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HOMOGENIZATION OF NONLINEAR MONOTONE OPERATORS BEYOND THE PERIODIC SETTING

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Abstract. We study the homogenization of nonlinear monotone operators beyond the classical periodic setting. The usual periodicity hypothesis is here replaced by an abstract assumption covering a wide range of concrete behaviours such as the periodicity, the almost periodicity, the convergence at infinity, and many more besides. Our approach is based on the recent theory of homogenization structures by the first author. The exactness of the results confirms the major role the homogenization structures are destined to play in a general deterministic homogenization theory equipped to consider the physical problems in their true perspective.

1. Introduction

Let $1 < p \leq 2$. Let $(y, \lambda) \to a(y, \lambda)$ be a function from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}^N$ with the following properties:

For each fixed $\lambda \in \mathbb{R}^N$, the function $y \to a(y, \lambda)$ (denoted by $a(\cdot, \lambda)$) from $\mathbb{R}^N$ to $\mathbb{R}^N$ is measurable (1.1)

$a(y, \omega) = \omega$ almost everywhere (a.e.) in $y \in \mathbb{R}^N$, where ω denotes the origin in $\mathbb{R}^N$ (1.2)

There exist two constants $\alpha, c > 0$ such that, a.e. in $y \in \mathbb{R}^N$:
(i) $(a(y,\lambda) - a(y,\mu)) \cdot (\lambda - \mu) \geq \alpha|\lambda - \mu|^p$
(ii) $|a(y,\lambda) - a(y,\mu)| \leq c|\lambda - \mu|^{p-1}$ (1.3)
for all $\lambda, \mu \in \mathbb{R}^N$, where the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$.

Let Ω be a bounded open set in $\mathbb{R}^N_x$ (the space $\mathbb{R}^N$ of variables $x = (x_1, \cdots, x_N)$), and let $f \in W^{-1,p'}(\Omega; \mathbb{R})$, $p' = \frac{p}{p-1}$. For each given real $\varepsilon > 0$, we consider the boundary value problem

$$-\operatorname{div} a\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big) = f \quad \text{in } \Omega,\ u_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R}), \tag{1.4}$$

which uniquely determines u_ε (see Section 6 of [18]). Here D denotes the usual gradient, i.e., $D = (D_{x_i})_{1 \leq i \leq N}$, where $D_{x_i} = \frac{\partial}{\partial x_i}$.

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We are interested in the homogenization of (1.4) (i.e., the analysis of the behaviour of u_ε when $\varepsilon \to 0$) under a suitable assumption on the behaviour of $a(y,\lambda)$ in $y \in \mathbb{R}^N$. Such an assumption will be referred to as a *structure hypothesis* [19, 20].

The common structure hypothesis is the so-called periodicity hypothesis, that is, the assumption that $a(y+k,\lambda) = a(y,\lambda)$ a.e. in $y \in \mathbb{R}^N$, where k is any arbitrary point in $\mathbb{Z}^N$ ($\mathbb{Z}$ denotes the integers). The homogenization of (1.4) under the periodicity hypothesis has been widely studied (see, e.g., [9, 10, 17, 18, 26]).

However, there is no doubt that in a great number of physical situations the periodicity hypothesis is inappropriate and should be therefore substituted by a realistic structure hypothesis. A few examples of such structure hypotheses will perhaps help us to have a clear idea of what a non periodic homogenization setting (for problem (1.4)) may look like. In what follows, a_i denotes the i^{th} component of the function $(y,\lambda) \to a(y,\lambda)$.

Example 1.1. Let $\mathcal{B}(\mathbb{R}^N)$ denote the space of all bounded continuous complex functions on $\mathbb{R}^N$, and let $\mathcal{B}_\infty(\mathbb{R}^N)$ be the space of all $\varphi \in \mathcal{B}(\mathbb{R}^N)$ such that $\varphi(y)$ has a (finite) limit when $|y| \to +\infty$. We define $\mathcal{B}_{\infty,\mathrm{per}}(Y)$ (with $Y = (0,1)^N$) to be the closure in $\mathcal{B}(\mathbb{R}^N)$ (with the supremum norm) of the set of all functions $u \in \mathcal{B}(\mathbb{R}^N)$ of the form

$$u(y) = \sum_{k\in F} \varphi_k(y) e^{2i\pi k\cdot y} \quad (y \in \mathbb{R}^N) \text{ with } \varphi_k \in \mathcal{B}_\infty(\mathbb{R}^N)$$

where F is any arbitrary finite subset of $S = \mathbb{Z}^N$. Then one natural structure hypothesis is that

$$a_i(\cdot,\lambda) \in \mathcal{B}_{\infty,\mathrm{per}}(Y) \quad \text{for any } \lambda \in \mathbb{R}^N \quad (1 \le i \le N). \tag{1.5}$$

Remark 1.1. The structure hypothesis (1.5) includes two particular cases of major interest: 1) the case where $a_i(\cdot,\lambda) \in \mathcal{B}_\infty(\mathbb{R}^N)$ for any $\lambda \in \mathbb{R}^N$ $(1 \le i \le N)$, and 2) the case of the periodicity hypothesis. However, (1.5) does not reduce to these two particular cases (see Remark 3.1 of [21]).

Example 1.2. Let $\mathcal{C}_{\mathrm{per}}(Y')$ (with $Y' = (0,1)^{N-1}$, $N \ge 2$) be the space of all continuous complex functions u on $\mathbb{R}^{N-1}$ such that $u(y'+k') = u(y')$ for all $y' = (y_1,\cdots,y_{N-1}) \in \mathbb{R}^{N-1}$ and all $k' \in \mathbb{Z}^{N-1}$ (such a function is said to be Y'-periodic). Let $\mathcal{B}_\infty(\mathbb{R};\mathcal{C}_{\mathrm{per}}(Y'))$ denote the space of all continuous functions $u : \mathbb{R} \to \mathcal{C}_{\mathrm{per}}(Y')$ such that $u(y_N)$ converges in $\mathcal{C}_{\mathrm{per}}(Y')$ (with the supremum norm) when $|y_N| \to +\infty$. Then, we may consider the homogenization of (1.4) under the structure hypothesis

$$a_i(\cdot,\lambda) \in \mathcal{B}_\infty(\mathbb{R};\mathcal{C}_{\mathrm{per}}(Y')) \quad \text{for any } \lambda \in \mathbb{R}^N \quad (1 \le i \le N). \tag{1.6}$$

Example 1.3. One may as well investigate the behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.4)) under the structure hypothesis

$$a_i(\cdot,\lambda) \in AP(\mathbb{R}^N) \quad \text{for any } \lambda \in \mathbb{R}^N \quad (1 \le i \le N) \tag{1.7}$$

where $AP(\mathbb{R}^N)$ denotes the usual space of all almost periodic continuous complex functions on $\mathbb{R}^N$ [3, 14, 16]. Let us point out two cases of practical interest that reduce to (1.7).

(1) Suppose there exists a family of networks S_i in $\mathbb{R}^N$ $(1 \le i \le N)$ such that $a_i(\cdot,\lambda)$ is S_i-periodic for any $\lambda \in \mathbb{R}^N$. Then (1.7) is fulfilled.

(2) Suppose that to each $\lambda \in \mathbb{R}^N$ there is assigned a network S_λ in $\mathbb{R}^N$ such that

$a_i(\cdot,\lambda)$ is S_λ-periodic ($1 \leq i \leq N$). This assumption naturally leads to (1.7), once again.

Remark 1.2. Given a function f on $\mathbb{R}^N$ and a network S in $\mathbb{R}^N$ (S is also termed a réseau [4, 11]), by f to be S-periodic we mean that $f(y+k) = f(y)$ for $y \in \mathbb{R}^N$ and $k \in S$. In the literature of periodic homogenization one often says f is Y-periodic in place of f is S-periodic, Y being a suitable parallelepiped attached to S (see [22]).

Example 1.4. As will be seen later, it is also possible to study the homogenization of (1.4) for $p = 2$ under the more general almost periodicity hypothesis

$$a_i(\cdot,\lambda) \in L^2_{AP}(\mathbb{R}^N) \quad \text{for any } \lambda \in \mathbb{R}^N \quad (1 \leq i \leq N) \tag{1.8}$$

provided the following condition is fulfilled:

$$\begin{gathered}\text{For } \Psi \in AP(\mathbb{R}^N;\mathbb{R})^N = AP(\mathbb{R}^N;\mathbb{R}) \times \cdots \times AP(\mathbb{R}^N;\mathbb{R}) \ (N \text{ times}),\\ \text{we have } \sup_{k\in\mathbb{Z}^N} \int_{k+Y} |a(y-r,\Psi(y)) - a(y,\Psi(y))|^2 dy \to 0 \text{ as } |r| \to 0\end{gathered} \tag{1.9}$$

where $Y = (0,1)^N$. It should be recalled that $L^2_{AP}(\mathbb{R}^N)$ denotes the space of all functions in $L^2_{\rm loc}(\mathbb{R}^N)$ that are almost periodic in the sense of Stepanoff (cf. [3, 22]).

The novelty of this article is to study the homogenization of (1.4) under diverse structure hypotheses such as (1.5)-(1.9) instead of the classical periodicity hypothesis. In fact, the real scope of the present study is much wider. We investigate here the behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.4)) under an abstract assumption on $a(\cdot,\lambda)$ (for fixed λ) covering a variety of concrete structure hypotheses beyond the periodic setting. Our main tool is the recent theory of homogenization structures [19, 20] and our basic approach is an adaptation of the two-scale convergence method [1, 18]. The achieved results are quite similar to those provided by periodic homogenization theory (see, e.g., [9, 17]), which confirms, as was anticipated in [19], that the recent homogenization approach earlier presented in [19, 20] fits nonlinear partial differential equations as well. Thus, this work falls within the scope of the new deterministic homogenization theory especially framed in [19, 20] to bridge the gap between periodic and stochastic homogenization [2, 8].

The layout of the paper is as follows: Section 2 is devoted to some preliminary results and remarks about the justification of such expressions as $a(\frac{x}{\varepsilon},\mathbf{v}(x))$. Among other things this permits us to give a rigorous meaning to the left-hand side of (1.4). In Section 3 we recall the fundamentals of homogenization structures and point out the main results underlying our homogenization approach. Attention is drawn to the particular case of proper homogenization structures. In Section 4 we prove a homogenization result for problem (1.4) under an abstract structure hypothesis (in place of the usual periodicity hypothesis). Finally, Section 5 deals with the homogenization of (1.4) under various concrete structure hypotheses such as those presented in the preceding examples.

Except where otherwise stated, the vector spaces throughout are assumed to be complex vector spaces, and the scalar functions are assumed to take values in $\mathbb{C}$ (the complex field). This permits us to make use of basic tools provided by the classical Banach algebras theory. For basic concepts and notations about integration theory we refer to [5, 6]. We shall always assume that the N-dimensional numerical space $\mathbb{R}^N$ and its open sets are each equipped with the Lebesgue measure.

2. Preliminaries

Throughout this section, Ω denotes a bounded open set in $\mathbb{R}_x^N$ and ε is a positive real number. Let us begin by recalling a standard notation. For $u \in L^1_{\text{loc}}(\Omega \times \mathbb{R}_y^N)$, we set

$$u^\varepsilon(x) = u\left(x, \frac{x}{\varepsilon}\right) \quad (x \in \Omega) \tag{2.1}$$

whenever the right-hand side has meaning. On this point it seems useful to recall the main cases in which (2.1) actually determines a function (for further details see [18, 22].

The most evident two cases are when u is a continuous (real or complex) function on $\Omega \times \mathbb{R}_y^N$ (or $\overline{\Omega} \times \mathbb{R}_y^N$, $\overline{\Omega}$ the closure of Ω) and when u lies in $L^p_{\text{loc}}(\Omega) \otimes L^{p'}_{\text{loc}}(\mathbb{R}_y^N)$ $(1 \leq p \leq \infty,\ \frac{1}{p'} = 1 - \frac{1}{p})$, i.e., is of the form

$$u = \sum_{\text{finite}} \varphi_i \otimes u_i \quad \text{with } \varphi_i \in L^p_{\text{loc}}(\Omega) \text{ and } u_i \in L^{p'}_{\text{loc}}(\mathbb{R}_y^N).$$

Next comes the less obvious case of the spaces $L^p(\Omega; A)$ $(1 \leq p \leq \infty)$ where A denotes a closed vector subspace of the space $\mathcal{B}(\mathbb{R}_y^N)$ (equipped with the supremum norm) of bounded continuous complex functions on $\mathbb{R}_y^N$. Given $u \in L^p(\Omega; A)$, there clearly exists a negligible set $\mathcal{N} \subset \Omega$ such that for each $x \in \Omega \backslash \mathcal{N}$ the mapping $y \to u(x, y)$ lies in A. Hence the complex number $u(x, \frac{x}{\varepsilon})$ is well defined and so we may consider the function $x \to u(x, \frac{x}{\varepsilon})$ defined almost everywhere in Ω and belonging to $L^p(\Omega)$. Thus, $u^\varepsilon \in L^p(\Omega)$ is well defined by (2.1) when $u \in L^p(\Omega; A)$. This yields a linear mapping $u \to u^\varepsilon$ that sends continuously $L^p(\Omega; A)$ into $L^p(\Omega)$ and has norm exactly 1.

We turn finally to the still less obvious case of the space $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N))$. By all probability, in the present case the right-hand side of (2.1) cannot be apprehended directly as we did before. One can nevertheless define $u(x, \frac{x}{\varepsilon})$, $x \in \Omega$, by extension by continuity thanks to two facts : on one hand, the space $\mathcal{C}(\overline{\Omega}) \otimes L^\infty(\mathbb{R}_y^N)$ is dense in $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N))$ (cf. [5, p.46]), on the other hand, as pointed out thereinbefore, the right-hand side of (2.1) is well defined for $u \in \mathcal{C}(\overline{\Omega}) \otimes L^\infty(\mathbb{R}_y^N)$. More precisely, we have the following lemma whose proof can be found in [22].

Lemma 2.1. *The transformation $u \to u^\varepsilon$ (see (2.1)) considered as a mapping of $\mathcal{C}(\overline{\Omega}) \otimes L^\infty(\mathbb{R}_y^N)$ into $L^\infty(\Omega)$ extends by continuity to a linear mapping, still denoted by $u \to u^\varepsilon$, of $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N))$ into $L^\infty(\Omega)$ with*

$$\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \sup_{x \in \overline{\Omega}} \|u(x)\|_{L^\infty(\mathbb{R}^N)}, \quad u \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N)).$$

The next result will prove to be of great interest.

