\sqrt{15\alpha^8} - |x|^2}$, $\alpha > 0$ (here we have dropped the requirement $u(0) = 1$).*

Unlike the entire solution situation, we do not know how to uniquely select the special solutions of Theorem 5.1(c). In fact, the requirement of negative scalar curvature is not enough:

Corollary 5.2. *For $q = 7$ there are infinitely many solutions with support $[0, R]$ which generate a metric with negative scalar curvature. Each of them can be pulled back by stereographic projection to a metric on hyperbolic space $\mathbb{H}^3_R$.*

One might conjecture that exact square root growth near its zero uniquely selects the explicit solution of Theorem 5.1(c). So far, we do not know whether this holds true. Geometrically the square root condition means that $\bar g = u^{-4}\delta_{ij}$ can be pulled back to $\mathbb{H}^3$ via inverse stereographic projection and the resulting metric $\hat g$ on $\mathbb{H}^3$ has the property that $\hat g/g_{\mathbb{H}^3} \to$ const. at ∞.

Lemma 5.3. *Suppose $n = 3$. Let $u(r)$ be a compact support solution of (3.1)-(3.2) with at least square-root growth. Then there exists a constant $C > 0$ such that*

$$|u'| \le \begin{cases} C(R-r)^{3-\frac{q}{2}} & \text{if } q > 6,\\ C|\log(R-r)| & \text{if } q = 6,\\ C & \text{if } 1 < q < 6,\end{cases}$$

$$|u''| \le \begin{cases} C(R-r)^{2-\frac{q}{2}} & \text{if } q > 4,\\ C|\log(R-r)| & \text{if } q = 4,\\ C & \text{if } 1 < q < 4,\end{cases}$$

$$|u'''| \le \begin{cases} C(R-r)^{1-\frac{q}{2}} & \text{if } q > 2,\\ C|\log(R-r)| & \text{if } q = 2,\\ C & \text{if } 1 < q < 2.\end{cases}$$

Proof. We begin with the estimate for u'''. As in Lemma 4.5 we see that $u''' \le 0$. Hence

$$0 \ge u'''(r) = \frac{-1}{r^4}\int_0^r s^4 u(s)^{-q}\,ds \ge -C\int_0^r (R-r)^{-q/2}\,ds.$$

The estimate follows by performing the integration. Likewise,

$$|u''(r)| \le C + \int_0^r |u'''(t)|\,dt$$

leads to the estimate on u'', and a further integration leads to

$$|u'(r)| \le \int_0^r |u''(t)|\,dt < \infty$$

and its subsequent estimate. □

Proof of Theorem 5.1. Part (b): If $1 < q < 6$ then u' is bounded, which is incompatible with the assumption that u has at least square root growth. The same holds if $q = 6$ since integration of the u' estimates yields $|u| \le C(R-r)|\log(R-r)|$, which is again incompatible with the square root lower bound.

Part (a): Assume $q > 6$. We use again Pohožaev's identity for $\rho \in (0, R)$. For the terms in the right-hand side we find $(u'')^2 \le C(R-\rho)^{4-q}$, $|u''u'| \le C(R-\rho)^{5-q}$, $|u'u'''| \le C(R-\rho)^{4-q}$, $|uu''| \le C(R-\rho)^{(5-q)/2}$, $|uu'''| \le C(R-\rho)^{(3-q)/2}$. It turns out that the most singular term is $(R-\rho)^{4-q}$. However, the remaining term $u^{1-q} \approx (R-\rho)^{(1-q)/2}$ is more singular provided $q < 7$. Here we use that u is bounded above by multiples of $\sqrt{R-r}$. Hence, the right-hand side converges to $-\infty$ with rate $-(R-\rho)^{(1-q)/2}$ as $\rho \to R$. The left-hand side is also negative for $q < 7$. If we use that u is bounded below by multiples of $\sqrt{R-r}$ then we see that the left hand side converges to $-\infty$ with the less singular rate $-(R-\rho)^{(3-q)/2}$. Hence there is no solution with square root growth for $6 < q < 7$. □

Proof of Corollary 5.2. If we prescribe $u(0) = \alpha > 0$ then by Theorem 3.1 there exists $\delta_0 > 0$ such that every solution with $u''(0) = \delta < \delta_0$ has compact support. Moreover, the zero $R(\delta)$ ranges continuously between 0 and $+\infty$ if δ ranges between $-\infty$ and δ_0. Hence for α and $R > 0$ there exists δ^* such that the solution with $u''(0) = \delta^*(\alpha)$ has exactly support $[0, R]$. If $\alpha \to \infty$ then necessarily $\delta^*(\alpha) \to -\infty$, i.e. the solutions are concave and decreasing. Hence $u'r^2/u^2$ is decreasing and thus u generates a metric with negative scalar curvature, cf. (4.3). □

6. Open questions

We finish with a selection of questions which remain open.

(1) For $n \ge 4$ the equation $\Delta^2 u = -u^{-q}$ in $\mathbb{R}^n$ has non-radial positive entire solutions given by $u(x', x_n) = v(|x'|)$, where $v(r)$ is a radial positive entire solution satisfying $\Delta^2 v = -v^{-q}$ in $\mathbb{R}^{n-1}$. This leaves the question whether in $\mathbb{R}^3$ non-radial positive entire solution exist. The above construction does not work, since Theorem 3.1 requires $n-1 \ge 3$ for the existence of a solution v.

(2) Do there exist positive entire solutions of $\Delta^2 u = -u^{-q}$ in $\mathbb{R}^2$? A positive answer would resolve question (1).

(3) Can one find an explicit formula for the growth rate of the separatrix in terms of q?

(4) For $n = 3$, can one drop the assumption $q > 4$ in Theorem 4.2?

(5) Suppose $q = 7$ and $n = 3$. Are the explicit compact support solutions $\alpha\sqrt{1/\sqrt{15\alpha^8} - |x|^2}$ unique in the class of solutions having square root growth near their zero?

(6) In $\mathbb{R}^3$ the equation $\Delta^2 u = u^{-7}$ arises from (2.2) by assuming $Q_{\bar{g}} = \text{const.} < 0$. The function $U(r) = \sqrt[4]{4/3}\sqrt{r}$ is a solution, and the generated metric $\bar{g} = U^{-4}\delta_{ij}$ has constant scalar curvature $8/3$. In what class of solutions is $U(r)$ unique?

(7) In what class of solutions are $u(r) = \left(\frac{2a}{a^2-r^2}\right)^{\frac{n-4}{2}}$ unique boundary-blow up solutions of $\Delta^2 u = \frac{n}{16}(n-4)(n^2-4)u^{\frac{n+4}{n-4}}$ in balls $B_a(0) \subset \mathbb{R}^n$ for $n \geq 5$?

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