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EXISTENCE AND STABILITY FOR SOME PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. We study the existence, regularity, and stability of solutions for some partial functional differential equations with infinite delay. We assume that the linear part is not necessarily densely defined and satisfies the Hille-Yosida condition on a Banach space X. The nonlinear term takes its values in space larger than X, namely the extrapolated Favard class of the extrapolated semigroup corresponding to the linear part. Our approach is based on the theory of the extrapolation spaces.

1. Introduction

In this work, we consider the partial functional differential equation with infinite delay:

$$\begin{gathered} \frac{d}{dt}x(t) = Ax(t) + F(x_t), \quad \text{for } t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{gathered} \tag{1.1}$$

where A is a nondensely defined linear operator on a Banach space X and satisfies the Hille-Yosida condition, this means that A satisfies the usual assumption of Hille-Yosida's theorem characterizing the generators of strongly continuous semigroups except the density of $D(A)$ in X: there exist $N_0 \geq 0$ and $\omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda - \omega_0)^n |(\lambda - A)^{-n}| : n \in \mathbb{N},\ \lambda > \omega_0\right\} \leq N_0,$$

where $\rho(A)$ is the resolvent set of A. $\mathcal{B}$ is a linear space of functions from $(-\infty, 0]$ into X satisfying some axioms which will be described in the sequel. For every $t \geq 0$, the history function $x_t \in \mathcal{B}$ is defined by

$$x_t(\theta) = x(t + \theta), \quad \text{for } \theta \in (-\infty, 0].$$

F is a continuous function from $\mathcal{B}$ with values in F_0 larger space than X, namely the extrapolated Favard class of the extrapolated semigroup corresponding to the linear part A, (see section 2). Note that the non-density occurs in many situations

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due to the restrictions on the space where the equation is considered (for example, periodic continuous functions, Hölder continuous functions) or due to boundary conditions (for example, the space C^1 with null value on the boundary is non dense in the space of continuous functions) (see examples in [5]). In the literature devoted to equations with finite delay, the state space is the space of continuous functions on $[-r, 0]$ with values in X, for more details we refer to the book by Wu [16]. When the delay is unbounded, the selection of the state space $\mathcal{B}$ plays an important role in the study of both quantitative and qualitative studies. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [10]. Concerning to the theory of functional differential equations with infinite delay, we refer to the book by Hino, Murakami and Naito [12]. In recent years, the theory of partial functional differential equations with infinite delay have been the subject of considerable activity. In [11], it has been proved the existence and regularity of solutions of Equation (1.1) when F takes its values in X and A is the infinitesimal generator of analytic semigroup on X. This in particular contains the density of the domain $D(A)$ in X by Hille-Yosida's theorem. More recently, it has been shown in [1] that the density condition is not necessary to deal with partial functional differential equations. In [3] and [4], it has been proved the existence, regularity of solutions and stability for Equation (1.1) when F takes its values in X and A is nondensely defined and satisfies the Hille-Yosida condition. There are many approaches to deal with partial differential equations with non dense domain, one of them is based on the extrapolation approach. For more details about the extrapolation approach, we cite the book by Engel and Nagel [7] and [13]. This work was motivated by [8] where the authors have proved the existence and regularity of solutions for the following partial functional differential equation

$$\frac{d}{dt}x(t) = Ax(t) + G(t, x_t), \text{ for } t \geq 0$$
$$x_0 = \varphi \in C([-r, 0]; X),$$

where A generates a strongly continuous semigroup on X, $C([-r, 0]; X)$ is the space of continuous function from $[-r, 0]$ into X endowed with the uniform norm topology and G is a continuous function on $\mathbb{R}^+ \times C([-r, 0]; X)$ with values in F_0.

The purpose of this work is to discuss the existence, regularity of the mild solutions of Equation (1.1) and the asymptotic behavior of solutions near an equilibrium. In our context we will use the extrapolation approach. The obtained results of this work would be an extension of the results in [3], [4], [11] and [8]. The organization is as follows: In section 2, we recall some preliminary results about the extrapolation spaces and Favard class which will be used in the whole of the work. In section 3, we start with our main results, in which we prove the existence and regularity of the mild solutions of Equation (1.1). In section 4, we prove the linearized stability and finally we propose an application.

2. Extrapolation spaces and Favard class

Here and hereafter we assume that

(H1) A is a Hille-Yosida operator on a Banach space X.

