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BOUNDARY-VALUE PROBLEMS FOR FIRST AND SECOND ORDER FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. This paper presents sufficient conditions for the existence of solutions to boundary-value problems of first and second order multi-valued differential equations in Banach spaces. Our results obtained using fixed point theorems, and lead to new existence principles.

1. Introduction

This paper is concerned with the existence of solutions for the multi-valued functional differential systems

$$\begin{gathered} x' \in F(t, x_t), \quad \text{a.e. } t \in [0, T] \\ x_0 = x_T \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} x'' \in F(t, x_t, x'(t)), \quad \text{a.e. } t \in [0, T] \\ x(t) = \varphi(t), \quad t \in [-r, 0], \quad x(T) = \eta, \end{gathered} \tag{1.2}$$

where $F : J \times C([-r, 0], E) \to \mathcal{P}(E)$ is a multi-valued map, $J = [0, T]$ is a compact real interval, E is a Banach space with norm $|\cdot|$, $\varphi \in C([-r, 0], E)$, $\eta \in E$ and $\mathcal{P}(E)$ is the family of all subsets of E.

For a continuous function x defined on the interval $[-r, T]$ and any $t \in J$, we denote by x_t the element of $C([-r, 0], E)$ defined by

$$x_t(s) = x(t + s), \quad s \in [-r, 0].$$

Here $x_t(\cdot)$ represents the history of the state from time $t - r$ to the time t.

The existence of solutions for functional differential equations in Banach space has been widely investigated. We refer for instance to [4-6, 9, 10]. Existence results for functional differential inclusions received much attention in the recent years. We refer to [1-3]. For instance, Benchohra and Ntouyas have studied initial and boundary problems for functional differential inclusions in [1] on a compact interval with the map F satisfying Lipschitz's contractive conditions of multivalued map and for Neutral functional differential and integrodifferential inclusions in [2].

This paper is organized as follows. In section 2, we introduce some definitions and preliminary facts from multivalued analysis which are used later. In section 3,

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we give existence results of positive and negative solutions on compact intervals for the first order boundary value problem (1.1). In section 4, some existence theorems are given for the second order boundary value problem (1.2).

The fundamental tools used in the existence proofs of all the above mentioned works are essentially fixed point theorems: Covitz and Nadler's in [1], Martelli's in [2]. Here we use a fixed point theorem (Lemma 2.2) in ordered Banach space. However, the hypotheses imposed on the multivalued map F and methods of the proof in this paper are different from all the above cited works.

2. Preliminaries

In this section, we introduce notations, definitions and preliminaries facts from multi-valued analysis which are used throughout this paper.

Let $(E, |\cdot|)$ be a Banach space with a partial order introduced by a cone P of E, that is, $x \le y$ if and only if $y - x \in P$, $x < y$ if and only if $x \le y$ and $x \ne y$. A cone P is said to be normal if there exists a constant $N > 0$ such that $|x| \le N|y|$ for any $x, y \in P$ with $x \le y$.

The set $C([-r, 0], E)$ is a Banach space consisting of all continuous functions from $[-r, 0]$ to E with the norm

$$\|x\| = \sup\{|x(t)| : -r \le t \le 0\}.$$

For any $x, y \in C([-r, b], E)$ for $b \ge 0$, define $x \le y$ if and only if $x(t) \le y(t)$ for each $t \in [-r, b]$, $x < y$ if and only if $x \le y$ and there exists some $t \in [-r, b]$ such that $x(t) \ne y(t)$.

Let $L^1(J, E)$ denote the Banach space of measurable functions $x : J \to E$ which are Bochner integrable with norm

$$\|x\|_1 = \int_0^T |x(t)| dt.$$

The partial order in $L^1(J, E)$ is defined as $x \le y \Leftrightarrow x(t) \le y(t)$ a.e. for $t \in J$.

$AC(J, E)$ denotes the Banach space of absolutely continuous functions defined on J with values in E.

We denote by $bcf(E)$ the set of all bounded, closed, convex and nonempty subsets of E.