Lemma 2.2. *Let $u \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N))$. Suppose that for each $x \in \overline{\Omega}$ we have $u(x, y) \geq 0$ a.e in $y \in \mathbb{R}^N$. Then $u^\varepsilon(x) \geq 0$ a.e. in $x \in \Omega$, u^ε (given by (2.1)) being defined in the sense of Lemma 2.1.*

Proof. Let u_n (integers $n \geq 0$) be a sequence in $\mathcal{C}(\overline{\Omega}) \otimes L^\infty(\mathbb{R}_y^N)$ such that $u_n \to u$ in $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}^N))$ when $n \to \infty$. Set $q_n = \sup_{x \in \overline{\Omega}} \|u_n(x) - u(x)\|_{L^\infty(\mathbb{R}^N)}$ (integers $n \geq 0$). Fix an integer $n \geq 0$ and let $x \in \overline{\Omega}$. Then $u_n(x, y) + q_n \geq 0$ a.e. in $y \in \mathbb{R}^N$. According to Lemma 1.2 of [22], we deduce that there is a negligible set $\mathcal{N}_n \subset \mathbb{R}_y^N$ ($\mathcal{N}_n$ independent of x) such that $u_n(x, y) + q_n \geq 0$ for all $y \in \mathbb{R}^N \backslash \mathcal{N}_n$.

Hence, letting $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$, it follows $u_n(x,y) + q_n \geq 0$ for all $y \in \mathbb{R}^N \backslash \mathcal{N}$, $x \in \overline{\Omega}$ and $n \in \mathbb{N}$. Therefore $u_n^\varepsilon(x) + q_n \geq 0$ for any $x \in \Omega \backslash \varepsilon\mathcal{N}$ and any integer $n \geq 0$. But, as $n \to \infty$, we have $u_n^\varepsilon \to u^\varepsilon$ in $L^\infty(\Omega)$ (Lemma 2.1) and $q_n \to 0$. Hence, by extraction of a suitable subsequence it follows that $u^\varepsilon(x) \geq 0$ a.e. in $x \in \Omega$, as claimed. □

Now, let $G : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a function with the following properties :

For each $\lambda \in \mathbb{R}^N$, the function $G(\cdot, \lambda)$ is measurable (2.2)

$G(y, \omega) = 0$ a.e. in $y \in \mathbb{R}^N$ (2.3)

There is a positive constant c such that $|G(y,\lambda) - G(y,\mu)| \leq c|\lambda - \mu|^{p-1}$ for all $\lambda, \mu \in \mathbb{R}^N$ and for almost all $y \in \mathbb{R}^N$, where $1 < p \leq 2$. (2.4)

Given $\Psi \in \mathcal{C}(\overline{\Omega};\mathbb{R})^N = \mathcal{C}(\overline{\Omega};\mathbb{R}) \times \cdots \times \mathcal{C}(\overline{\Omega};\mathbb{R})$ (N times), it is an easy task to check, using (2.2)-(2.4), that the function $(x,y) \to u(x,y) = G(y, \Psi(x))$ of $\overline{\Omega} \times \mathbb{R}_y^N$ into $\mathbb{R}$ belongs to $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N))$. Hence, the function $x \to G(\frac{x}{\varepsilon}, \Psi(x))$ of Ω into $\mathbb{R}$, denoted below by $G^\varepsilon(\cdot, \Psi)$, is well defined as a function in $L^\infty(\Omega)$ (cf. Lemma 2.1). This leads us to the following proposition and corollary.

Proposition 2.1. *The transformation $\Psi \to G^\varepsilon(\cdot,\Psi)$ of $\mathcal{C}(\overline{\Omega};\mathbb{R})^N$ into $L^\infty(\Omega)$ extends by continuity to a mapping, still denoted by $\Psi \to G^\varepsilon(\cdot,\Psi)$, of $L^p(\Omega;\mathbb{R})^N$ into $L^{p'}(\Omega)$ with the property:*

$$\|G^\varepsilon(\cdot,\Psi) - G^\varepsilon(\cdot,\Phi)\|_{L^{p'}(\Omega)} \leq c\|\Psi - \Phi\|_{L^p(\Omega)^N}^{p-1} \tag{2.5}$$

for all $\Psi, \Phi \in L^p(\Omega;\mathbb{R})^N$, where $1 < p \leq 2$ and $p' = \frac{p}{p-1}$.

Proof. Let $\Psi, \Phi \in \mathcal{C}(\overline{\Omega};\mathbb{R})^N$. By applying Lemma 2.2 with

$$u(x,y) = c|\Psi(x) - \Phi(x)|^{p-1} - |G(y,\Psi(x)) - G(y,\Phi(x))|$$

(cf. (2.4)), we get

$$|G\big(\frac{x}{\varepsilon}, \Psi(x)\big) - G\big(\frac{x}{\varepsilon}, \Phi(x)\big)| \leq c|\Psi(x) - \Phi(x)|^{p-1} \text{ a.e. in } x \in \Omega.$$

Hence (2.5) follows immediately and that for all $\Psi, \Phi \in \mathcal{C}(\overline{\Omega};\mathbb{R})^N$. Therefore, since $\mathcal{C}(\overline{\Omega};\mathbb{R})^N$ is dense in $L^p(\Omega;\mathbb{R})^N$, the proposition follows by extension by continuity. □

Corollary 2.1. *Under the preceding notation, we have*

$$\big[a\big(\frac{x}{\varepsilon}, Du(x)\big) - a\big(\frac{x}{\varepsilon}, Dv(x)\big)\big] \cdot (Du(x) - Dv(x)) \geq \alpha|Du(x) - Dv(x)|^p \tag{2.6}$$

a.e. in $x \in \Omega$ and

$$\|a^\varepsilon(\cdot, Du) - a^\varepsilon(\cdot, Dv)\|_{L^{p'}(\Omega)^N} \leq c\|Du - Dv\|_{L^p(\Omega)^N}^{p-1} \tag{2.7}$$

for all $u, v \in W^{1,p}(\Omega;\mathbb{R})$, where $a^\varepsilon(\cdot, Du) = (a_i^\varepsilon(\cdot, Du))_{1\leq i\leq N}$.

This corollary follows by Proposition 2.1 with $G = a_i$ (the i^{th} component of the function a in Section 1) and use of (1.1)-(1.3) together with Lemma 2.2.

Remark 2.1. Thanks to Corollary 2.1, the left-hand side of (1.4) is now justified.

We also need to define $G^\varepsilon(\cdot, \Psi)$ for $\Psi \in \mathcal{B}(\mathbb{R}^N;\mathbb{R})^N$ instead of $\Psi \in \mathcal{C}(\overline{\Omega};\mathbb{R})^N$.

Proposition 2.2. *For $\Psi \in \mathcal{B}(\mathbb{R}^N_x;\mathbb{R})^N$, the function $x \to G(\frac{x}{\varepsilon},\Psi(x))$ of $\mathbb{R}^N_x$ into $\mathbb{R}$ can be defined as an element of $L^\infty(\mathbb{R}^N_x)$ denoted by $G^\varepsilon(\cdot,\Psi)$.*

Proof. For each integer $n \geq 1$, let $B_n \subset \mathbb{R}^N$ denote the open ball of center ω (the origin of $\mathbb{R}^N$) and of radius n. Let $G_n(x) = G(\cdot,\Psi(x))$ for $x \in \overline{B}_n$, which gives a function $G_n \in \mathcal{C}(\overline{B}_n; L^\infty(\mathbb{R}^N_y))$, thanks to (2.2)-(2.4). Thus, we can define, in the sense of Lemma 2.1, the function $x \to G_n(\frac{x}{\varepsilon},\Psi(x))$ of B_n into $\mathbb{R}$ as an element of $L^\infty(B_n)$ denoted by $G^\varepsilon_n(\cdot,\Psi)$. This yields a sequence of functions $G^\varepsilon_n(\cdot,\Psi) \in L^\infty(B_n)$ (integers $n \geq 1$) verifying $G^\varepsilon_n(\cdot,\Psi) = G^\varepsilon_{n+1}(\cdot,\Psi)\big|_{B_n}$ (the restriction of $G^\varepsilon_{n+1}(\cdot,\Psi)$ to B_n). Let $G^\varepsilon(\cdot,\Psi)$ denote the function in $L^\infty(\mathbb{R}^N_x)$ uniquely defined as $G^\varepsilon(\cdot,\Psi)\big|_{B_n} = G^\varepsilon_n(\cdot,\Psi)$ (integers $n \geq 1$). Clearly $G^\varepsilon(\cdot,\Psi)(x) = G(\frac{x}{\varepsilon},\Psi(x))$ a.e. in $x \in B_n$ (integers $n \geq 1$). Therefore, the proposition follows. □

As a consequence of this, we have the following

Corollary 2.2. *Let $\mathbf{w} \in \mathcal{C}(\overline{\Omega};\mathcal{B}(\mathbb{R}^N;\mathbb{R})^N)$. Then one can define, in the sense of Lemma 2.1, the function $x \to a_i(\frac{x}{\varepsilon},\mathbf{w}(x,\frac{x}{\varepsilon}))$ of Ω into $\mathbb{R}$ as an element of $L^\infty(\Omega)$ denoted by $a^\varepsilon_i(\cdot,\mathbf{w}^\varepsilon)$.*

Proof. For each $x \in \overline{\Omega}$, let $a_i(\cdot,\mathbf{w}(x,\cdot))$ denote the function $y \to a_i(y,\mathbf{w}(x,y))$ defined as in Proposition 2.2 with $G = a_i$, $\Psi = \mathbf{w}(x,\cdot)$ and $\varepsilon = 1$. Then $a_i(\cdot,\mathbf{w}(x,\cdot)) \in L^\infty(\mathbb{R}^N_y)$ and (by applying Lemma 2.2 as in the proof of Proposition 2.1)

$$\|a_i(\cdot,\mathbf{w}(x,\cdot)) - a_i(\cdot,\mathbf{w}(x',\cdot))\|_{L^\infty(\mathbb{R}^N_y)} \leq c\|\mathbf{w}(x,\cdot) - \mathbf{w}(x',\cdot)\|^{p-1}_\infty$$

for all $x,x' \in \overline{\Omega}$. Hence it follows that the function $x \to a_i(\cdot,\mathbf{w}(x,\cdot))$ lies in $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}^N_y))$. Therefore the corollary follows by Lemma 2.1. □

We conclude the present section with one further result.

Proposition 2.3. *Let $\psi \in \mathcal{B}(\mathbb{R}^N_x; L^\infty(\mathbb{R}^N_y))$. One can suitably define $\psi^\varepsilon(x) = \psi(x,\frac{x}{\varepsilon})$, $x \in \mathbb{R}^N$, as a function $\psi^\varepsilon \in L^\infty(\mathbb{R}^N_x)$ such that $\|\psi^\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq \sup_{x\in\mathbb{R}^N} \|\psi(x)\|_{L^\infty(\mathbb{R}^N_y)}$. Furthermore, if for each $x \in \mathbb{R}^N$ we have $\psi(x,y) \geq 0$ a.e. in $y \in \mathbb{R}^N$, then $\psi^\varepsilon(x) \geq 0$ a.e. in $x \in \mathbb{R}^N$.*

Proof. The first part of the proposition proceeds by the same line of argument as in the proof of Proposition 2.2 [consider the sequence of functions $\psi_n = \psi\big|_{B_n} \in \mathcal{C}(\overline{B}_n; L^\infty(\mathbb{R}^N_y))$], and the next part is a direct consequence of Lemma 2.2. □

3. Proper homogenization structures

3.1. Fundamentals of homogenization structures.

By a structural representation on $\mathbb{R}^N_y$ is meant any subset Γ of $\mathcal{B}(\mathbb{R}^N_y)$ with the following properties:

- (HS1) Γ is a group under multiplication in $\mathcal{B}(\mathbb{R}^N_y)$
- (HS2) Γ is countable
- (HS3) If $\gamma \in \Gamma$ then $\overline{\gamma} \in \Gamma$ ($\overline{\gamma}$ the complex conjugate of γ)
- (HS4) $\Gamma \subset \Pi^\infty$

where Π^∞ denotes the space of all $u \in \mathcal{B}(\mathbb{R}^N_y)$ with the property that $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}^N_x)$-weak$*$ as $\varepsilon \to 0$ ($\varepsilon > 0$), $M(u)$ a complex number and

$$u^\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right) \quad \text{for } x \in \mathbb{R}^N \quad (\varepsilon > 0). \tag{3.1}$$

It is worth recalling that Π^∞ contains the constants and is translation invariant, and further the mapping $u \to M(u)$ of Π^∞ into $\mathbb{C}$ is a positive linear form with $M(1) = 1$ and $M(\tau_a u) = M(u)$ for $u \in \Pi^\infty$ and $a \in \mathbb{R}^N$, where $\tau_a u(y) = u(y-a)$ for $y \in \mathbb{R}^N$. See [21] for further details.

Now, by an *H-structure* on $\mathbb{R}^N_y$ (H stands for *homogenization*) is meant any equivalence class modulo $\sim$, where the equivalence relation $\sim$ in the collection of all structural representations on $\mathbb{R}^N_y$ is defined as $\Gamma \sim \Gamma'$ *if and only if* $CLS(\Gamma) = CLS(\Gamma')$, $CLS(\Gamma)$ standing for the closed vector subspace of $\mathcal{B}(\mathbb{R}^N_y)$ spanned by Γ. If Σ is a given H-structure on $\mathbb{R}^N_y$, we let $A = CLS(\Gamma)$, where Γ is any equivalence class representative of Σ (such a Γ is termed a *representation* of Σ). A is a so-called H-algebra on $\mathbb{R}^N_y$, that is, a closed subalgebra of $\mathcal{B}(\mathbb{R}^N_y)$ with the properties:

(HA1) A with the supremum norm is separable
(HA2) A contains the constants
(HA3) $u \in A$ implies $\overline{u} \in A$
(HA4) $A \subset \Pi^\infty$.

Furthermore, A depends only on Σ and not on the chosen representation Γ of Σ. Thus, we may set $A = \mathcal{J}(\Sigma)$ (image of Σ). This yields a mapping $\Sigma \to \mathcal{J}(\Sigma)$ that carries the collection of all H-structures bijectively over the collection of all H-algebras on $\mathbb{R}^N_y$ (see Theorem 3.1 of [19]).