Let A_0 be the part of A in $X_0 = \overline{D(A)}$ which is defined by

$$D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\} \quad A_0x = Ax, \quad \text{for } x \in D(A_0).$$

Lemma 2.1 ([7]). *A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on X_0 and $|T_0(t)| \leq N_0 e^{\omega t}$, for $t \geq 0$. Moreover $\rho(A) \subset \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)/X_0$, for $\lambda \in \rho(A)$.*

For a fixed $\lambda_0 \in \rho(A)$, we introduce on X_0 a new norm defined by

$$\|x\|_{-1} = |R(\lambda_0, A_0)x| \quad \text{for } x \in D(A_0)\,.$$

The completion X_{-1} of $(X_0, \|.\|_{-1})$ is called the extrapolation space of X associated with A Note that $\|.\|_{-1}$ and the norm on X_0 given by $|R(\lambda, A_0)x|$ for $\lambda \in \rho(A)$ are equivalent. The operator $T_0(t)$ has a unique bounded linear extension $T_{-1}(t)$ to the Banach space X_{-1} and $(T_{-1}(t))_{t\geq 0}$ is a strongly continuous semigroup on X_{-1}. $(T_{-1}(t))_{t\geq 0}$ is called the extrapolated semigroup of $(T_0(t))_{t\geq 0}$, we denote its generator by $(A_{-1}, D(A_{-1})$. We have some fundamental results.

Lemma 2.2 ([8]). *The following properties hold:*

(i) $|T_{-1}(t)|_{L(X_{-1})} = |T_0(t)|_{L(X_0)}$
(ii) $D(A_{-1}) = X_0$
(iii) *$A_{-1} : X_0 \to X_{-1}$ is the unique continuous extension of A_0: $D(A_0) \subset (X_0, |.|) \to (X_{-1}, \|.\|_{-1})$ and $(\lambda_0 - A_{-1})$ is an isometry from $(X_0, |.|)$ to $(X_{-1}, \|.\|_{-1})$,*
(iv) *If $\lambda \in \rho(A_0)$, then $(\lambda - A_{-1})$ is invertible and $(\lambda - A_{-1})^{-1} \in L(X_{-1})$. In particular $\lambda \in \rho(A_{-1})$ and $R(\lambda, A_{-1})/X_0 = R(\lambda, A_0)$.*
(v) *The space $X_0 = \overline{D(A)}$ is dense in $(X_{-1}, \|.\|_{-1})$. Hence the extrapolation space X_{-1} is also the completion of $(X, \|.\|_{-1})$ and $X \hookrightarrow X_{-1}$.*
(vi) *The operator A_{-1} is an extension of A. In particular if $\lambda \in \rho(A)$, then $R(\lambda, A_{-1})/X = R(\lambda, A)$ and $(\lambda, A_{-1})X = D(A)$.*

Next we introduce the Favard class corresponding to semigroup.

Definition 2.3 ([8]). Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $(B, D(B))$ on a Banach space Z such that $|S(t)| \leq N e^{\nu t}$ for some $N \geq 1$ and $\nu \in \mathbb{R}$. The Favard class of $(S(t))_{t\geq 0}$ is the Banach space

$$\mathbb{F} = \Big\{ x \in Z : \sup_{t>0} \frac{1}{t} |e^{-\nu t} S(t)x - x| < \infty \Big\}$$

equipped with the norm

$$|x|_{\mathbb{F}} = |x| + \sup_{t>0} \frac{1}{t} |e^{-\nu t} S(t)x - x|.$$

We can see that $\mathbb{F}$ is invariant under $(S(t))_{t\geq 0}$ and $D(B) \subset \mathbb{F}$. If Z is reflexive then $\mathbb{F} = D(B)$. Furthermore if we denote by $|.|_B$ the graph norm of B, then $|.|_B$ and $|.|_{\mathbb{F}}$ are equivalent norms on $D(B)$.

For the rest of the paper we denote by $F_1 \subset X_0$ the Favard class of the C_0 semigroup $(T_0(t))_{t\geq 0}$ and $F_0 \subset X_{-1}$ the Favard class of $(T_{-1}(t))_{t\geq 0}$.