A multi-valued map $G : E \to 2^E$ is said to be convex (closed) if $G(x)$ is convex (closed) for all $x \in E$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in E for any bounded set B of E (i.e. $\sup_{x \in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

the function G is called upper semi-continuous (u.s.c.) on E if for each $u \in E$ the set $G(u)$ is a nonempty, closed subset of E, and if for each open set B of E containing $G(u)$, there exists an open neighbourhood V of u such that $G(V) \subseteq B$.

The function G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq E$.

For two points x and y of E, we write $G(x) \le G(y)$ if for any $u \in G(x)$ there exists $v \in G(y)$ such that $u \le v$.

The function G has a fixed point if there is $x \in E$ such that $x \in G(x)$.

The function $G : J \to bcf(E)$ is said to be measurable if for each $x \in E$ the distance between x and $G(t)$ is a measurable function on J.

Throughout this paper θ stands for the zero element of E.

Our results are based on Lemma 2.2 which will be obtained by the following fixed point theorem [7] for multi-valued operators.

Theorem 2.1. *Let E be a Banach space and $G : E \to bcf(E)$ be a condensing map. If the set*
$$M := \{y \in E : \lambda y \in Gy \text{ for some } \lambda > 1\}$$
is bounded, then G has a fixed point.

Lemma 2.2. *Let P be a closed and convex cone of E and $G : P \to bcf(P)$ a u.s.c. and completely continuous multi-valued map. If*
$$\alpha := \sup\{|x| : x \in P \text{ and there exists } \lambda \in (0,1) \text{ such that } x \in \lambda G(x)\} < \infty, \quad (2.1)$$
where $\lambda G(x) = \{\lambda g : g \in G(x)\}$, then G has a fixed point $x \in P$.

Proof. Define the map $\tilde{G} : E \to bcf(P)$ by
$$\tilde{G}(x) = \begin{cases} G(x) & \text{if } x \in P, \\ G(\theta) & \text{if } x \notin P\,. \end{cases}$$
Evidently, $\tilde{G}$ is u.s.c. and completely continuous on E, therefore, $\tilde{G}$ is condensing. Let $\beta = \sup\{|y| : y \in G(\theta)\}$, then for any y belongs to M given in the above theorem, we have $y \in \lambda\tilde{G}(y)$ for some $\lambda \in (0,1)$. If $y \in P$, then $|y| \le \alpha$. Otherwise, $y \in \lambda G(\theta)$, which yields that $|y| \le \beta$. Hence, M is bounded. By the theorem $\tilde{G}$ has a fixed point x. From $\tilde{G}(x) \in bcf(P)$ it follows that $x \in P$. □

3. First Order Boundary Value Problems

In this section we consider the existence of positive and negative solutions for first order boundary value problems of the functional differential inclusion (1.1).

Definition A function $x : [-r,0] \to E$ is a solution of (1.1) if $x \in C([-r,T],E) \cap AC([0,T],E)$ and satisfies the differential inclusion (1.1) a.e. on $[0,T]$.

Let us impose the following hypotheses on the multi-valued map $F : J \times E \to bcf(E)$.

(H1) $(t,u) \to F(t,u)$ is measurable with respect to t for each $u \in C([-r,0],E)$, u.s.c. with respect to u for each $t \in J$ and for each fixed $u \in C([-r,0],E)$ the set
$$S_{F(u)} = \{g \in L^1(J,E) : g(t) \in F(t,u) \quad \text{a.e. } t \in J\}$$
is nonempty.

(H2) There exist functions $\alpha \in L^1(J,\mathbb{R}_+)$, $\beta \in L^1(J,E)$ and $\delta \in [0,1]$ such that $|\alpha(t)| > 0$ for all $t \in J$ and
$$\beta(t)[|\psi(0)|^\delta + 1] \le f \le \alpha(t)\psi(0)$$
for all $t \in J$, $\psi \in C([-r,0],P)$ and $f \in S_{F(\psi)}$.

(H3) There exists a real number $k > 0$ such that $\int_0^T f(t)dt > \theta$ for any $f \in S_{F(x)}$ with
$$S_{F(x)} = \{g \in L^1(J,E) : g(t) \in F(t,x_t) \quad \text{a.e. } t \in J\}$$
for all $x \in C([-r,T],P)$ with $\sup_{t\in J}|x(t)| > k$.