Given an H-algebra A on $\mathbb{R}^N_y$, we will denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A, i.e., the mapping $u \to \mathcal{G}(u)$ of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ for $s \in \Delta(A)$ and $u \in A$, where $\langle , \rangle$ stands for the duality between A' (the topological dual of A) and A. It is worth noting that $\Delta(A)$ is here a metrizable compact space and $\mathcal{G}$ is an isometric isomorphism of the $\mathcal{C}^*$-algebra A onto the $\mathcal{C}^*$-algebra $\mathcal{C}(\Delta(A))$ (see, e.g., [16, p. 277]). The appropriate measure on $\Delta(A)$ is the so-called M-measure for A, that is, the Radon measure β on $\Delta(A)$ such that $M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta$ for $u \in A$.

Let $A^1 = \{\psi \in \mathcal{C}^1(\mathbb{R}^N_y) : \psi, D_{y_i}\psi \in A \ (1 \le i \le N)\}$ where $D_{y_i}\psi = \frac{\partial \psi}{\partial y_i}$. The partial derivative of index i $(1 \le i \le N)$ on $\Delta(A)$ is defined to be the mapping $\partial_i = \mathcal{G} \circ D_{y_i} \circ \mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^1\}$ into $\mathcal{C}(\Delta(A))$, where $\mathcal{G}^{-1}$ (the inverse of $\mathcal{G}$) is viewed as defined on $\mathcal{D}^1(\Delta(A))$. Higher order derivatives are defined analogously (see [19]). Now, let A^∞ be the space of all $\psi \in \mathcal{C}^\infty(\mathbb{R}^N_y)$ such that $D^\alpha_y \psi = \frac{\partial^{|\alpha|}\psi}{\partial y_1^{\alpha_1} \cdots \partial y_N^{\alpha_N}} \in A$ for each multi-index $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$. Let $\mathcal{D}(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^\infty\}$. Endowed with a suitable locally convex topology (see [19]), A^∞ (resp. $\mathcal{D}(\Delta(A))$) is a Fréchet space and, further, $\mathcal{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$.

Any continuous linear form on $\mathcal{D}(\Delta(A))$ is referred to as a *distribution* on $\Delta(A)$. The space $\mathcal{D}'(\Delta(A))$ (topological dual of $\mathcal{D}(\Delta(A))$) of all distributions on $\Delta(A)$ is endowed with the strong dual topology. If we assume that A^∞ is dense in A (this amounts to assuming that $\mathcal{D}(\Delta(A))$ is dense in $\mathcal{C}(\Delta(A))$), then we have $L^p(\Delta(A)) \subset \mathcal{D}'(\Delta(A))$ $(1 \le p \le \infty)$ with continuous embedding. Hence we may define

$$W^{1,p}(\Delta(A)) = \{u \in L^p(\Delta(A)) : \ \partial_i u \in L^p(\Delta(A)), \ 1 \le i \le N\}$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$ [19]. We equip $W^{1,p}(\Delta(A))$ with the norm

$$||u||_{W^{1,p}(\Delta(A))} = ||u||_{L^p(\Delta(A))} + \sum_{i=1}^{N} ||\partial_i u||_{L^p(\Delta(A))} \quad (u \in W^{1,p}(\Delta(A))),$$

Which makes it a Banach space. However, we will be mostly concerned with the space

$$W^{1,p}(\Delta(A))/\mathbb{C} = \big\{u \in W^{1,p}(\Delta(A)) : \int_{\Delta(A)} u(s)d\beta(s) = 0\big\}$$

provided with the seminorm

$$||u||_{W^{1,p}(\Delta(A))/\mathbb{C}} = \sum_{i=1}^{N} ||\partial_i u||_{L^p(\Delta(A))} \quad (u \in W^{1,p}(\Delta(A))/\mathbb{C}).$$

So topologized, $W^{1,p}(\Delta(A))/\mathbb{C}$ is in general nonseparated and noncomplete. We denote by $W^{1,p}_{\#}(\Delta(A))$ the separated completion of $W^{1,p}(\Delta(A))/\mathbb{C}$ and by J the canonical mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into its separated completion (see, e.g., chapter II of [7] and page 29 of [12]). $W^{1,p}_{\#}(\Delta(A))$ is a Banach space and $W^{1,2}_{\#}(\Delta(A))$ is a Hilbert space. Furthermore, as pointed out in [19], the distribution derivative ∂_i viewed as a mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into $L^p(\Delta(A))$ extends to a unique continuous linear mapping, still denoted by ∂_i, of $W^{1,p}_{\#}(\Delta(A))$ into $L^p(\Delta(A))$ such that $\partial_i J(v) = \partial_i v$ for $v \in W^{1,p}(\Delta(A))/\mathbb{C}$ and

$$||u||_{W^{1,p}_{\#}(\Delta(A))} = \sum_{i=1}^{N} ||\partial_i u||_{L^p(\Delta(A))} \quad \text{for } u \in W^{1,p}_{\#}(\Delta(A)).$$

To an H-structure Σ on $\mathbb{R}^N$ there are attached the important concepts of weak and strong Σ-convergence in L^p $(1 \le p < \infty)$, see [19]. Likewise it is possible and even desirable to introduce the concept of weak Σ-convergence in $W^{1,p}$.

Let Σ be an H-structure on $\mathbb{R}^N$. Let $A = \mathcal{J}(\Sigma)$. We assume that Σ is of class $\mathcal{C}^\infty$ [19], i.e., A^∞ is dense in A. Let $1 \le p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^N_x$. First of all, we set

$$W^{1,p}(\Omega; L^p(\Delta(A))) = \{u \in L^p(\Omega \times \Delta(A)) : D_{x_i} u \in L^p(\Omega \times \Delta(A)),\ 1 \le i \le N\}$$

where the derivatives $D_{x_i} u = \frac{\partial u}{\partial x_i}$ are taken in the sense of vector distributions $\mathcal{D}'(\Omega; L^p(\Delta(A)))$ [24] (see also [25]), since $L^p(\Omega \times \Delta(A)) = L^p(\Omega; L^p(\Delta(A))) \subset \mathcal{D}'(\Omega; L^p(\Delta(A)))$. We equip $W^{1,p}(\Omega; L^p(\Delta(A)))$ with the norm

$$||u||_{W^{1,p}(\Omega; L^p\Delta(A)))} = ||u||_{L^p(\Omega\times\Delta(A))} + \sum_{i=1}^{N} ||D_{x_i} u||_{L^p(\Omega\times\Delta(A))},$$

which makes it a Banach space.

Before we can define the concept of weak Σ-convergence in $W^{1,p}$, we also need to give a meaning to $\partial_i u$ for $u \in L^p(\Omega; W^{1,p}_{\#}(\Delta(A)))$. This is straightforward. Indeed, considering ∂_i $(1 \le i \le N)$ as a mapping of $W^{1,p}_{\#}(\Delta(A))$ into $L^p\Delta(A))$ (as seen above) and using a classical result (see Theorem 4 of page 132 in [5]), we see that $\partial_i \circ u \in L^p(\Omega \times \Delta(A))$ for $u \in L^p(\Omega; W^{1,p}_{\#}(\Delta(A)))$, and the transformation $u \to \partial_i \circ u$ (usual composition) maps continuously and linearly $L^p(\Omega; W^{1,p}_{\#}(\Delta(A)))$ into $L^p(\Omega \times \Delta(A))$. In the sequel we set $\partial_i u = \partial_i \circ u$ for $u \in L^p(\Omega; W^{1,p}_{\#}(\Delta(A)))$.

Finally, the letter E throughout will denote exclusively a family of positive real numbers admitting 0 as an accumulation point. For example $E = \mathbb{R}_+^*$ (the positive real numbers) or $E = (\varepsilon_n)$ (integers $n \geq 0$), where $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ when $n \to +\infty$. In the last case E is referred to as a *fundamental sequence*.

Definition 3.1. A sequence of functions $u_\varepsilon \in W^{1,p}(\Omega)$ ($\varepsilon \in E$) is said to be *weakly Σ-convergent* in $W^{1,p}(\Omega)$ to some $u_0 \in W^{1,p}(\Omega; L^p(\Delta(A)))$ if there exists a function $u_1 \in L^p(\Omega; W^{1,p}_{\#}(\Delta(A)))$ such that as $E \ni \varepsilon \to 0$, we have:

(i) $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ

(ii) $D_{x_i} u_\varepsilon \to D_{x_i} u_0 + \partial_i u_1$ in $L^p(\Omega)$-weak Σ, $1 \leq i \leq N$.

We then write $u_\varepsilon \to u_0$ in $W^{1,p}(\Omega)$-weak Σ and we refer to u_0 (which is necessarily unique) as the weak Σ-limit in $W^{1,p}$ of the sequence $(u_\varepsilon)_{\varepsilon \in E}$. The function u_1 is called a *corrector* for $(u_\varepsilon)_{\varepsilon \in E}$.

Remark 3.1. The concept of weak Σ-convergence in $W^{1,p}(\Omega)$ is a natural generalization of the usual notion of weak convergence in $W^{1,p}(\Omega)$. Indeed, a sequence $u_\varepsilon \in W^{1,p}(\Omega)$ ($\varepsilon \in E$) is weakly convergent in $W^{1,p}(\Omega)$ to some $u_0 \in W^{1,p}(\Omega)$ as $\varepsilon \to 0$ if and only if as $\varepsilon \to 0$, we have $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak and $D_{x_i} u_\varepsilon \to D_{x_i} u_0$ in $L^p(\Omega)$-weak for $i = 1, \cdots, N$. But then it amounts to saying that $u_\varepsilon \to u_0$ in $W^{1,p}(\Omega)$-weak Σ_0, where Σ_0 is the trivial H-structure on $\mathbb{R}^N$ (see Example 3.1 of [19]).

Proposition 3.1. *Suppose that $\mathcal{D}(\Delta(A))$ is dense in $W^{1,p}(\Delta(A))$. If the sequence $u_\varepsilon \in W^{1,p}(\Omega)$ ($\varepsilon \in E$) is weakly Σ-convergent in $W^{1,p}(\Omega)$ to some function $u_0 \in W^{1,p}(\Omega;\ L^p(\Delta(A)))$, then $u_\varepsilon \to \widetilde{u}_0$ in $W^{1,p}(\Omega)$-weak as $E \ni \varepsilon \to 0$, where $\widetilde{u}_0$ is given by $\widetilde{u}_0(x) = \int_{\Delta(A)} u_0(x,s)\,d\beta(s)$ for $x \in \Omega$.*

Proof. By Definition 3.1 and use of Proposition 4.4 of [19] we have, in the weak topology in $L^p(\Omega)$, $u_\varepsilon \to \widetilde{u}_0$ and $D_{x_i} u_\varepsilon \to (D_{x_i} u_0)^\sim + (\partial_i u_1)^\sim$ ($i = 1, \cdots, N$) when $E \ni \varepsilon \to 0$. But $(D_{x_i} u_0)^\sim = D_{x_i} \widetilde{u}_0$ (this can be easily shown) and $(\partial_i u_1)^\sim = 0$ (this follows by the same line of reasoning as in the case of Proposition 4.8 in [19]). Hence the proposition follows. $\square$

This proposition has two useful corollaries.

Corollary 3.1. *Let the hypotheses be as in Proposition 3.1. Assume moreover that $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$. Then $u_0 \in W^{1,p}(\Omega)$.*

Proof. By Proposition 3.1 and use of the above compactness hypothesis we have $u_\varepsilon \to \widetilde{u}_0$ in $L^p(\Omega)$ as $E \ni \varepsilon \to 0$, hence $u_\varepsilon \to \widetilde{u}_0$ in $L^p(\Omega)$-weak Σ (use Example 4.2 and Proposition 4.6 of [19]). Therefore $u_0 = \widetilde{u}_0$, according to the unicity of the weak Σ-limit. The corollary follows. $\square$

Corollary 3.2. *Let the hypotheses be as in Proposition 3.1, and let us assume further that each u_ε lies in $W^{1,p}_0(\Omega)$ (the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$). Then $u_0 \in W^{1,p}_0(\Omega)$.*

Proof. Indeed, since Ω is bounded, we have that $W^{1,p}_0(\Omega)$ is compactly embedded in $L^p(\Omega)$ (this is classical). Therefore the corollary follows in the same way as above. $\square$

Remark 3.2. If Σ is an almost periodic H-structure (cf.Example 3.3 of [19]) then we arrive at the conclusion of Corollary 3.1 without assuming that $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$ (proceed as in the proof of Theorem 4.1 of [22]).

3.2. **Proper H-structures.** In what follows, Σ denotes an H-structure on $\mathbb{R}^N$. We set $A = \mathcal{J}(\Sigma)$. We assume that Σ is of class $\mathcal{C}^\infty$ (see Subsection 3.1). Let $1 < p < \infty$.

Definition 3.2. Given a bounded open set Ω in $\mathbb{R}^N_x$, the Sobolev space $W^{1,p}(\Omega)$ is said to be Σ-reflexive if the following holds: Given a fundamental sequence E and a sequence $(u_\varepsilon)_{\varepsilon \in E}$ which is bounded in $W^{1,p}(\Omega)$, a subsequence E' can be extracted from E such that the sequence $(u_\varepsilon)_{\varepsilon \in E'}$ is weakly Σ-convergent in $W^{1,p}(\Omega)$ (Definition 3.1).

Remark 3.3. The notion of Σ-reflexivity (for $W^{1,2}(\Omega)$) stated in [19] (Definition 4.4) turns out to be restricted because the corresponding weak Σ-limit u_0 was straight off subject to lie in $W^{1,2}(\Omega)$, which is not the case in Definition 3.2. No doubt, the concept of Σ-reflexivity framed above (in Definition 3.2) is both general and better.

Remark 3.4. Assuming $1 < p < \infty$ implies that $W^{1,p}(\Omega)$ is reflexive, as is classical. But this is equivalent to saying that $W^{1,p}(\Omega)$ is Σ_0-reflexive (Σ_0 as in Remark 3.1).

We are now in a position to define the notion of a proper H-structure when $1 < p < \infty$.

Definition 3.3. The H-structure (of class $\mathcal{C}^\infty$) Σ on $\mathbb{R}^N$ is said to be proper for some given real $p > 1$ if the following two conditions are satisfied:

(P1) Σ is total (for p), i.e., $\mathcal{D}(\Delta(A))$ is dense in $W^{1,p}(\Delta(A))$

(P2) For each bounded open set $\Omega \subset \mathbb{R}^N_x$, $W^{1,p}(\Omega)$ is Σ-reflexive.