Lemma 2.4 ([8]). *For the Favard classes F_0 and F_1 the following hold:*

(i) $(\lambda_0 - A_{-1})F_1 = F_0$
(ii) $T_{-1}(t)F_0 \subset F_0$, $t \geq 0$,
(iii) *$D(A_0) \hookrightarrow D(A) \hookrightarrow F_1 \hookrightarrow X_0 \hookrightarrow X \hookrightarrow F_0 \hookrightarrow X_{-1}$, where $D(A)$ is equipped with the graph norm.*

Proposition 2.5 ([13]). *For $f \in L^1_{loc}(\mathbb{R}^+, F_0)$, we define*

$$(T_{-1} * f)(t) = \int_0^t T_{-1}(t-s)f(s)ds, \ t \geq 0 .$$

Then

(i) $(T_{-1} * f)(t) \in \overline{D(A)}$, *for all* $t \geq 0$,

(ii) $|(T_{-1} * f)(t)| \leq \overline{M} \int_0^t e^{w(t-s)}|f(s)|_{F_0} ds$, *for some* $\overline{M}$ *independent of* f *and* t,

(iii) $\lim_{t\to 0} |(T_{-1} * f)(t)| = 0$.

Remark 2.6. *Condition (iii) in Proposition 2.5 implies that the function* $t \to \int_0^t T_{-1}(t-s)f(s)ds$ *is continuous from* $\mathbb{R}^+$ *to* X_0.

3. Existence and regularity of solutions

We assume that the phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X, endowed with a norm $|.|_{\mathcal{B}}$ and satisfying the following fundamental axioms introduced at first by Hale and Kato in [10]:

(A1) There exist a positive constant H and functions $K, M : [0, +\infty) \to [0, +\infty)$, with K is continuous and M is locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, \sigma + a] \to X$, $x_\sigma \in \mathcal{B}$ and x is continuous on $[\sigma, \sigma + a]$, then for all t in $[\sigma, \sigma + a]$ the following conditions hold:
  (i) $x_t \in \mathcal{B}$,
  (ii) $|x(t)| \leq H\,|x_t|_{\mathcal{B}}$,
  (iii) $|x_t|_{\mathcal{B}} \leq K(t-\sigma) \sup_{\sigma\leq s\leq t} |x(s)| + M(t-\sigma)|x_\sigma|_{\mathcal{B}}$.

(A2) For a function x satisfying (A1), $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.

(B1) The space $\mathcal{B}$ is complete.

For the remaining of this work, we use the notations: for $a > 0$, we define $K_a = \max_{0\leq t\leq a} K(t)$ and $M_a = \max_{0\leq t\leq a} M(t)$.

(H2) We assume that F takes its values in F_0 and satisfies the Lipschitz condition $|F(\varphi_1) - F(\varphi_2)|_{F_0} \leq L|\varphi_1 - \varphi_2|_{\mathcal{B}}$, for $\varphi_1, \varphi_2 \in \mathcal{B}$.

Definition 3.1. *A function* $x : (-\infty, \infty) \to X$ *is called a mild solution of (1.1) if* x *is continuous on* $[0, \infty)$ *and satisfies*

$$x(t) = T_0(t)\varphi(0) + \int_0^t T_{-1}(t-s)F(x_s)ds, \quad t \geq 0$$
$$x_0 = \varphi .$$

When F maps into X, the mild solution coincides with the integral solution given in [3] and [4]. By Proposition 2.5, the mild solution (if it exists) takes values in $\overline{D(A)}$ only if $\varphi(0) \in \overline{D(A)}$.

Theorem 3.2. *Assume that (H1)–(H2) hold. Then for* $\varphi \in \mathcal{B}$ *such that* $\varphi(0) \in \overline{D(A)}$, *Equation (1.1) has a unique mild solution* $x(., \varphi)$ *which is defined for all* $t \geq 0$. *Moreover for every* $a > 0$, *there exists* $\beta > 0$ *such that for* φ_1, $\varphi_2 \in \mathcal{B}$ *with* $\varphi_1(0)$, $\varphi_2(0) \in \overline{D(A)}$, *we have*

$$|x_t(., \varphi_1) - x_t(., \varphi_2)|_{\mathcal{B}} \leq \beta|\varphi_1 - \varphi_2|_{\mathcal{B}}, \quad \text{for } t \in [0, a]. \tag{3.1}$$