(H4) For each bounded $B \subset C([-r,T],E)$, $u \in B$ and $t \in J$ the set
$$\Big\{ \int_0^T [f(s) - \alpha(s)u(s)]ds : f \in S_{F(u)} \Big\}$$
is relatively compact.

Remark 3.1. If F is measurable, then the function $Y : J \to \mathbb{R}$, defined by

$$Y(t) = \inf\{|v| : v \in F(t,u)\},$$

belongs to $L^1(J,(R))$. So $S_{F(u)}$ is nonempty [12].

Remark 3.2. Let $E = \mathbb{R}$, $h(t) \in L^1(J,\mathbb{R}_+)$, $G : \mathbb{R} \to bcf([0,\ \rho])$, $F(t,\psi) = h(t)\psi(0)G(\psi(-r))$ for $t \in J$, $\psi \in C([-r,0],\mathbb{R})$ and G continuous, then F satisfies the condition (H1). Take $\alpha(t) = \rho h(t)$, $\beta(t) = 0$, $\delta = 0$, then F satisfies the condition (H2). For any $f \in S_{F(x)}$, by Fatou's theorem we obtain that condition (H3) holds.

Theorem 3.3. *Assume that closed convex cone P is normal. If the conditions (H1)–(H4) hold, then* (1.1) *has at least one (positive) solution x on $[-r,T]$ with $x(t) \in P$.*

Proof. Step 1. Let $X = \{x \in C(J,E) : x(0) = x(T)\}$ with the norm

$$\|x\|_J = \sup\{|x(t)| : 0 \le t \le T\}$$

and $X_+ = \{x \in X : x(t) \in P \text{ for } t \in J\}$. It is obvious that X is a Banach space and X_+ is a closed convex cone of X, moreover, $x \in X_+$ if $x(t) \ge \theta$ for every $t \in J$. Let us introduce the differential operator $L : AC(J,X) \to L^1(J,E)$ by

$$Lx = x' - \alpha(t)x$$

with α given in (H2). From the well known results of ordinary differential equations it follows that for any $y \in L^1(J,E)$ the boundary value problem

$$Lx(t) = y(t), \quad x(0) = x(T)$$

has an unique solution $x := Ky \in AC(J,X)$ with the operator K defined by

$$(Ky)(t) = \int_0^T G(t,s)y(s)ds \quad \text{for } t \in J, \tag{3.1}$$

where the Green function $G(t,s)$ satisfies

$$(\tilde{\alpha}(T)-1)\tilde{\alpha}(t)G(t,s) = \begin{cases} \tilde{\alpha}(T)\tilde{\alpha}(s), & s \le t \\ \tilde{\alpha}(s), & s > t \end{cases} \tag{3.2}$$

with $\tilde{\alpha}(t) = \exp(-\int_0^t \alpha(s)ds)$. Thus we have that $K = L^{-1}$ and (3.1) guarantees that K is a bounded linear operator from $L^1(J,E)$ to X.

Step 2. For any $x \in X_+$, from $x(0) = x(T)$ it follows that x can uniquely be extended to a $T-$periodic function on $\mathbb{R}$, written as x^*. Let $\tilde{x} = x^*|_{[-r,T]}$ and $x_t = \tilde{x}_t$ for each $t \in J$. It immediately follows that $x_t \in C([-r,0],P)$, $x_0 = x_T$, $x_t(0) = x(t)$, $\|x_t\| \le \|x\|_J$ and $t \to x_t$ is continuous for $t \in J$.