Remark 3.5. If Σ is total (for p), then $J(\mathcal{D}(\Delta(A))/\mathbb{C})$ is dense in $W^{1,p}_{\#}(\Delta(A))$, where $\mathcal{D}(\Delta(A))/\mathbb{C} = \{\varphi \in \mathcal{D}(\Delta(A)) : \int_{\Delta(A)} \varphi d\beta = 0\}$. Furthermore, $\int_{\Delta(A)} \partial_i u d\beta = 0$ $(1 \le i \le N)$ for $u \in W^{1,p}_{\#}(\Delta(A))$. Indeed, this follows by the same arguments as in the proof of Proposition 4.8 in [19].

Several examples of proper H-structures for $p = 2$ are available in [19, 20]. However, in the present study we need to discuss the properness for a wide range of reals $p > 1$. Indeed, the more the proper H-structures (for p) at our disposal, the wider the range of those homogenization problems (for (1.4)) that can be worked out beyond the classical periodic setting. In [19] our quest of proper H-structures led us to a general properness result (see Theorem 4.2 of [19]) whose practicality has been established in the case $p = 2$. Of course, there is much to be gained by extending such a result to $1 < p < \infty$.

To this end, let Σ_2 be a further$H$-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}^N_y$, and let $A_2 = \mathcal{J}(\Sigma_2)$. We assume that hypothesis (H) below is satisfied.

(H) There exist an isometric isomorphism L of $L^p(\Delta(A))$ onto $L^p(\Delta(A_2))$, a dense vector subspace $\mathcal{V}$ of A, a surjective linear mapping $\ell : \mathcal{V} \to A_2$ and a vector subspace $\mathcal{V}^\infty$ of $A^\infty \cap \mathcal{V}$ such that:

$$L(\mathcal{G}(v)) = \mathcal{G}(\ell v) \quad \text{for } v \in \mathcal{V}, \tag{3.2}$$

where $\mathcal{G}$ is the Gelfand transformation on A and on A_2.

$$L(\widehat{v}u) = L(\widehat{v})L(u) \quad \text{for } v \in \mathcal{V} \text{ and } u \in L^p(\Delta(A))), \text{ where } \widehat{v} = \mathcal{G}(v) \tag{3.3}$$

$$(v - \ell v)^\varepsilon \to 0 \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N_x) \text{ as } \varepsilon \to 0 \ (v \in \mathcal{V}) \tag{3.4}$$

$$\text{If } v \in \mathcal{V}^\infty \text{ then } D^\alpha_y v \in \mathcal{V} \quad (\alpha \in \mathbb{N}^N) \tag{3.5}$$

$$\text{The restriction of } \ell \text{ to } \mathcal{V}^\infty \text{ maps } \mathcal{V}^\infty \text{ onto } A_2^\infty \tag{3.6}$$

$$D_y^\alpha(\ell v) = \ell(D_y^\alpha v) \quad \text{for } v \in \mathcal{V}^\infty \quad (\alpha \in \mathbb{N}^N). \tag{3.7}$$

Our goal is to prove that under this hypothesis if Σ_2 is proper for some given $p > 1$, then so also is Σ. Before we can do this, however, we need a few notations and lemmas. To begin with, if $u \in L^p(\Omega; L^p(\Delta(A)))$, where Ω is as thereinbefore, we set $L_\Omega u(x) = L(u(x))$ $(x \in \Omega)$, which defines an isometric isomorphism L_Ω of $L^p(\Omega; L^p(\Delta(A)))$ onto $L^p(\Omega; L^p(\Delta(A_2)))$, according to (H). We will denote by J_2 the canonical mapping of $W^{1,p}(\Delta(A_2))/\mathbb{C}$ into its separated completion $W_\#^{1,p}(\Delta(A_2))$, whereas J denotes the canonical mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into $W_\#^{1,p}(\Delta(A))$, as stated above.

Each of the following three lemmas can be obtained by simple adaptation of the proof of its analog in [19]. The details are left to the reader.

Lemma 3.1. *Suppose Σ_2 is total (for p). Then the following assertions are true:*

(i) *If $u \in W^{1,p}(\Delta(A))$, then $Lu \in W^{1,p}(\Delta(A_2))$ and further $\partial_i(Lu) = L(\partial_i u)$ $(1 \le i \le N)$.*

(ii) *The restriction of the operator L to $W^{1,p}(\Delta(A))$ is an isometric isomorphism of $W^{1,p}(\Delta(A))$ onto $W^{1,p}(\Delta(A_2))$.*

(iii) *Σ is total (for p).*

Lemma 3.2. *Suppose Σ_2 is total (for p). Then there exists an isometric isomorphism $L_\# : W_\#^{1,p}(\Delta(A)) \to W_\#^{1,p}(\Delta(A_2))$ such that $L_\#(Jf) = J_2(Lf)$ for $f \in W^{1,p}(\Delta(A))/\mathbb{C}$ and $\partial_i L_\#(u) = L(\partial_i u)$ for $u \in W_\#^{1,p}(\Delta(A))$ $(1 \le i \le N)$.*

Lemma 3.3. *Suppose E is a fundamental sequence, and let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(\Omega)$ (Ω as in Definition 3.2) such that $u_\varepsilon \to v_0$ in $L^p(\Omega)$-weak Σ_2 as $E \ni \varepsilon \to 0$, where $v_0 \in L^p(\Omega \times \Delta(A_2))$. Then $u_\varepsilon \to L_\Omega^{-1} v_0$ in $L^p(\Omega)$-weak Σ (L_Ω^{-1} the inverse isomorphism of L_Ω).*

We turn now to the statement and proof of the desired result.

Theorem 3.1. *Suppose Σ_2 is proper (for p). Then Σ is proper (for p).*

Proof. Since Σ is total for p (according to Lemma 3.1), the whole problem amounts to verifying that $W^{1,p}(\Omega)$ is Σ-reflexive for each given bounded open set $\Omega \subset \mathbb{R}_x^N$. To do this, let $(u_\varepsilon)_{\varepsilon\in E}$ (E a fundamental sequence) be a bounded sequence in $W^{1,p}(\Omega)$. According to the Σ_2-reflexivity of $W^{1,p}(\Omega)$, there exist a subsequence E' extracted from E and two functions $v_0 \in W^{1,p}(\Omega; L^p(\Delta(A_2)))$ and $v_1 \in L^p(\Omega, W_\#^{1,p}(\Delta(A_2)))$ such that if $E' \ni \varepsilon \to 0$, then $u_\varepsilon \to v_0$ in $L^p(\Omega)$-weak Σ_2 and $D_{x_i} u_\varepsilon \to D_{x_i} v_0 + \partial_i v_1$ in $L^p(\Omega)$-weak Σ_2 $(1 \le i \le N)$. Let $u_0 = L_\Omega^{-1} v_0$ and $u_1 = L_{\#\Omega}^{-1} v_1$, where $L_{\#\Omega}$ denotes the isometric isomorphism of $L^p(\Omega, W_\#^{1,p}(\Delta(A)))$ onto $L^p(\Omega, W_\#^{1,p}(\Delta(A_2)))$ defined by $L_{\#\Omega} u = L_\# \circ u$, $u \in L^p(\Omega, W_\#^{1,p}(\Delta(A)))$. By applying [5, Theorem 1 of page 142] with $X = \Omega$, $F = L^p(\Delta(A_2))$, $G = L^p(\Delta(A))$ and $u = L^{-1}$, we see immediately that $L_\Omega^{-1} D_{x_i} v_0 = D_{x_i} L_\Omega^{-1} v_0$ $(i = 1, \cdots, N)$. Hence $u_0 \in W^{1,p}(\Omega; L^p(\Delta(A)))$ with $D_{x_i} u_0 = L_\Omega^{-1} D_{x_i} v_0$ $(i = 1, \cdots, N)$. On the other hand, it is clear that $u_1 \in L^p(\Omega, W_\#^{1,p}(\Delta(A)))$ with $\partial_i u_1 = L_\Omega^{-1}(\partial_i v_1)$ $(i = 1, \cdots, N)$. Hence, recalling Lemma 3.3, it follows that as $E' \ni \varepsilon \to 0$, we have $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ and $D_{x_i} u_\varepsilon \to D_{x_i} u_0 + \partial_i u_1$ $(i = 1, \cdots, N)$ in $L^p(\Omega)$-weak Σ, which completes the proof. $\square$

In the next subsection we will use the preceding theorem to establish the properness of some specific H-structures.

3.3. **Examples of proper H-structures.** We present here five basic examples of proper H-structures.

Example 3.1. Periodic H-structures. Let Σ_S be the periodic H-structure on $\mathbb{R}^N$ represented by the network $S = \mathbb{Z}^N$ (cf.Example 3.2 of [19]). We intend showing that Σ_S is proper for each $p > 1$. We will set $Y = (0,1)^N$ (the open unit cube in $\mathbb{R}^N_y$). We recall that the image $\mathcal{J}(\Sigma_S)$ is here $A = \mathcal{C}_{\rm per}(Y)$ (the space of Y-periodic continuous complex functions on $\mathbb{R}^N_y$). On the other hand, we have (up to an isometric isomorphism) $L^p(\Delta(A)) \equiv L^p_{\rm per}(Y) = \{v \in L^p_{\rm loc}(\mathbb{R}^N_y)$: v Y-periodic$\}$, $W^{1,p}(\Delta(A)) \equiv W^{1,p}_{\rm per}(Y) = \{v \in W^{1,p}_{\rm loc}(\mathbb{R}^N_y)$: v Y-periodic$\}$ and $W^{1,p}_{\#}(\Delta(A)) \equiv W^{1,p}_{\#}(Y) = \{v \in W^{1,p}_{\rm per}(Y) : \int_Y v(y)dy = 0\}$. This follows by simple arguments that can be found in [22]. We also draw attention to the fact that Σ_S-convergence in the present context is nothing else but two-scale convergence [1, 18]. Each of the spaces $L^p_{\rm per}(Y)$, $W^{1,p}_{\rm per}(Y)$ and $W^{1,p}_{\#}(Y)$ is a Banach space under a standard norm:

$$\|u\|_{L^p(Y)} = \Big(\int_Y |u(y)|^p dy\Big)^{1/p} \quad (u \in L^p_{\rm per}(Y)),$$

$$\|u\|_{W^{1,p}(Y)} = \|u\|_{L^p(Y)} + \sum_{i=1}^N \|D_{y_i}u\|_{L^p(Y)} \quad (u \in W^{1,p}_{\rm per}(Y))$$

and

$$\|u\|_{W^{1,p}_{\#}(Y)} = \sum_{i=1}^N \|D_{y_i}u\|_{L^p(Y)} \quad (u \in W^{1,p}_{\#}(Y)).$$

Finally, we conclude this preliminary step by recalling that Σ_S is of class $\mathcal{C}^\infty$.

Having made this point, let us begin by showing that Σ_S is total for any arbitrary real $p > 1$.

Proposition 3.2. *Σ_S is total for $p > 1$.*

Proof. Let $\theta \in \mathcal{D}(\mathbb{R}^N_y) = \mathcal{C}^\infty_0(\mathbb{R}^N_y)$ with $\theta \geq 0$, $\int \theta(y)dy = 1$, θ having support in the closed unit ball of $\mathbb{R}^N$. For each integer $n \geq 1$, we put $\theta_n(x) = n^N\theta(nx)$, $x \in \mathbb{R}^N$, which gives a sequence of functions $\theta_n \in \mathcal{D}(\mathbb{R}^N_x)$. This being so, let $u \in L^p_{\rm per}(Y)$, where $1 < p < +\infty$. By applying Hölder's inequality to the functions $x \to \theta_n(x)^{\frac{1}{p'}}$ and $x \to \theta_n(x)^{\frac{1}{p}}(u(y-x) - u(y))$ for fixed $y \in \mathbb{R}^N$, where $\frac{1}{p'} = 1 - \frac{1}{p}$, we get

$$\|u * \theta_n - u\|^p_{L^p(Y)} \leq \int \theta_n(x)\|u - \tau_x u\|^p_{L^p(Y)}dx.$$

Hence $u * \theta_n \to u$ in $L^p_{\rm per}(Y)$ when $n \to \infty$, as is easily seen by noting that θ_n has support in $\frac{1}{n}\bar{B}_N$ ($\bar{B}_N$ the closed unit ball of $\mathbb{R}^N$) and using the fact that for $f \in L^p_{\rm loc}(\mathbb{R}^N)$, $\tau_x f \to f$ in $L^p_{\rm loc}(\mathbb{R}^N)$ when $|x| \to 0$.

We deduce immediately that for $u \in W^{1,p}_{\rm per}(Y)$, $u*\theta_n \to u$ in $W^{1,p}_{\rm per}(Y)$ as $n \to \infty$. This shows that Σ_S is total, since $u * \theta_n \in \mathcal{C}^\infty_{\rm per}(Y) = \mathcal{C}^\infty(\mathbb{R}^N) \cap \mathcal{C}_{\rm per}(Y)$. ☐

The next point is to show that for each bounded open set $\Omega \subset \mathbb{R}^N$, the space $W^{1,p}(\Omega)$ is Σ_S-reflexive for any arbitrary real $p > 1$. We need one basic lemma.

Lemma 3.4. *Let* $\mathbf{f} = (f_i) \in L^p_{\rm per}(Y)^N$. *Suppose*

$$\int_Y \mathbf{f} \cdot \Psi dy \equiv \sum_{i=1}^{N} \int_Y f_i \psi_i dy = 0 \quad (\Psi \in \mathcal{V}_{\rm per}) \tag{3.8}$$

where $\mathcal{V}_{\rm per} = \{\Psi = (\psi_i) \in \mathcal{C}^\infty_{\rm per}(Y)^N : \operatorname{div}_y \Psi = 0\}$. *Then there is a unique* $q \in W^{1,p}_{\#}(Y)$ *such that*

$$D_y q = \mathbf{f}. \tag{3.9}$$

Proof. Consider again the sequence (θ_n) in the proof of Proposition 3.2. Let $\varphi_n = \mathbf{f} * \theta_n$ (integers $n \geq 1$), and bear in mind that $\varphi_n \in \mathcal{C}^\infty_{\rm per}(Y)^N$ and $\varphi_n \to \mathbf{f}$ in $L^p_{\rm per}(Y)^N$ when $n \to \infty$. Now, for $\Psi \in \mathcal{V}_{\rm per}$, note that

$$\int_Y \varphi_n \cdot \Psi dy = \int \theta_n(x) \left(\int_{Y-x} \mathbf{f}(y) \cdot \Psi_x(y) dy \right) dx$$

where $\Psi_x(y) = \Psi(y + x)$. But

$$\int_{Y-x} \mathbf{f}(y) \cdot \Psi_x(y) dy = \int_Y \mathbf{f}(y) \cdot \Psi_x(y) dy = 0 \text{ (use (3.8)), since } \Psi_x \in \mathcal{V}_{\rm per}.$$

Hence (3.8) still holds when $\mathbf{f}$ is replaced by φ_n. Consequently, by a well-known result there exists $q_n \in W^{1,2}_{\#}(Y)$ such that $D_y q_n = \varphi_n$. Furthermore, thanks to a classical regularity result (see [23, page 61]), q_n belongs to $\mathcal{C}^1_{\rm per}(Y) = \mathcal{C}^1(\mathbb{R}^N) \cap \mathcal{C}_{\rm per}(Y)$, thus $q_n \in W^{1,p}_{\#}(Y)$ and that for any integer $n \geq 1$. But then the sequence (q_n) is Cauchy in $W^{1,p}_{\#}(Y)$. Therefore $q_n \to q$ in $W^{1,p}_{\#}(Y)$ as $n \to \infty$, and it is clear that q is the sole function in $W^{1,p}_{\#}(Y)$ satisfying (3.9). □

This leads to the claimed result.