Proof. Let $\varphi \in \mathcal{B}$ such that $\varphi(0) \in \overline{D(A)}$ and for $a > 0$, we introduce the set $Z_a(\varphi)$ by

$$Z_a(\varphi) := \{y \in C([0,a];X) : y(0) = \varphi(0)\},$$

provided with the uniform norm topology which will be denoted by

$$|y|_\infty = \sup_{s\in[0,a]} |y(s)|, \quad \text{for } y \in Z_a(\varphi).$$

For $x \in Z_a(\varphi)$, the function $\widetilde{x} : (-\infty, a] \to X$ is given by

$$\widetilde{x}(t) = \begin{cases} x(t), & t \in [0,a], \\ \varphi(t), & -\infty < t \le 0. \end{cases}$$

By virtue of conditions (H2) and (A2), we deduce that the mapping $s \to F(\widetilde{x}_s)$ is continuous from $[0,a]$ to F_0. Let $\mathcal{P}$ be defined on $Z_a(\varphi)$ by

$$(\mathcal{P}x)(t) = T_0(t)\varphi(0) + \int_0^t T_{-1}(t-s)F(\widetilde{x}_s)ds, \quad t \in [0,a].$$

By Proposition 2.5, we get that the function $t \to \int_0^t T_{-1}(t-s)F(\widetilde{x}_s)ds$ is continuous from $[0,a]$ to X, which implies that $\mathcal{P}x \in Z_a(\varphi)$, if $x \in Z_a(\varphi)$. Therefore, by Proposition 2.5, we have

$$\Big|\int_0^t T_{-1}(t-s)(F(\widetilde{x}_s) - F(\widetilde{y}_s))ds\Big| \le \overline{M}\int_0^t e^{w(t-s)}|F(\widetilde{x}_s) - F(\widetilde{y}_s)|_{F_0} ds,$$

this implies by assumption (H2) that

$$\Big|\int_0^t T_{-1}(t-s)(F(\widetilde{x}_s) - F(\widetilde{y}_s))ds\Big| \le \overline{M}L\int_0^t e^{w(t-s)}|\widetilde{x}_s - \widetilde{y}_s|_{\mathcal{B}} ds.$$

Without loss of generality we assume that $\omega > 0$ and then by (A1) part (iii), we obtain

$$\begin{aligned} |(\mathcal{P}x)(t) - (\mathcal{P}y)(t)| &\le \overline{M}Le^{\omega a}\int_0^t K(s)\sup_{0\le\xi\le s}|x(\xi) - y(\xi)|\,ds \\ &\le \overline{M}Le^{\omega a}K_a\int_0^t \sup_{0\le\xi\le s}|x(\xi) - y(\xi)|ds. \end{aligned}$$

Arguing as above we can see that

$$\begin{aligned} |(\mathcal{P}^2x)(t) - (\mathcal{P}^2y)(t)| &\le \overline{M}Le^{\omega a}K_a\int_0^t \sup_{0\le\xi\le s}|(\mathcal{P}x)(\xi) - (\mathcal{P}y)(\xi)|ds, \\ &\le (\overline{M}Le^{\omega a}K_a)^2\int_0^t \sup_{0\le\xi\le s}\int_0^\xi \sup_{0\le\alpha\le p}|x(\alpha) - y(\alpha)|dp\,ds, \\ &\le (\overline{M}Le^{\omega a}K_a)^2\int_0^t\int_0^s dp\,ds|x-y|_\infty, \\ &\le \frac{(\overline{M}Le^{\omega a}K_a)^2}{2}a^2|x-y|_\infty. \end{aligned}$$

Then for every $n \in \mathbb{N}^*$ we have

$$|(\mathcal{P}^nx)(t) - (\mathcal{P}^ny)(t)| \le \frac{(\overline{M}Le^{\omega a}K_a)^n}{n!}a^n|x-y|_\infty.$$

Since there exists $n \in \mathbb{N}$ such that $\frac{(\overline{M}Le^{\omega a}K_a)^n}{n!}a^n < 1$, it follows that $\mathcal{P}^n$ is a strict contraction on the closed subset $Z_a(\varphi)$ of the Banach space $C([0,a];X)$. Consequently $\mathcal{P}$ has a unique fixed point in $Z_a(\varphi)$, this holds for every $a > 0$. We conclude that Equation (1.1) has a unique mild solution $x(.,\varphi)$ on $(-\infty,\infty)$. Moreover the solution depends continuously on the initial data. In fact, if we consider two solutions $x := x(.,\varphi_1)$ and $y := y(.,\varphi_2)$ for $\varphi_1,\varphi_2 \in \mathcal{B}$ such that $\varphi_1(0),\varphi_2(0) \in \overline{D(A)}$, then for every $t \in [0,a]$ with $a > 0$ fixed, we have