For any $x \in X_+$, define the multi-valued map as follows:

$$H(t,x) = \{f(t) - \alpha(t)x(t) : f \in S_{F(x)},\ t \in J\}$$

with $S_{F(x)}$ given in (H3). By (H1) we have that $H(t,x)$ is measurable with respect to t. For each $g \in H(t,x)$, by (H2) we have that

$$\beta(t)[|x(t)|^\delta + 1] - \alpha(t)x(t) \le g(t) \le \theta, \tag{3.3}$$

This inequality and the normality of P imply that

$$|g(t)| \le N|\beta(t)|[|x(t)|^\delta + 1] + N\alpha(t)|x(t)| \le N|\beta(t)|(\|x\|_J^\delta + 1) + N\alpha(t)\|x\|_J, \tag{3.4}$$

here N is the normal constant of P. This implies that $H : X_+ \to \mathcal{P}(L^1(J,E))$ is bounded.

Step 3. Let $A = KH$ be a multivalued map from X_+ to X defined by

$$A(t,x) = KH(t,x) = \Big\{ \int_0^T G(t,s)g(s)ds : g \in H(t,x)\Big\}$$

for $x \in X_+$ and $t \in J$. It is clear that A is bounded. Moreover, for any $x \in X_+$ and $h(t) \in A(t,x)$, by (3.1), (3.3), and $G(t,s) \le 0$, there exists $g(t) \in H(t,x)$ for $t \in J$ such that

$$h(t) = \int_0^T G(t,s)g(s)ds \ge \theta.$$

This implies that $h \in X_+$, i.e., $A(t,x) \subset X_+$. It is easy to see that $A(t,x) \subset AC(J,X)$. Thus, $AX_+ \subset AC(J,X) \cap X_+$.

Now, we are in a position to prove that A is a u.s.c. and completely continuous multi-valued map with convex closed values.

$A(t,x)$ is convex for each $x \in X_+$. In fact, if $h_1, h_2 \in A(t,x)$, then there exist $f_1, f_2 \in S_{F(x)}$ such that for each $t \in J$ we have

$$h_1(t) = \int_0^T G(t,s)[f_1(s) - \alpha(s)x(s)]ds\,,$$
$$h_2(t) = \int_0^T G(t,s)[f_2(s) - \alpha(s)x(s)]ds\,.$$

Let $0 \le k \le 1$. Then for each $t \in J$ we have

$$(kh_1 + (1-k)h_2)(t) = \int_0^T G(t,s)[kf_1(s) + (1-k)f_2(s) - \alpha(s)x(s)]ds.$$

Since $S_{F(x)}$ is convex (because F has convex values), so $kh_1 + (1-k)h_2 \in A(t,x)$.

We next shall prove that A is a completely continuous operator. For any bounded set $M \subset X_+$, let $Q = AM$, $m = \sup_{x\in M} \|x\|_J$, $q = \sup_{z\in Q} \|z\|_J$. For any $t, \tau \in [0,T]$ with $t < \tau$ and $x \in M$, if $z \in A(t,x)$, then there exists $g \in H(t,x)$ such that $z = \int_0^T G(t,s)g(s)ds$, hence $z' = \alpha(t)z + g$. By means of (3.4), we have

$$\begin{aligned}
|z(\tau) - z(t)| &\le \int_t^\tau |z'(s)|ds \\
&\le \int_t^\tau [\alpha(s)\|z\|_J + |g(s)|\,]ds \\
&\le \int_t^\tau [(m + Nq)\alpha(s) + N|\beta(s)|(M^\delta + 1)]ds,
\end{aligned}$$

which shows that Q is equi-continuous on J. In virtue of (H4) together with the Ascoli-Arzela theorem we can conclude that Q is a relatively compact subset in X, therefore, A is completely continuous.

Finally, similar to [8] we can prove that A has closed graph. Hence, A is u.s.c. (see [8]).