Proposition 3.3. *For each* $\Omega \subset \mathbb{R}^N_x$ *as above,* $W^{1,p}(\Omega)$ *is* Σ_S*-reflexive for any arbitrary real* $p > 1$.

Proof. Once we have Lemma 3.4 at our disposal, the proposition follows by a classical way (see, e.g., [18, Theorem 13]). □

We are now justified in stating the final conclusion.

Proposition 3.4. Σ_S *is proper for any arbitrary* $p > 1$.

Example 3.2. Any almost periodic H-structure on $\mathbb{R}^N$ is proper for $p = 2$ (cf. Theorem 4.1 and Proposition 4.3 of [22]).

Example 3.3. Let $\Sigma_{\infty,S}$ ($S = \mathbb{Z}^N$) be the H-structure (of class $\mathcal{C}^\infty$) on $\mathbb{R}^N$ defined by $\mathcal{J}(\Sigma_{\infty,S}) = \mathcal{B}_{\infty,S}(\mathbb{R}^N)$, where $\mathcal{B}_{\infty,S}(\mathbb{R}^N)$ denotes the closure in $\mathcal{B}(\mathbb{R}^N)$ of the space of all finite sums

$$\sum_{\text{finite}} \varphi_i u_i \quad \text{with } \varphi_i \in A_1 = \mathcal{B}_\infty(\mathbb{R}^N) \text{ and } u_i \in A_2 = \mathcal{C}_{\rm per}(Y) \quad (Y = (0,1)^N).$$

The H-structure $\Sigma_{\infty,S}$ is proper for $p = 2$ (cf. [19, Corollary 4.2]). Our purpose is to show that $\Sigma_{\infty,S}$ is actually proper for any $p > 1$. Let $\mathcal{V} = \mathcal{B}_0(\mathbb{R}^N) \oplus \mathcal{C}_{\rm per}(Y)$ (direct sum), where $\mathcal{B}_0(\mathbb{R}^N)$ denotes the space of those functions in $\mathcal{C}(\mathbb{R}^N)$ that vanish at infinity. It is worth noting that $\mathcal{V}$ is dense in $A = \mathcal{B}_{\infty,S}(\mathbb{R}^N)$ (cf.[19]). In the sequel we use the same $\mathcal{G}$ to denote the Gelfand transformation on A and on A_2, as well. This being so, define the mapping $\ell_2 : \mathcal{V} \to \mathcal{V}$ as being the projection on A_2

along $\mathcal{B}_0(\mathbb{R}^N)$. Thus, each $\psi \in \mathcal{V}$ admits the unique decomposition $\psi = \psi_0 + \ell_2(\psi)$ with $\psi_0 \in \mathcal{B}_0(\mathbb{R}^N)$. Let L_2 be the mapping of $\mathcal{G}(\mathcal{V})$ into $\mathcal{C}(\Delta(A_2))$ defined by $L_2(\mathcal{G}(\psi)) = \mathcal{G}(\ell_2(\psi))$ for $\psi \in \mathcal{V}$. In view of Proposition 3.4, the desired properness result will follow by Theorem 3.1 exactly in the same way as Corollary 4.2 of [19] was obtained by Theorem 4.2 of the same reference provided we check that L_2 extends by continuity to an isometric isomorphism L of $L^p(\Delta(A))$ onto $L^p(\Delta(A_2))$. This is straightforward by the classical inequality $||a|^p - |b|^p| \leq p|a-b|(|a|+|b|)^{p-1}$ $(a, b \in \mathbb{C})$. Indeed, if $\psi \in \mathcal{V}$ then by choosing $a = \psi(y)$ and $b = \ell_2(\psi)(y)$ for fixed $y \in \mathbb{R}^N$, we get $|M(|\psi|^p) - M(|\ell_2(\psi)|^p)| \leq CM(|\psi_0|)$, where C denotes the supremum of the function $p(|\psi| + |\ell_2(\psi)|)^{p-1}$. Since $M(|\psi_0|) = 0$, we deduce $M(|\psi|^p) = M(|\ell_2(\psi)|^p)$, i.e., $\|L_2(\widehat{\psi})\|_{L^p(\Delta(A_2))} = \|\widehat{\psi}\|_{L^p(\Delta(A))}$ with $\widehat{\psi} = \mathcal{G}(\psi)$, and that for all $\psi \in \mathcal{V}$. Therefore, thanks to the density of $\mathcal{G}(\mathcal{V})$ in $L^p(\Delta(A))$, the isometric isomorphism L follows.

Example 3.4. For any countable subgroup $\mathcal{R}$ of $\mathbb{R}^N$, the H-structure (of class $\mathcal{C}^\infty$) $\Sigma_{\infty,\mathcal{R}}$ on $\mathbb{R}^N$ [19, Example 3.5] is proper for $p = 2$ (cf. [19, Corollary 4.2]). In particular the H-structure Σ_∞ [19, Example 3.4] is proper for $p = 2$, since it coincides with $\Sigma_{\infty,\mathcal{R}}$ for $\mathcal{R} = \{\omega\}$ (ω the origin of $\mathbb{R}^N$).

Example 3.5. Let Σ_∞ be the H-structure of the convergence at infinity on $\mathbb{R}$, and let $\Sigma_{\mathcal{R}'}$ be the almost periodic H-structure on $\mathbb{R}^{N-1}$ represented by a countable subgroup $\mathcal{R}'$ of $\mathbb{R}^{N-1}$. Then the product H-structure $\Sigma = \Sigma_{\mathcal{R}'} \times \Sigma_\infty$ on $\mathbb{R}^N$ is proper for $p = 2$ (see [19, Example 4.4]).

4. The abstract homogenization problem

Throughout this section, Σ denotes an H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}^N$. We put $A = \mathcal{J}(\Sigma)$ and we denote by $\mathcal{G}$ the Gelfand transformation on A and by β the M-measure (on $\Delta(A)$) for A. Finally, Ω denotes a bounded open set in $\mathbb{R}^N_x$.

4.1. **Preliminaries.**

Let $1 \leq p < \infty$. We begin by introducing the space Ξ^p of all $u \in L^p_{\rm loc}(\mathbb{R}^N_y)$ for which the sequence $(u^\varepsilon)_{0<\varepsilon\leq 1}$ is bounded in $L^p_{\rm loc}(\mathbb{R}^N_x)$ (u^ε defined in (2.1)). We provide Ξ^p with the norm

$$\|u\|_{\Xi^p} = \sup_{o<\varepsilon\leq 1} \Big(\int_{B_N} |u(\frac{x}{\varepsilon})|^p dx \Big)^{1/p} \quad (u \in \Xi^p)$$

where B_N denotes the open unit ball in $\mathbb{R}^N_x$, which makes it a Banach space. This being so, we define $\mathfrak{X}^p_\Sigma$ to be the closure of A in Ξ^p. Provided with the Ξ^p-norm, $\mathfrak{X}^p_\Sigma$ is a Banach space. Furthermore, the Gelfand transformation on A extends by continuity to a continuous linear mapping, still denoted by $\mathcal{G}$, of $\mathfrak{X}^p_\Sigma$ into $L^p(\Delta(A))$ (cf. [19]). This is referred to as the *canonical mapping* of $\mathfrak{X}^p_\Sigma$ into $L^p(\Delta(A))$.

Given a locally compact space X (equipped with a positive Radon measure), we will most of the time put $L^p_\mathbb{R}(X) = L^p(X;\mathbb{R})$, $\mathcal{C}_\mathbb{R}(X) = \mathcal{C}(X;\mathbb{R})$. In particular we will write $\mathcal{D}_\mathbb{R}(\Omega) = \mathcal{D}(\Omega;\mathbb{R})$. Likewise we will put $A_\mathbb{R} = A \cap \mathcal{C}_\mathbb{R}(\mathbb{R}^N)$ and $A^\infty_\mathbb{R} = A^\infty \cap \mathcal{C}_\mathbb{R}(\mathbb{R}^N)$.

The main purpose of this section is to investigate the behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.4)) under the abstract structure hypothesis

$$a_i(\cdot, \Psi) \in \mathfrak{X}^{p'}_\Sigma \quad \text{for all } \Psi \in (A_\mathbb{R})^N \quad (1 \leq i \leq N) \tag{4.1}$$

where $p' = \frac{p}{p-1}$ with $1 < p \leq 2$, and where $a_i(\cdot, \Psi)$ denotes the function $y \to a_i(y, \Psi(y))$ of $\mathbb{R}^N$ into $\mathbb{R}$ (cf.Proposition 2.2).

This problem will be referred to as the *abstract homogenization problem for* (1.4). We will see that it is quite solvable provided the H-structure Σ is proper. However, before embarking upon the analysis of this homogenization problem as such, we need further notation and basic results.

We define

$$W^{1,p}_{\#}(\Delta(A);\mathbb{R}) = \{u \in W^{1,p}_{\#}(\Delta(A)) : \partial_j u \in L^p_{\mathbb{R}}(\Delta(A)) \ (1 \le j \le N)\}.$$

Equipped with the $W^{1,p}_{\#}(\Delta(A))$-norm, $W^{1,p}_{\#}(\Delta(A);\mathbb{R})$ is a Banach space. Next, we set

$$\mathbb{F}^{1,p}_0 = W^{1,p}_0(\Omega;\mathbb{R}) \times L^p(\Omega; W^{1,p}_{\#}(\Delta(A);\mathbb{R})),$$

and we provide $\mathbb{F}^{1,p}_0$ with the norm

$$\|\mathbf{v}\|_{\mathbb{F}^{1,p}_0} = \sum_{i=1}^{N} \left[\|D_{x_i} v_0\|_{L^p(\Omega)} + \|\partial_i v_1\|_{L^p(\Omega\times\Delta(A))}\right] \quad (\mathbf{v} = (v_0, v_1) \in \mathbb{F}^{1,p}_0),$$

which makes it a Banach space. Furthermore, if we assume that Σ is total (for p), then the space

$$F^{\infty}_0 = \mathcal{D}_{\mathbb{R}}(\Omega) \times [\mathcal{D}_{\mathbb{R}}(\Omega) \otimes J(\mathcal{D}_{\mathbb{R}}(\Delta(A))/\mathbb{C})]$$

is dense in $\mathbb{F}^{1,p}_0$ (use Remark 3.5).

Now, let the index $1 \le i \le N$ be arbitrarily fixed. For $\varphi = (\varphi_j)_{1\le j\le N} \in \mathcal{C}_{\mathbb{R}}(\Delta(A))^N$, let

$$b_i(\varphi) = \mathcal{G}(a_i(\cdot, \mathcal{G}^{-1}\varphi)) \tag{4.2}$$

where $\mathcal{G}^{-1}\varphi = (\mathcal{G}^{-1}\varphi_j)_{1\le j\le N}$, and where we recall that for $\theta \in (A_{\mathbb{R}})^N$, $a_i(\cdot,\theta)$ denotes the function $y \to a_i(y, \theta(y))$ (cf.Proposition 2.2) of $\mathbb{R}^N$ into $\mathbb{R}$, which belongs to $\mathfrak{X}^{p',\infty}_{\Sigma} = \mathfrak{X}^{p'}_{\Sigma} \cap L^{\infty}(\mathbb{R}^N_y)$ provided (4.1) holds. Thus, under hypothesis (4.1) we see that (4.2) defines a mapping b_i of $\mathcal{C}_{\mathbb{R}}(\Delta(A))^N$ into $L^{\infty}(\Delta(A))$ (see Corollary 2.2 of [19]).

Proposition 4.1. *Let $1 < p \le 2$. Suppose (4.1) holds. For $\Psi = (\psi_j)_{1\le j\le N}$ in $\mathcal{C}(\overline{\Omega}; (A_{\mathbb{R}})^N)$, let $b_i \circ \widehat{\Psi} = \mathcal{G}(a_i(\cdot,\Psi))$, i.e., $b_i(\widehat{\Psi}(x)) = \mathcal{G}(a_i(\cdot,\Psi(x)))$ for $x \in \overline{\Omega}$, where $\widehat{\Psi} = (\mathcal{G} \circ \psi_j)_{1\le j\le N}$. The following assertions are true :*

(i) *We have $b_i \circ \widehat{\Psi} \in \mathcal{C}(\overline{\Omega}; L^{\infty}(\Delta(A)))$ and*

$$a^{\varepsilon}_i(\cdot, \Psi^{\varepsilon}) \to b_i \circ \widehat{\Psi} \text{ in } L^{p'}(\Omega)\text{-weak } \Sigma \text{ when } \varepsilon \to 0. \tag{4.3}$$

(ii) *The mapping $\Phi \to b(\Phi) = (b_i \circ \Phi)_{1\le i\le N}$ of $\mathcal{C}(\overline{\Omega}; \mathcal{C}_{\mathbb{R}}(\Delta(A))^N)$ into $L^{p'}(\Omega\times\Delta(A))^N$ extends by continuity to a mapping, still denoted by b, of $L^p(\Omega; L^p_{\mathbb{R}}(\Delta(A))^N)$ into $L^{p'}(\Omega\times\Delta(A))^N$ such that*

$$\|b(\mathbf{u}) - b(\mathbf{v})\|_{L^{p'}(\Omega\times\Delta(A))^N} \le c\|\mathbf{u}-\mathbf{v}\|^{p-1}_{L^p(\Omega;L^p(\Delta(A))^N)} \tag{4.4}$$

and

$$(b(\mathbf{u}) - b(\mathbf{v}))\cdot(\mathbf{u}-\mathbf{v}) \ge \alpha|\mathbf{u}-\mathbf{v}|^p \text{ a.e. in } \Omega\times\Delta(A) \tag{4.5}$$

for all $\mathbf{u},\mathbf{v} \in L^p(\Omega; L^p_{\mathbb{R}}(\Delta(A))^N)$.