Step 4. To prove that the equations (1.1) has solutions, we show that A satisfies (2.1). Suppose that this is not the case, then there exist $(\lambda_n, x_n) \in (0,1) \times X_+$ such that $x_n \in \lambda_n A(t, x_n)$, $\mu_n = \|x_n\|_J \ge n$ for $n = 1, 2, \dots$. In Step 3 we proved that

$x_n \in AC(J,X) \cap X_+$. There exist $f_n \in S_{F(x_n)}$ such that $x_n = \lambda_n K(f_n - \alpha x_n)$, i.e. $Lx_n = \lambda_n(f_n - \alpha x_n)$ for $n = 1, 2, \dots$, that is,

$$x_n'(t) = (1-\lambda_n)\alpha(t)x_n(t) + \lambda_n f_n(t,(x_n)_t). \tag{3.5}$$

By integrating (3.5) with respect to t we obtain

$$\theta = (1-\lambda_n)\int_0^T \alpha(t)x_n(t)dt + \lambda_n \int_0^T f_n(t,(x_n)_t)dt,$$

so

$$\int_0^T f_n(t,(x_n)_t)dt = \frac{\lambda_n - 1}{\lambda_n}\int_0^T \alpha(t)x_n(t)dt \le \theta \quad \text{for } n = 1, 2, \dots. \tag{3.6}$$

On the other hand, from the condition (H3), it follows that

$$\int_0^T f_n(t,(x_n)_t)dt > \theta$$

for large enough n. This contradicts (3.6), which completes the proof of (2.1). By Lemma 2.2, A has a fixed point $x \in X_+$, which is a solution to (1.1). The proof is complete. □

Similarly we can prove the next theorem under the following assumptions:

(H'2) There exist functions $\alpha \in L^1(J,\mathbb{R}_+)$, $\beta \in L^1(J,E)$ and $\delta \in [0,1]$ satisfy that $|\alpha(t)| > 0$ for all $t \in J$ and
$$\beta(t)[|\psi(0)|^\delta + 1] \le f \le -\alpha(t)\psi(0)$$
for all $t \in J$, $-\psi \in C([-r,0],P)$ and $\in S_F(\psi)$

(H'3) There exists a real number $k > 0$ such that $\int_0^T f(t)dt > \theta$ for any $f \in S_{F(x)}$ (see (H3)) if $x = -y$ with $\sup_{t\in J}|x(t)| > k$, where $y \in C([-r,T],P)$.

Theorem 3.4. *Let the closed convex cone P be normal. Assume conditions (H1)–(H4) and (H'2), (H'3) hold. Then* (1.1) *has at least one (negative) solution x on $[-r,T]$ with $-x(t) \in P$.*

Remark 3.5. Let $r = 0$, $E = \mathbb{R}$, and F be a single valued function, then results for ordinary differential equations in [11] can be deduced from Theorems 3.3 and 3.4. Therefore, the results presented in this section are the generalization and improvement of the corresponding results in [11].

4. Second order Boundary Value Problems

In this section, we consider existence of solutions for (1.2). A function $x \in C([-r,T],E)$ is called the solution if $x_0 = \varphi$ $x(T) = \eta$, for any $t \in J$, $x'(t)$ exists and is absolutely continuous and (1.2) is satisfied. The following hypotheses will be used.

(H'1) The mapping $(t,\psi,u) \to F(t,\psi,u)$ is measurable with respect to t for each $(\psi,u) \in C([-r,0],E) \times E$, u.s.c. with respect to (ψ,u) for each $t \in J$ and for each fixed $(\psi,u) \in C([-r,0],E) \times E$ the set
$$S_{F(\psi,u)} = \{g \in L^1(J,E) : g(t) \in F(t,\psi,u) \text{ for a.e. } t \in J\}$$
is nonempty.

(H'4) For each bounded set $B \subset C([-r,T],E) \times E$, $x \in C([-r,T],E)$ with $(x_t, x'(t)) \in B$ for $t \in J$, then the set
$$\Big\{ \int_0^T [f(s) - \alpha(s)x(s)]ds : f \in S_{F(x)} \Big\}$$
is relatively compact, where
$$S_{F(x)} = \{g \in L^1(J,E) : g(t) \in F(t, x_t, x'(t)) \quad for\ a.e.\ t \in J\}.$$

(H5) There exist $\alpha > 0$, $\beta \in L^1(J,E)$, $\delta \in [0,1)$ such that
$$\beta(t)[|\psi(0) - \xi(t)|^\delta + |y - \mu|^\delta + 1] \le f \le \alpha[\psi(0) - \xi(t)],$$
where $f \in S_{F(\psi,y)}$, $\xi(t) = \varphi(0) + \frac{t}{T}[\eta - \varphi(0)]$, $\mu = \frac{\eta - \varphi(0)}{T}$, $t \in J$, $\psi \in C([-r,0],E)$, $y \in E$, $\psi(0) \ge \xi(t)$.

Theorem 4.1. *Assume that the closed convex cone P is normal. If the conditions (H'1), (H3),(H'4) and (H5) hold, then* (1.2) *has at least one solution x on* $[-r,T]$ *with $x(t) \ge \xi(t)$ $(t \in J)$.*

Proof. Step 1. Let $z = x - \xi$, then (1.2) is transformed into
$$z''(t) \in F(t, z_t + \xi_t, z'(t) + \mu) := \tilde{F}(t, z_t, z'(t)) \quad t \in J,$$
$$z_0 = \varphi - \xi_0 := \hat{\varphi}, \quad z(T) = \theta.$$
Here $\hat{\varphi} \in C([-r,0],E)$ and $\hat{\varphi}(0) = \theta$. The condition (H5) implies that $\tilde{F}(t,\psi,y) = F(t, \psi + \xi_t, y + \mu)$ satisfies
$$\beta(t)[|\psi(0)|^\delta + |y|^\delta + 1] \le \tilde{F}(t,\psi,y) \le \alpha\psi(0) \quad \psi \in C([-r,0],E),\ \psi(0) \in P.$$
Since $x(t) \ge \xi(t)$ is equivalent to $z(t) \ge \theta$, for the sake of convenience, we assume that $\varphi(0) = \eta = \theta$, which shows that $\xi(t) \equiv \theta$, $\mu = \theta$.

Let $X = \{x \in C^1(J,E) : x(0) = x(T) = \theta,\ x'(0) = x'(T)\}$ with the norm $\|x\|_X = \|x\|_J + \|x'\|_J$, $X_+ = \{x \in X : x \in P\}$, $Y = L^1(J,E)$, $Z = \{x \in X : x'$ is absolutely continuous$\}$. Defining
$$L : Z \to Y, \quad x \to x'' - \alpha x,$$
where α is given in (H5). Similar to the proof of Theorem 3.3, there exists the operator $K = L^{-1}$ defined by
$$(Ky)(t) = \int_0^T G(t,s)y(s)ds \quad \text{for } t \in J,\ y \in Y,$$
where Green's function $G(t,s)$ satisfies
$$\sqrt{\alpha}\,\mathrm{sh}\,\sqrt{\alpha}G(t,s) = \begin{cases} \mathrm{sh}\,\sqrt{\alpha}(t-T)\,\mathrm{sh}\,\sqrt{\alpha}\,s, & s \le t, \\ \mathrm{sh}\,\sqrt{\alpha}(s-T)\,\mathrm{sh}\,\sqrt{\alpha}\,t, & s > t. \end{cases}$$
Step 2. For $x \in X_+$, $t \in J$, let
$$x_t(s) = \begin{cases} x(t+s), & \max\{-r,-t\} \le s \le 0, \\ \varphi(t+s), & -r \le s \le -t. \end{cases}$$
Since $x(0) = \varphi(0) = \theta$, we have that $x_t \in C([-r,0],E)$ and $\|x_t\| \le \|x\|_J + \|\varphi\|$, $t \to x_t$ is continuous for $(t \in J)$.

For $x \in X_+$, define the multi-valued map
$$H(t,x) = \{f(t) - \alpha x(t) : f \in S_{F(x)},\ t \in J\}$$

with $S_{F(x)}$ given in (H'4). (H'1) guarantees that $H(t,x)$ is measurable with respect to $t \in J$. For each $g \in H(t,x)$, from the condition (H5) it follows that

$$\beta(t)[|x(t)|^\delta + |x'(t)|^\delta + 1] - \alpha x(t) \le g(t) \le \theta.$$

This, and the normality of P, implies that $|g(t)| \le N|\beta(t)|(2\|x\|_X^\delta + 1) + N\alpha\|x\|_X$. This shows that $H : X_+ \to 2^Y$ is bounded. Let $A = KH$ be a multi-valued map from X_+ to X defined by

$$A(t,x) = KH(t,x) = \big\{ \int_0^T G(t,s)g(s)ds : g \in H(t,x)\big\}$$

for $x \in X_+$ and $t \in J$. It is clear that A is bounded. Similar to Theorem 3.3 we can prove that $AX_+ \subset AC(J,X) \cap X_+$ and A is u.s.c., completely continuous and has convex closed values.