Proof. The function $x \to a_i(\cdot,\Psi(x))$ lies in $\mathcal{C}(\overline{\Omega}; \mathfrak{X}^{p',\infty}_{\Sigma})$ (see the proof of Corollary 2.2) ($\mathfrak{X}^{p',\infty}_{\Sigma}$ equipped with the L^{∞}-norm) and thus $b_i \circ \widehat{\Psi} \in \mathcal{C}(\overline{\Omega}; L^{\infty}(\Delta(A)))$, according to [19, Corollary 2.2]. Furthermore, thanks to [19, Corollary 4.1], we have

$a_i^\varepsilon(\cdot, \Psi^\varepsilon) \to b_i \circ \widehat{\Psi}$ in $L^{p'}(\Omega)$-weak Σ as $\varepsilon \to 0$. With this in mind, we now concentrate on (1.3) where we may assume that $|\cdot|$ denotes precisely that norm on $\mathbb{R}^N$ which is given by $|\zeta| = \sum_{j=1}^N |\zeta_j|$ for $\zeta = (\zeta_j)$. Based on this, we next use Lemma 2.2 (proceed as in the proof of Proposition 2.1) to see that if $\mathbf{f}, \mathbf{g} \in \mathcal{C}(\overline{\Omega}; (A_\mathbb{R})^N)$, then

$$|a^\varepsilon(\cdot, \mathbf{f}^\varepsilon) - a^\varepsilon(\cdot, \mathbf{g}^\varepsilon)| \le c|\mathbf{f}^\varepsilon - \mathbf{g}^\varepsilon|^{p-1} \quad \text{a.e. in } \Omega$$

and

$$[a^\varepsilon(\cdot, \mathbf{f}^\varepsilon) - a^\varepsilon(\cdot, \mathbf{g}^\varepsilon)] \cdot (\mathbf{f}^\varepsilon - \mathbf{g}^\varepsilon) \ge \alpha|\mathbf{f}^\varepsilon - \mathbf{g}^\varepsilon|^p \quad \text{a.e. in } \Omega$$

for any $\varepsilon > 0$. Letting $\varepsilon \to 0$, it follows by [19, Corollary 4.1] and use of Remark 4.1 below that

$$|b(\widehat{\mathbf{f}}) - b(\widehat{\mathbf{g}})| \le c|\widehat{\mathbf{f}} - \widehat{\mathbf{g}}|^{p-1} \quad \text{a.e. in } \Omega \times \Delta(A) \tag{4.6}$$

and

$$[b(\widehat{\mathbf{f}}) - b(\widehat{\mathbf{g}})] \cdot (\widehat{\mathbf{f}} - \widehat{\mathbf{g}}) \ge \alpha|\widehat{\mathbf{f}} - \widehat{\mathbf{g}}|^p \quad \text{a.e. in } \Omega \times \Delta(A) \tag{4.7}$$

where $b(\widehat{\mathbf{f}}) = (b_i \circ \widehat{\mathbf{f}})_{1 \le i \le N}$. Immediately (4.6) yields

$$\|b(\widehat{\mathbf{f}}) - b(\widehat{\mathbf{g}})\|_{L^{p'}(\Omega \times \Delta(A))^N} \le c\|\widehat{\mathbf{f}} - \widehat{\mathbf{g}}\|^{p-1}_{L^p(\Omega; L^p(\Delta(A))^N)}$$

and that for any $\mathbf{f}, \mathbf{g} \in \mathcal{C}(\overline{\Omega}; (A_\mathbb{R})^N)$. Hence, thanks to the fact that $\mathcal{C}(\overline{\Omega}; \mathcal{C}_\mathbb{R}(\Delta(A))^N)$ is (identifiable with) a dense subspace of $L^p(\Omega; L^p_\mathbb{R}(\Delta(A))^N)$, it follows that b extends by continuity to a mapping, still denoted b, of $L^p(\Omega; L^p_\mathbb{R}(\Delta(A))^N)$ into $L^{p'}(\Omega \times \Delta(A))^N$ such that (4.4) holds for all $\mathbf{u}, \mathbf{v} \in L^p(\Omega; L^p_\mathbb{R}(\Delta(A))^N)$. Finally, (4.5) follows from (4.7) by the above density argument combined with the continuity of $b : L^p(\Omega; L^p_\mathbb{R}(\Delta(A))^N) \to L^{p'}(\Omega \times \Delta(A))^N$ and use of the fact that $L^p(\Omega; L^p_\mathbb{R}(\Delta(A))^N)$ may be identified with $L^p_\mathbb{R}(\Omega \times \Delta(A))^N$. This completes the proof. □

Remark 4.1. If $v_\varepsilon \in L^p(\Omega)$ ($\varepsilon > 0$) with $v_\varepsilon \to v_0$ in $L^p(\Omega)$-weak Σ (as $\varepsilon \to 0$) and if for each $\varepsilon > 0$ we have $v_\varepsilon \ge 0$ a.e. in Ω, then $v_0 \ge 0$ a.e. in $\Omega \times \Delta(A)$.

The preceding proposition has an important corollary.

Corollary 4.1. *Let $\phi_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon$, i.e., $\phi_\varepsilon(x) = \psi_0(x) + \varepsilon\psi_1(x, \frac{x}{\varepsilon})$, $x \in \Omega$, where $\psi_0 \in \mathcal{D}_\mathbb{R}(\Omega)$ and $\psi_1 \in \mathcal{D}_\mathbb{R}(\Omega) \otimes A^\infty_\mathbb{R}$. Then, when $\varepsilon \to 0$,*

$$a_i^\varepsilon(\cdot, D\phi_\varepsilon) \to b_i(D_x\psi_0 + \partial\widehat{\psi}_1) \text{ in } L^{p'}(\Omega)\text{-weak } \Sigma \quad (1 \le i \le N) \tag{4.8}$$

where $\partial\widehat{\psi}_1 = (\partial_i\widehat{\psi}_1)_{1 \le i \le N}$. Furthermore, if $(v_\varepsilon)_{\varepsilon \in E}$ is a sequence in $L^p(\Omega)$ such that $v_\varepsilon \to v_0$ in $L^p(\Omega)$-weak Σ as $E \ni \varepsilon \to 0$, then, as $E \ni \varepsilon \to 0$,

$$\int_\Omega a_i^\varepsilon(\cdot, D\phi_\varepsilon)v_\varepsilon dx \to \int\int_{\Omega \times \Delta(A)} b_i(D_x\psi_0 + \partial\widehat{\psi}_1)v_0 dx d\beta \quad (1 \le i \le N).$$

Proof. Since $D\phi_\varepsilon = D_x\psi_0 + \varepsilon(D_x\psi_1)^\varepsilon + (D_y\psi_1)^\varepsilon$, it is clear that

$$\|a^\varepsilon(\cdot, D\phi_\varepsilon) - a^\varepsilon(\cdot, D_x\psi_0 + (D_y\psi_1)^\varepsilon)\|_{L^{p'}(\Omega)} \le c\varepsilon^{p-1}\|D_x\psi_1\|^{p-1}_{L^p(\Omega; \mathcal{B}(\mathbb{R}^N))}.$$

Therefore (4.8) follows by Proposition 4.1 (see especially (4.3)). Finally, recalling that for $\Psi = D_x\psi_0 + D_y\psi_1$ the function $x \to a_i(\cdot, \Psi(x, \cdot))$ lies in $\mathcal{C}(\overline{\Omega}; \mathfrak{X}^{p',\infty}_\Sigma)$, we see that the last part of the corollary follows immediately by [19, Proposition 4.5]. □

4.2. **The abstract homogenization result.** The main purpose of this subsection is to state and prove the following theorem.

Theorem 4.1. *Let $1 < p \leq 2$. Suppose (4.1) holds and Σ is proper (for p). For each real $\varepsilon > 0$, let u_ε be the solution of (1.4). As $\varepsilon \to 0$, we have*

$$u_\varepsilon \to u_0 \quad \text{in } W_0^{1,p}(\Omega)\text{-weak} \tag{4.9}$$

and

$$D_{x_i} u_\varepsilon \to D_{x_i} u_0 + \partial_i u_1 \ \text{ in } L^p(\Omega)\text{-weak } \Sigma \quad (1 \leq i \leq N), \tag{4.10}$$

where $\mathbf{u} = (u_0, u_1)$ is uniquely defined by $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^{1,p}$ and

$$\iint_{\Omega\times\Delta(A)} b(D_x u_0 + \partial u_1) \cdot (D_x v_0 + \partial v_1) dx d\beta = \langle f, v_0 \rangle \tag{4.11}$$

for all $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^{1,p}$.

Proof. It is a routine exercise to verify that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore, given an arbitrary fundamental sequence E, the properness of Σ and the compactness of the embedding $W_0^{1,p}(\Omega) \to L^p(\Omega)$ yield a subsequence E' from E and a couple $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^{1,p}$ such that (4.9) and (4.10) hold when $E' \ni \varepsilon \to 0$ (use Corollary 3.2). If we prove that $\mathbf{u} = (u_0, u_1)$ verifies the variational equation in (4.11), since such an equation admits at most one solution by virtue of (4.5), then it will turn out that (4.9) and (4.10) hold not only when $E \ni \varepsilon \to 0$ but merely when $0 < \varepsilon \to 0$. To this end let $\Phi \in F_0^\infty$, that is, let $\Phi = (\psi_0, J(\widehat{\psi}_1))$ with $\psi_0 \in \mathcal{D}_{\mathbb{R}}(\Omega)$, $\psi_1 \in \mathcal{D}_{\mathbb{R}}(\Omega) \otimes (A_{\mathbb{R}}^\infty/\mathbb{C})$ and $J(\widehat{\psi}_1) = J \circ \widehat{\psi}_1$. Define ϕ_ε as in Corollary 4.1. Clearly

$$0 \leq \langle f, u_\varepsilon - \phi_\varepsilon \rangle - \int_\Omega a^\varepsilon(\cdot, D\phi_\varepsilon) \cdot (Du_\varepsilon - D\phi_\varepsilon) dx. \tag{4.12}$$

Indeed, the right-hand side is equal to

$$\int_\Omega [a^\varepsilon(\cdot, Du_\varepsilon) - a^\varepsilon(\cdot, D\phi_\varepsilon)] \cdot (Du_\varepsilon - D\phi_\varepsilon) dx$$

and the latter is nonnegative. Now, noting that $\phi_\varepsilon \to \psi_0$ in $W_0^{1,p}(\Omega)$-weak as $\varepsilon \to 0$, we next pass to the limit (as $E' \ni \varepsilon \to 0$) in (4.12) using (4.9), (4.10) and Corollary 4.1, and we obtain

$$0 \leq \langle f, u_0 - \psi_0 \rangle - \iint_{\Omega\times\Delta(A)} b(\mathbb{D}\Phi) \cdot \mathbb{D}(\mathbf{u} - \Phi) \, dx \, d\beta \tag{4.13}$$

where $\mathbb{D}\Phi = D_x\psi_0 + \partial\widehat{\psi}_1$ and $\mathbb{D}(\mathbf{u} - \Phi) = D_x(u_0 - \psi_0) + \partial(u_1 - \widehat{\psi}_1)$. Thanks to the density of F_0^∞ in $\mathbb{F}_0^{1,p}$, (4.13) still holds for any $\Phi \in \mathbb{F}_0^{1,p}$. Finally, take in (4.13) the particular $\Phi = \mathbf{u} - t\mathbf{v}$ with $t > 0$ and $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^{1,p}$, then divide both sides of the resultant inequality by t and, letting $t \to 0$, pass to the limit using (4.4). Hence (4.11) follows at once. $\square$

The variational problem (4.11) is referred to as the *global homogenized problem for* (1.4) under the structure hypothesis (4.1) (where Σ is assumed to be proper for p). It is immediate that the variational equation in (4.11) is equivalent to the system of the two equations :

$$\iint_{\Omega\times\Delta(A)} b(D_x u_0 + \partial u_1) \cdot \partial v_1 \, dx d\beta = 0 \quad \text{for all } v_1 \in L^p(\Omega; W_\#^{1,p}(\Delta(A); \mathbb{R})) \tag{4.14}$$

and

$$\int_\Omega \Big[\int_{\Delta(A)} b(D_x u_0 + \partial u_1)d\beta\Big] \cdot D_x v_0 dx = \langle f, v_0\rangle \quad \text{for all } v_0 \in W_0^{1,p}(\Omega;\mathbb{R}). \quad (4.15)$$

The next point is to derive the *macroscopic homogenized problem for* (1.4) under hypothesis (4.1). To this end, let $r \in \mathbb{R}^N$ be freely fixed, and let $\pi(r)$ be defined by the so-called *cell problem*: $\pi(r) \in W_\#^{1,p}(\Delta(A);\mathbb{R})$ and

$$\int_{\Delta(A)} b(r + \partial\pi(r)) \cdot \partial\theta \, d\beta = 0 \quad \text{for all } \theta \in W_\#^{1,p}(\Delta(A);\mathbb{R}). \quad (4.16)$$

Thanks to (4.4) and (4.5), the existence and unicity of $\pi(r)$ follow by adaptation of a classical line of argument (see, e.g., [15]). This yields a mapping π of $\mathbb{R}^N$ into $W_\#^{1,p}(\Delta(A);\mathbb{R})$.

Now, in (4.14) choose v_1 of the form $v_1(x) = \varphi(x)\theta$ ($x \in \Omega$) with $\varphi \in \mathcal{D}_\mathbb{R}(\Omega)$ and $\theta \in W_\#^{1,p}(\Delta(A);\mathbb{R})$. Then, almost everywhere in $x \in \Omega$,

$$\int_{\Delta(A)} b(Du_0(x) + \partial u_1(x)) \cdot \partial\theta \; d\beta = 0 \quad \text{for all } \theta \in W_\#^{1,p}(\Delta(A);\mathbb{R}).$$

Comparing with (4.16) for $r = Du_0(x)$ (x arbitrarily fixed), it follows (by the unicity argument) that $u_1 = \pi(Du_0)$, where the right-hand side stands for the function $x \to \pi(Du_0(x))$ of Ω into $W_\#^{1,p}(\Delta(A);\mathbb{R})$. Hence, letting

$$q(r) = \int_{\Delta(A)} b(r + \partial\pi(r))d\beta \quad (r \in \mathbb{R}^N),$$

we see by (4.15) that u_0 is a solution of the boundary value problem

$$-\operatorname{div} q(Du_0) = f \quad \text{in } \Omega, \quad u_0 \in W_0^{1,p}(\Omega;\mathbb{R}). \quad (4.17)$$

Remark 4.2. By a method similar to that which is commonly followed in the periodic case (see, e.g., [17]), it is possible to show that the homogenized operator, i.e., the operator $v \to \operatorname{div} q(Dv)$ of $W_0^{1,p}(\Omega)$ into $W_0^{-1,p'}(\Omega)$, is lipschitz continuous and strictly monotone as in (1.3), so that u_0 is uniquely defined by (4.17).

Application. By way of illustration, let us suppose that $a(y,\lambda)$ verifies the classical periodicity hypothesis (see Section 1). We want to show that the abstract hypothesis (4.1) is then verified so that Theorem 4.1 holds. Let $1 < p \leq 2$, and fix freely $1 \leq i \leq N$. For each $\lambda \in \mathbb{R}^N$, we have $a_i(\cdot,\lambda) \in L_{\text{per}}^{p'}(Y)$ (cf.Subsection 3.3), since $a_i(\cdot,\lambda) \in L^\infty(\mathbb{R}_y^N)$, as is quickly seen by using (1.1), (1.2) and part (ii) of (1.3). But $L_{\text{per}}^{p'}(Y) = \mathfrak{X}_\Sigma^{p'}$ (use [22, Lemma 1.3]) with $\Sigma = \Sigma_S$ (and $A_\mathbb{R} = \mathcal{C}_{\text{per}}(Y) \cap \mathcal{C}_\mathbb{R}(\mathbb{R}^N)$, of course). Thanks to Proposition 3.4, this leads us to the conclusion of Theorem 4.1 with $\Sigma = \Sigma_S$, with Y in place of $\Delta(A)$, dy in place of $d\beta$, and D_y in that of ∂ (see Subsection 3.3).

5. Concrete homogenization problems for (1.4)

5.1. **Introduction.** First of all, it is worth recalling that a homogenization problem is posed as soon as one has on one hand a suitable boundary value problem, on the other hand a so-called structure hypothesis. Thus, each of the structure hypotheses presented in Examples 1.1-1.4 (see also the periodicity hypothesis) determines a specific homogenization problem for the boundary value problem (1.4). Such a homogenization problem is said to be concrete because the associated structure hypothesis appears under a natural form, as opposed to the abstract hypothesis (4.1).

The present section deals with the study of a few concrete homogenization problems for (1.4). Thanks to the results achieved in the precedent section (see especially Theorem 4.1), the whole problem in each case will reduce to showing that the associated concrete structure hypothesis can write as (4.1) for a suitable proper H-structure Σ (to be determined).

5.2. **Problem I.** Here, our purpose is to work out the homogenization of (1.4) under the structure hypothesis : For each fixed $\lambda \in \mathbb{R}^N$, $a(\cdot,\lambda)$ lies in $\mathcal{C}(\mathbb{R}^N;\mathbb{R}^N)$ and further $a(y,\lambda)$ converges in $\mathbb{R}^N$ when $|y| \to +\infty$. Let us begin by noting that this structure hypothesis again writes as

$$a_i(\cdot,\lambda) \in \mathcal{B}_\infty(\mathbb{R}^N_y) \quad \text{for each fixed } \lambda \in \mathbb{R}^N \ (1 \le i \le N).$$

This suggests that we should introduce the H-structure $\Sigma = \Sigma_\infty$ of the convergence at infinity on $\mathbb{R}^N$ [19]. Accordingly $A = \mathcal{B}_\infty(\mathbb{R}^N_y)$ (the image of Σ_∞). Now, let $\Psi \in (A_\mathbb{R})^N$. Put $r = \lim_{|y|\to+\infty} \Psi(y)$. By (1.3) we have $|a_i(y,\Psi(y)) - a_i(y,r)| \le c|\Psi(y) - r|^{p-1}$ for $y \in \mathbb{R}^N$. Hence $a_i(y,\Psi(y)) = a_i(y,r) + \varphi(y)$ for $y \in \mathbb{R}^N$, where $\varphi \in \mathcal{B}_0(\mathbb{R}^N)$ [observe that $a_i \in \mathcal{C}(\mathbb{R}^N_\lambda;\mathcal{B}(\mathbb{R}^N_y))$, which implies that a_i is continuous on $\mathbb{R}^N \times \mathbb{R}^N$]. Therefore $a_i(\cdot,\Psi) \in A$ $(1 \le i \le N)$ for every $\Psi \in (A_\mathbb{R})^N$. Thus, (4.1) holds with $\Sigma = \Sigma_\infty$. Since the H-structure Σ_∞ is proper for $p = 2$ (Example 3.4), it follows that the conclusion of Theorem 4.1 and the subsequent developments hold when $p = 2$ and $\Sigma = \Sigma_\infty$. This solves the homogenization problem under consideration.

5.3. **Problem II.** The present subsection deals with the homogenization of (1.4) under the structure hypothesis (1.5). Noting that the space $\mathcal{B}_{\infty,S}(\mathbb{R}^N)$ in Example 3.3 is none other than $\mathcal{B}_{\infty,\mathrm{per}}(Y)$, we see that the appropriate H-structure is $\Sigma = \Sigma_{\infty,S}$, and the latter is a proper H-structure for each real $p > 1$ (Example 3.3). Thus, the solution of the homogenization problem under consideration is provided by Theorem 4.1 if we can check that (4.1) holds with $\Sigma = \Sigma_{\infty,S}$. In fact, we want to show a better result namely

Proposition 5.1. *Suppose (1.5) holds. Then, for all* $\Psi \in (A_\mathbb{R})^N$,

$$a_i(\cdot,\Psi) \in A = \mathcal{B}_{\infty,S}(\mathbb{R}^N) \quad (1 \le i \le N). \tag{5.1}$$

Proof. We begin by proving (5.1) for Ψ of the form

$$\Psi = \Psi_1 + \Psi_2 \quad \text{with } \Psi_1 \in (A_1)^N \text{ and } \Psi_2 \in (A_2)^N \tag{5.2}$$

where $A_1 = \mathcal{B}_\infty(\mathbb{R}^N) \cap \mathcal{C}_\mathbb{R}(\mathbb{R}^N)$ and $A_2 = \mathcal{C}_{\mathrm{per}}(Y) \cap \mathcal{C}_\mathbb{R}(\mathbb{R}^N)$.

This will be done in two steps.

1) For fixed $1 \le i \le N$, let us assume that a_i is of the form

$$a_i(y,\lambda) = \chi(\lambda)\varphi(y) \quad (y,\lambda \in \mathbb{R}^N) \tag{5.3}$$

where $\varphi \in A_\mathbb{R}$ and $\chi \in \mathcal{C}_\mathbb{R}(\mathbb{R}^N)$ (see (1.5)). Let $r_1 = \lim_{|y|\to\infty} \Psi_1(y)$. Clearly $a_i(\cdot, r_1 + \Psi_2) \in A$. Next, since Ψ_1 and Ψ_2 are bounded, we may consider a compact set $K \subset \mathbb{R}^N$ such that $\Psi(y) = \Psi_1(y) + \Psi_2(y) \in K$ for all $y \in \mathbb{R}^N$. On the other hand, let $c_1 > 0$ be a constant such that $\|\varphi\|_\infty \le c_1$. Finally, let $\eta > 0$. Since the function χ is uniformly continuous on K, there is some $\gamma > 0$ such that

$|\chi(\lambda)-\chi(\mu)|\leq\frac{\eta}{c_1}$ for all $\lambda,\mu\in K$ with $|\lambda-\mu|\leq\gamma$. Furthermore, let $\rho>0$ be such that $|\Psi_1(y)-r_1|\leq\gamma$ for $y\in\mathbb{R}^N$ with $|y|\geq\rho$. Then

$$|a_i(y,\Psi(y))-a_i(y,r_1+\Psi_2(y))|\leq\eta \text{ for } y\in\mathbb{R}^N \text{ with } |y|\geq\rho.$$

Therefore $a_i(\cdot,\Psi)=a_i(\cdot,r_1+\Psi_2)+\phi$ with $\phi\in\mathcal{B}_0(\mathbb{R}^N)$, which shows (5.1). But then, clearly (5.1) still holds true for a_i in $\mathcal{C}_{\mathbb{R}}(\mathbb{R}^N)\otimes A_{\mathbb{R}}$ (the finite sums $\sum\chi_j\otimes\varphi_j$ with $\varphi_j\in A_{\mathbb{R}}$ and $\chi_j\in\mathcal{C}_{\mathbb{R}}(\mathbb{R}^N)$).

2) Now, we consider the general case where we know merely that a (in addition to (1.1)-(1.3)) satisfies (1.5). Let Ψ be given with (5.2), and let K be as above. For convenience we still denote the restriction $a_i\big|_K$ by a_i. Then, clearly $a_i\in\mathcal{C}(K;A_{\mathbb{R}})$. Thanks to the density of $\mathcal{C}_{\mathbb{R}}(K)\otimes A_{\mathbb{R}}$ in $\mathcal{C}(K;A_{\mathbb{R}})$ (see, e.g., page 46 of [5]), there exists a sequence of functions $\zeta_n\in\mathcal{C}_{\mathbb{R}}(K)\otimes A_{\mathbb{R}}$ (integers $n\geq 0$) such that

$$\|\zeta_n-a_i\|_{\mathcal{C}(K;A)}\equiv\sup_{y\in\mathbb{R}^N,\,\lambda\in K}|\zeta_n(y,\lambda)-a_i(y,\lambda)|\to 0 \quad \text{as } n\to\infty.$$

It follows immediately that $\zeta_n(\cdot,\Psi)\to a_i(\cdot,\Psi)$ in $\mathcal{B}(\mathbb{R}^N)$ (with the supremum norm) as $n\to\infty$. But $\zeta_n(\cdot,\Psi)\in A$, according to step 1). Hence (5.1) follows for Ψ of the form (5.2). This completes step 2).

Finally, let $\Psi\in(A_{\mathbb{R}})^N$ be arbitrarily fixed. Remarking that $\mathcal{B}_\infty(\mathbb{R}^N)+\mathcal{C}_{\mathrm{per}}(Y)$ is dense in A, we see that $(A_1)^N+(A_2)^N$ is dense in $(A_{\mathbb{R}})^N$ so that we may consider a sequence of functions $\mathbf{f}_n$ of the form (5.2) such that $\mathbf{f}_n\to\Psi$ in $\mathcal{B}(\mathbb{R}^N)^N$ as $n\to\infty$. Therefore (5.1) follows by $\|a_i(\cdot,\Psi)-a_i(\cdot,\mathbf{f}_n)\|_\infty\leq c\|\Psi-\mathbf{f}_n\|_\infty^{p-1}$ (according to (1.3)) and use of the fact that $a_i(\cdot,\mathbf{f}_n)\in A$, as established previously. □

5.4. **Problem III.** In this subsection we assume that the family $\{a(\cdot,\lambda)\}_{\lambda\in\mathbb{R}^N}$ is uniformly equicontinuous, i.e.,

(UE) Given $\eta>0$, there exists $\rho>0$ such that $|a(y-r,\lambda)-a(y,\lambda)|\leq\eta$ for all $y,\lambda\in\mathbb{R}^N$ provided $|r|\leq\rho$.

Our purpose is to investigate the behaviour (as $\varepsilon\to 0$) of u_ε (the solution of (1.4)) under the structure hypothesis (1.6). To this end we first note that $\mathcal{B}_\infty(\mathbb{R};\mathcal{C}_{\mathrm{per}}(Y'))$ is the image of the product H-structure $\Sigma=\Sigma_{\mathcal{R}'}\times\Sigma_\infty$ ($\mathcal{R}'=\mathbb{Z}^{N-1}$) on $\mathbb{R}^N=\mathbb{R}^{N-1}\times\mathbb{R}$ and further that the latter is proper for $p=2$ (Example 3.5). Thus, for $p=2$, the homogenization problem under consideration will have been solved through Theorem 4.1 if we can show that (4.1) holds (with $p=2$). But this will be a direct consequence of a stronger result.

Proposition 5.2. *Suppose (1.6) holds. Then*

$$a_i(\cdot,\Psi)\in A=\mathcal{B}_\infty(\mathbb{R};\mathcal{C}_{\mathrm{per}}(Y')) \quad \textit{for all } \Psi\in(A_{\mathbb{R}})^N \quad (1\leq i\leq N). \tag{5.4}$$

Proof. Let $\Psi\in(A_{\mathbb{R}})^N$. Let $\theta=\lim_{|y_N|\to\infty}\Psi(\cdot,y_N)$, where the limit is taken in $\mathcal{B}(\mathbb{R}^{N-1})$ (with the supremum norm) and where $\Psi(\cdot,y_N)$ stands for the function $y'=(y_1,\cdots,y_{N-1})\to\Psi(y',y_N)$ of $\mathbb{R}^{N-1}$ into $\mathbb{R}$. Let us begin by verifying that the function $y\to a_i(y,\theta(y'))$ of $\mathbb{R}^N$ into $\mathbb{R}$, denoted by $a_i(\cdot,\theta')$, lies in A. For fixed $y_N\in\mathbb{R}$, let $a_i(\cdot,y_N,\theta)$ denote the function $y'\to a_i(y',y_N,\theta(y'))$ of $\mathbb{R}^{N-1}$ into $\mathbb{R}$. It is an easy matter to check, using (UE), that the mapping $y_N\to a_i(\cdot,y_N,\theta)$ sends continuously $\mathbb{R}$ into $\mathcal{C}_{\mathrm{per}}(Y')$. Thus, we will claim that $a_i(\cdot,\theta')$ lies in A if we have shown that $a_i(\cdot,y_N,\theta)$ has a limit in $\mathcal{B}(\mathbb{R}^{N-1})$ when $|y_N|\to\infty$. We proceed in two steps: 1) Suppose a_i is of the form (5.3). Let $h\in\mathcal{C}_{\mathrm{per}}(Y')$ be the limit of $\varphi(\cdot,y_N)$

in $\mathcal{B}(\mathbb{R}^{N-1})$ as $|y_N| \to \infty$. The mapping $y' \to h(y')\chi(\theta(y'))$ of $\mathbb{R}^{N-1}$ into $\mathbb{R}$ lies in $\mathcal{C}_{\mathrm{per}}(Y')$ and

$$|a_i(y,\theta(y')) - h(y')\chi(\theta(y'))| \leq C|\varphi(y) - h(y')| \quad (y \in \mathbb{R}^N)$$

where $C = \max_{\lambda \in K} |\chi(\lambda)|$, K being a compact set in $\mathbb{R}^N$ such that $\theta(y') \in K$ for all $y' \in \mathbb{R}^{N-1}$. This shows that when $|y_N| \to \infty$, $a_i(\cdot, y_N, \theta)$ converges in $\mathcal{B}(\mathbb{R}^{N-1})$ to the preceding function. 2) We now consider the general case where we know merely that, in addition to (1.1)-(1.3) and (UE), a satisfies (1.6). For convenience we still write a_i for $a_i\big|_K$, K being as above. Then a_i belongs to $\mathcal{C}(K; A_{\mathbb{R}})$. By the same line of argument as followed in proving Proposition 5.1, we are immediately led to the desired result.