Step 3. We will now show that A satisfies (2.1). Suppose that this is not the case, then there exist $\lambda_n \in (0,1)$, $x_n \in \lambda_n A(t,x_n) \in Z \cap X_+$ such that $\mu_n = \|x_n\|_X \ge n$ for $n = 1,2,\dots$. In Step 2 we proved that $x_n \in AC(J,X) \cap X_+$. There exists $f_n \in S_{F(x_n)}$ such that $x_n = \lambda_n K(f_n - \alpha x_n)$, i.e. $Lx_n = \lambda_n(f_n - \alpha x_n)$ for $n = 1,2,\dots$, that is,

$$x_n''(t) = (1-\lambda_n)\alpha x_n(t) + \lambda_n f_n(t,(x_n)_t, x_n'(t)). \tag{4.1}$$

By integrating this expression with respect to t we obtain

$$\theta = (1-\lambda_n)\int_0^T \alpha x_n(t)dt + \lambda_n \int_0^T f_n(t,(x_n)_t, x_n'(t))dt,$$

so

$$\int_0^T f_n(t,(x_n)_t, x'(t))dt = \frac{\lambda_n - 1}{\lambda_n}\int_0^T \alpha x_n(t)dt \le \theta \quad \text{for } n = 1,2,\dots. \tag{4.2}$$

On the other hand, the condition (H3) guarantees that

$$\int_0^T f_n(t,(x_n)_t, x'(t))dt > \theta$$

for large enough n. This contradicts (4.2), which completes the proof of (2.1). By Lemma 2.2, A has a fixed point $x \in X_+$, which is a solution to (1.2). The proof is completed. □

Remark 4.2. *In fact, we can allow that* $0 \le \delta \le 1$ *in (H5).*

(H6) If $B \subset W(t) := \{x(t) : x \in C^1(J,E), x(0) = x(T) = \theta\}$ is bounded, then B is relatively compact.

Theorem 4.3. *Assume that the closed convex cone P is normal. If the conditions (H'1), (H'4), (H5) and (H6) hold, then* (1.2) *has at least one solution x on* $[-r,T]$ *with $x(t) \ge \xi(t)$ $(t \in J)$.*

Proof. According to Theorem 4.1 it suffices to prove that (2.1) is true. Suppose that this is not the case, then there exist $\lambda_n \in (0,1)$, $x_n \in \lambda_n A(t,x_n) \in Z \cap X_+$ such that $\mu_n = \|x_n\|_X \ge n$ for $n = 1,2,\dots$. Let $y_n = \frac{1}{\mu_n}x_n$, $\rho_n = \frac{\lambda_n}{\mu_n}$, then $\|y_n\|_X = 1$. Similar to (12) we obtain

$$y_n''(t) = (1-\lambda_n)\alpha y_n(t) + \rho_n f_n(t,(x_n)_t, x_n'(t)). \tag{4.3}$$

This and (H5) guarantee

$$\rho_n\beta(t)[|x_n(t)|^\delta + |x_n'(t)|^\delta + 1] \le y_n''(t) - (1-\lambda_n)\alpha y_n(t) \le \lambda_n \alpha y_n(t),$$

that is

$$\begin{aligned} &|y_n''(t)| \\ &\le N(1-\lambda_n)\alpha|y_n(t)| + N\lambda_n\alpha|y_n(t)| + 2N\rho_n|\beta(t)|[|x_n(t)|^\delta + |x_n'(t)|^\delta + 1] \quad (4.4) \\ &\le N\alpha + 6N|\beta(t)|. \end{aligned}$$