We are now in a position to show that $a_i(\cdot,\Psi)$ lies in A. This is straightforward. Indeed, by

$$|a_i(y,\Psi(y)) - a_i(y,\theta(y'))| \leq c|\Psi(y) - \theta(y')|^{p-1} \quad (y \in \mathbb{R}^N)$$

we are quickly led to $a_i(\cdot,\Psi) = a_i(\cdot,\theta') + \phi$, where $\phi \in \mathcal{B}_0(\mathbb{R}; \mathcal{C}_{\mathrm{per}}(Y'))$ [it is an easy matter to show, using (1.3)(ii) and (UE), that $a_i(\cdot,\Psi) \in \mathcal{C}(\mathbb{R}; \mathcal{C}_{\mathrm{per}}(Y'))$]. Hence (5.4) follows. $\square$

5.5. **Problem IV.** The matter in hand is the homogenization of (1.4) under the structure hypothesis (1.7). For this purpose, starting from (1.7) we begin by considering (exactly as in Lemma 5.1 of [22]) a countable subgroup $\mathcal{R}$ of $\mathbb{R}^N$ such that [19, 22]

$$a_i(\cdot,\lambda) \in A = AP_{\mathcal{R}}(\mathbb{R}^N) \quad \text{for any } \lambda \in \mathbb{R}^N \ (1 \leq i \leq N).$$

Let $\Sigma_{\mathcal{R}}$ be the almost periodic H-structure on $\mathbb{R}^N$ represented by $\mathcal{R}$. We have $\mathcal{J}(\Sigma_{\mathcal{R}}) = A = AP_{\mathcal{R}}(\mathbb{R}^N)$ and $\Sigma_{\mathcal{R}}$ is proper for $p = 2$, as pointed out in Example 3.2. Therefore, if we show that

$$a_i(\cdot,\Psi) \in A \text{ for all } \Psi \in (A_{\mathbb{R}})^N \ (1 \leq i \leq N), \tag{5.5}$$

which implies (4.1) with $\Sigma = \Sigma_{\mathcal{R}}$, then for $p = 2$ the solution of the homogenization problem under examination follows at once by Subsection 4.2.

To do this, let us fix freely $\Psi \in (A_{\mathbb{R}})^N$. Let K be a compact set in $\mathbb{R}^N$ such that $\Psi(y) \in K$ for all $y \in \mathbb{R}^N$. For an obvious reason we may here view a_i as belonging to $\mathcal{C}(K; A_{\mathbb{R}})$. Let us first assume that a_i is of the form

$$a_i(\lambda) = \chi(\lambda)\varphi \quad (\lambda \in K) \text{ with } \chi \in \mathcal{C}_{\mathbb{R}}(K) \text{ and } \varphi \in A_{\mathbb{R}}.$$

By the Stone-Weierstrass theorem (see chapter X, page 37 of [4]) there is a sequence (f_n) of polynomials in $\lambda = (\lambda_1, \cdots, \lambda_N)$ such that $f_n \to \chi$ in $\mathcal{C}(K)$ when $n \to \infty$. Thus, $f_n(\Psi) \to \chi(\Psi)$ in $\mathcal{B}(\mathbb{R}^N)$ as $n \to \infty$. It follows that $\chi(\Psi) \in A_{\mathbb{R}}$, since $f_n(\Psi) \in A_{\mathbb{R}}$. Hence $a_i(\cdot,\Psi) \in A$. But then this is still manifestly true if $a_i(\lambda) = \sum \chi(\lambda)\varphi$ for $\lambda \in K$, where χ (resp. φ) ranges over a finite subset of $\mathcal{C}_{\mathbb{R}}(K)$ (resp. $A_{\mathbb{R}}$). Hence the same routine argument as used in Step 2) of the proof of Proposition 5.1 leads us to (5.5).

5.6. **Problem V.** We assume here that $a(y,\lambda)$ satisfies (1.8) and (1.9), and we want to study the homogenization of (1.4) for $p = 2$. As is now well known, this problem reduces to the abstract problem of Section 4 if for some proper H-structure Σ, condition (4.1) is fulfilled. To achieve this, let $\theta \in \mathcal{D}(\mathbb{R}^N)$ with $\theta \geq 0$, $Supp\theta \subset \bar{B}_N$ (closed unit ball of $\mathbb{R}^N$) and $\int \theta(y)dy = 1$. For each integer $n \geq 1$, set $\theta_n(y) = n^N\theta(ny)$ $(y \in \mathbb{R}^N)$, which gives a sequence of functions $\theta_n \in \mathcal{D}(\mathbb{R}^N)$. Now, define

$$\zeta_n^i(y,\lambda) = \int \theta_n(r)a_i(y-r,\lambda)dr \quad \text{for } y,\lambda \in \mathbb{R}^N \quad (1 \le i \le N).$$

It is easily checked that $\zeta_n^i(\cdot,\lambda) \in AP(\mathbb{R}_y^N)$ for $\lambda \in \mathbb{R}^N$ and further

$$|\zeta_n(y,\lambda) - \zeta_n(y,\mu)| \le c|\lambda-\mu|^{p-1} \quad (\lambda,\mu,y \in \mathbb{R}^N) \tag{5.6}$$

where $\zeta_n = (\zeta_n^i)_{1\le i\le N}$. Therefore, according to Subsection 5.5, there is a countable subgroup $\mathcal{R}$ of $\mathbb{R}^N$ such that $\zeta_n^i(\cdot,\Psi) \in A = AP_{\mathcal{R}}(\mathbb{R}_y^N)$ for $\Psi \in (A_{\mathbb{R}})^N$, $1 \le i \le N$, and all integers $n \ge 1$. Furthermore, given $\Psi \in (A_{\mathbb{R}})^N$ and $1 \le i \le N$, a quick calculation reveals that

$$\begin{aligned} &\|\zeta_n^i(\cdot,\Psi) - a_i(\cdot,\Psi)\|_{2,\infty}^2 \\ &\le \int_{\frac{1}{n}B_N} \theta_n(r)\Big[\sup_{k\in\mathbb{Z}^N}\int_{k+Y} |a_i(y-r,\Psi(y)) - a_i(y,\Psi(y))|^2dy\Big]dr \end{aligned} \tag{5.7}$$

for any integer $n \ge 1$, where $\|\cdot\|_{2,\infty}$ is the norm in the amalgam space $(L^2,\ell^\infty)(\mathbb{R}^N)$ [13, 22] (see also [19, Example 5.4]). Hence, if $\eta > 0$ is given, then by (1.9) we are led to some integer $\gamma \ge 1$ such that $\|\zeta_n^i(\cdot,\Psi) - a_i(\cdot,\Psi)\|_{2,\infty} \le \eta$ for any $n \ge \gamma$. Since $(L^2,\ell^\infty)(\mathbb{R}^N)$ is continuously embedded in $\Xi^2(\mathbb{R}^N)$ [19] (use [22, Lemma 1.3]), we deduce that (4.1) follows with $\Sigma = \Sigma_{\mathcal{R}}$ (as in Subsection 5.5) and $p = 2$.

Remark 5.1. Condition (1.9) is fulfilled if the following holds : Given $\eta > 0$, there exists $\rho > 0$ such that $|a(y-r,\lambda) - a(y,\lambda)| \le \eta$ for all $\lambda \in \mathbb{R}^N$ and for almost all $y \in \mathbb{R}^N$ provided $|r| \le \rho$.

5.7. **Problem VI.** This subsection is intended to study the homogenization of (1.4) for $p = 2$ under the following hypotheses:

$$a_i(\cdot,\lambda) \in \mathcal{B}_\infty(\mathbb{R}; L^\infty_{\mathrm{per}}(Y')) \quad \text{for each } \lambda \in \mathbb{R}^N \ (1 \le i \le N) \tag{5.8}$$

$$\begin{gathered} \text{Given } \eta > 0, \text{ a real } \rho > 0 \text{ exists such that } |a(y-r,\lambda) - a(y,\lambda)| \le \eta \\ \text{for all } \lambda \in \mathbb{R}^N \text{ and for almost all } y \in \mathbb{R}^N \text{ provided } |r| \le \rho \end{gathered} \tag{5.9}$$

where Y' is as in (1.6). As we are now familiar with the approach, the whole problem reduces to verifying that (4.1) holds with $p = 2$ and $\Sigma = \Sigma_{\mathcal{R}'=\mathbb{Z}^{N-1}} \times \Sigma_\infty$, hence $A = \mathcal{B}_\infty(\mathbb{R}; \mathcal{C}_{\mathrm{per}}(Y'))$. This proceeds by adaptation of what we did earlier in Subsection 5.6.

Let $(\zeta_n)_{n\ge1}$ with $\zeta_n = (\zeta_n^i)_{1\le i\le N}$ for each integer $n \ge 1$, where ζ_n^i is defined in Subsection 5.6. Let $\lambda \in \mathbb{R}^N$ be fixed. Clearly

$$a_i(y,\lambda) = t_i(y',\lambda) + g_i(y,\lambda) \quad \text{for } y \in \mathbb{R}^N \ (1 \le i \le N)$$

where the functions $y' = (y_1,\cdots,y_{N-1}) \to t_i(y',\lambda)$ from $\mathbb{R}^{N-1}$ to $\mathbb{R}$ and $y \to g_i(y,\lambda)$ from $\mathbb{R}^N$ to $\mathbb{R}$ lie in $L^\infty_{\mathrm{per}}(Y')$ and $\mathcal{B}_0(\mathbb{R}; L^\infty_{\mathrm{per}}(Y'))$ (those functions in $\mathcal{B}_\infty(\mathbb{R}; L^\infty_{\mathrm{per}}(Y'))$ that vanish at infinity), respectively. Hence

$$\zeta_n^i(y,\lambda) = \varphi_n^i(y',\lambda) + \gamma_n^i(y,\lambda) \text{ for } y \in \mathbb{R}^N \quad (1 \le i \le N)$$

where

$$\varphi_n^i(y',\lambda) = \int \theta_n(r)t_i(y-r,\lambda)dr \quad (y \in \mathbb{R}^N)$$

and

$$\gamma_n^i(y,\lambda) = \int \theta_n(r)g_i(y-r,\lambda)dr \quad (y \in \mathbb{R}^N), \tag{5.10}$$

$t_i(\cdot,\lambda)$ being considered as a function in $L^\infty(\mathbb{R}^N)$ independent of the N^{th} variable. It is immediate that the function $y' \to \varphi_n^i(y',\lambda)$ lies in $\mathcal{C}_{\rm per}(Y')$. On the other hand, whereas $\gamma_n^i(\cdot,\lambda)$, considered as a function of the variable y_N, belongs to $\mathcal{B}(\mathbb{R};\mathcal{C}_{\rm per}(Y'))$ (this follows at once by classical properties of the convolution), it is not clear, however, that the latter function vanishes at infinity. To see this, let $\eta > 0$. Let $\rho > 0$ be such that

$$\|g_i(z_N,\lambda)\|_{L^\infty(\mathbb{R}^{N-1})} \le \eta \tag{5.11}$$

for all $z_N \in \mathbb{R}$ verifying $|z_N| \ge \rho$, where $g_i(\cdot,\lambda)$ is viewed as a function of $z_N \in \mathbb{R}$ with values in $L^\infty_{\rm per}(Y')$, of course. Fix $y = (y', y_N) \in \mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$ where y' is arbitrary and y_N is subject to the condition $|y_N| \ge 1+\rho$. Since the variable $r = (r_1,\cdots,r_N)$ in (5.10) actually runs through $\frac{1}{n}\bar{B}_N$, it follows from (5.11) that

$$\|g_i(y_N - r_N,\lambda)\|_{L^\infty(\mathbb{R}^{N-1})} \le \eta$$

for $|y_N| \ge 1+\rho$ provided r lies in $\frac{1}{n}\bar{B}_N$. Based on (5.10), we deduce that $\|\gamma_n^i(y_N,\lambda)\|_\infty \le \eta$ for $|y_N| \ge 1+\rho$ and so $\gamma_n^i(\cdot,\lambda)$ lies in $\mathcal{B}_0(\mathbb{R};\mathcal{C}_{\rm per}(Y'))$. Hence $\zeta_n^i(\cdot,\lambda) \in A = \mathcal{B}_\infty(\mathbb{R};\mathcal{C}_{\rm per}(Y'))$.

On the other hand, it is clear that (5.6) holds true. Finally, for each fixed integer $n \ge 1$, the family $\{\zeta_n(\cdot,\lambda)\}_{\lambda\in\mathbb{R}^N}$ is uniformly equicontinuous (see Subsection 5.4), as is easily seen by using (5.9). Consequently, we are justified in replacing $a(\cdot,\lambda)$ by $\zeta_n(\cdot,\lambda)$ in Proposition 5.2, so that $\zeta_n^i(\cdot,\Psi) \in A$ $(1 \le i \le N)$ for all $\Psi \in (A_\mathbb{R})^N$.

With this in mind, fix freely $\Psi \in (A_\mathbb{R})^N$. Clearly (5.7) holds true for $1 \le i \le N$. On the other hand, by (5.9) one easily gets

$$\sup_{k\in\mathbb{Z}^N} \int_{k+Y} |a(y-r,\Psi(y)) - a(y,\Psi(y))|^2 dy \to 0 \text{ as } |r| \to 0.$$

Therefore, given $\eta > 0$, there is an integer $\gamma \ge 1$ such that $\|\zeta_n^i(\cdot,\Psi) - a_i(\cdot,\Psi)\|_{2,\infty} \le \eta$ for any $n \ge \gamma$. Hence the desired conclusion follows in the same manner as in Subsection 5.6.

Remark 5.2. Hypothesis (5.8) generalizes (1.6). Condition (5.9) is equivalent to $\lim_{|r|\to 0} \|a_r(\cdot,\lambda) - a(\cdot,\lambda)\|_{L^\infty(\mathbb{R}^N)} = 0$ for any $\lambda \in \mathbb{R}^N$, where $a_r(\cdot,\lambda)$ denotes the function $y \to a(y-r,\lambda)$.

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