This inequality implies that $\{y_n'\}_{n=1}^\infty$ is equicontinuous on J. Note that $|y_n(t) - y_n(s)| \le |t-s|\,\|y_n'\|_J \quad (t,s \in J)$, we have that $\{y_n\}_{n=1}^\infty$ is also equicontinuous on J. For each $t \in J$, since $\{y_n(t)\}_{n=1}^\infty \subset W(t)$ is bounded, by (H6) $\{y_n(t)\}_{n=1}^\infty$ is relatively compact. By Arzelá -Ascoli's theorem, one has that $\{y_n\}_{n=1}^\infty$ is relatively compact in X. Without loss of generality, let $y_n \to y$ with some $y \in X$ and $\lambda_n \to \lambda \in J$ for $n \to \infty$.

Integrating (4.3) with respect to t, we obtain

$$y_n'(t) = y_n(0) + (1-\lambda_n)\alpha \int_0^t y_n(s)ds + \rho_n \int_0^t f_n(s,(x_n)_s, x_n'(s))ds.$$

Letting n approach ∞, we obtain

$$y'(t) = y(0) + (1-\lambda)\alpha \int_0^t y(s)ds + g(t) \tag{4.5}$$

with $g(t) = \lim_{n\to\infty} \rho_n \int_0^t f_n(s,(x_n)_s, x_n'(s))ds$, which exists for $0 \le t < \tau \le T$. By (H5) we have that

$$\rho_n \int_t^\tau f_n(s,(x_n)_s, x_n'(s))ds \ge \rho_n \int_t^\tau \beta(s)[|(x_n)(s)|^\delta + |x_n'(s)|^\delta + 1)]ds$$

and

$$\begin{aligned} &|\rho_n \int_t^\tau \beta(s)[|(x_n)(s)|^\delta + |x_n'(s)|^\delta + 1)]ds| \\ &\le \rho_n \int_t^\tau |\beta(s)|(2\mu_n^\delta + 1)ds \\ &\le (2\mu_n^{\delta-1} + \rho_n) \int_t^\tau |\beta(s)|\,ds \to 0 \quad (n \to \infty), \end{aligned}$$

which yields $g(\tau) - g(t) \ge \theta$, that is, $g(t)$ is monotone increasing on J. Especially, $g(t) \ge g(0) = \theta$ for each $t \in J$. For any $t \in J$, $y(t) \in X_+$ deduces that $y(t) \ge \theta = y(0)$, which implies that $y'(0) \ge \theta$. Summing up, from (4.5), it follows that $y'(t) \ge \theta$ $(t \in J)$. This shows that y is a increasing function on J. Note that $y(0) = y(T) = \theta$, we have $y(t) \equiv \theta$ on J, which contradicts $\|y\|_X = 1$. The proof is completed. $\square$

Corollary 4.4. *Let $E = \mathbb{R}^n$. If the conditions (H'1) and (H5) hold, then* (1.2) *has at least one solution x on $[-r,T]$ with $x(t) \ge \xi(t)$ $(t \in J)$.*

Corollary 4.5. *. Let $r = 0$ and $F : J\times\mathbb{R}^n\times\mathbb{R}^n \to 2^{\mathbb{R}^n}$. If F satisfies the condition (H'1), and there exist $\alpha > 0$, $\beta \in L^1(J,\mathbb{R}^n)$, $\delta \in [0,1)$ such that*

$$\beta(t)\Big[|x-\xi(t)|^\delta + \big|y - \frac{B-A}{T}\big|^\delta + 1\Big] \le f(t,x,y) \le \alpha[x-\xi(t)]$$

for $t \in J$, $x, y \in \mathbb{R}^n$, $x \geq \xi(t) := A + (t/T)(B - A)$ *and* $f \in \{g \in L^1(J, E) : g(t) \in F(t, x, y)$ for a.e. $t \in J\}$, *then second order ordinary differential inclusion*

$$x''(t) \in F(t, x(t), x'(t)) \ (t \in J),$$
$$x(0) = A, \quad x(T) = B,$$

has at least a solution $x \in C^1(J, \mathbb{R}^n)$, *with* $x(t) \geq \xi(t)$ *on* J.

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