$$\begin{aligned}
|x(t)-y(t)| &\le |T_0(t)(\varphi_1(0)-\varphi_2(0))| + \overline{M}Le^{\omega a}\int_0^t |x_s-y_s|_{\mathcal{B}}ds,\\
&\le N_0e^{\omega a}|\varphi_1(0)-\varphi_2(0)|\\
&\quad + \overline{M}Le^{\omega a}\int_0^t \Big(K(s)\max_{0\le\xi\le s}|x(\xi)-y(\xi)| + M(s)|\varphi_1-\varphi_2|_{\mathcal{B}}\Big)ds,\\
&\le HN_0e^{\omega a}|\varphi_1-\varphi_2|_{\mathcal{B}} + \overline{M}Le^{\omega a}K_a\int_0^t \max_{0\le\xi\le s}|x(\xi)-y(\xi)|ds\\
&\quad + a\overline{M}Le^{\omega a}M_a|\varphi_1-\varphi_2|_{\mathcal{B}}.
\end{aligned}$$

By Gronwall's lemma, it follows that

$$\max_{0\le s\le t}|x(s)-y(s)| \le \beta_0|\varphi_1-\varphi_2|_{\mathcal{B}}, \quad \text{for } t \in [0,a],$$

where $\beta_0 = e^{\omega a}(HN_0 + a\overline{M}LM_a)\exp(a\overline{M}LK_ae^{\omega a})$. Therefore, by (A1) part (iii),

$$|x_t(.,\varphi_1)-x_t(.,\varphi_2)|_{\mathcal{B}} \le K(t)\sup_{0\le s\le t}|x(s,\varphi_1)-x(s,\varphi_2)| + M(t)|\varphi_1-\varphi_2|_{\mathcal{B}},$$

which implies the desired estimate (3.1). □

For the regularity of the mild solution, we need to compute the integral in $\mathcal{B}$ from the integral in X. For that we suppose that $\mathcal{B}$ satisfies the condition:

(C1) If $(\phi_n)_{n\ge0}$ is a Cauchy sequence in $\mathcal{B}$ and if $(\phi_n)_{n\ge0}$ converges compactly to ϕ on $(-\infty,0]$, then ϕ is in $\mathcal{B}$ and $|\phi_n-\phi|_{\mathcal{B}} \to 0$, as $n \to \infty$.

Lemma 3.3 ([14])**.** *Let $\mathcal{B}$ satisfy (C1) and $f : [0,a] \to \mathcal{B}$, $a > 0$, be a continuous function such that $f(t)(\theta)$ is continuous for $(t,\theta) \in [0,a]\times(-\infty,0]$. Then*

$$\Big[\int_0^a f(t)dt\Big](\theta) = \int_0^a f(t)(\theta)dt, \quad \theta \in (-\infty,0].$$

One can obtain the similar result, by assuming that $\mathcal{B}$ satisfies

(D1) For a sequence $(\phi_n)_{n\ge0}$ in $\mathcal{B}$, if $|\phi_n|_{\mathcal{B}} \to 0$, as $n \to \infty$, then $|\phi_n(\theta)| \to 0$, as $n \to \infty$, for each $\theta \in (-\infty,0]$.

Lemma 3.4. *Let $\mathcal{B}$ satisfy (D1) and $f : [0,a] \to \mathcal{B}$ be a continuous function. Then for all $\theta \in (-\infty,0]$, the function $f(.)(\theta)$ is continuous and*

$$\Big[\int_0^a f(t)dt\Big](\theta) = \int_0^a f(t)(\theta)dt, \quad \theta \in (-\infty,0].$$

Proof. Since the function f is continuous, it follows that

$$\int_0^a f(t)dt = \lim_{n\to+\infty}\frac{a}{n}\sum k = 1^n f(\frac{ka}{n}) \quad \text{in } \mathcal{B}.$$

By assumption (D1),

$$\Big[\int_0^a f(t)dt\Big](\theta) = \lim_{n\to+\infty} \frac{a}{n}\sum_{k=1}^{n} f(\frac{kn}{a})(\theta), \quad \theta\in(-\infty,0].$$

Moreover by (D1), the function $f(.)(\theta)$ is continuous on $[0,a]$, from what we infer that for all $\theta\in(-\infty,0]$, the function $f(.)(\theta)$ is integrable on $[0,a]$ and

$$\int_0^a f(t)(\theta)dt = \lim_{n\to+\infty} \frac{a}{n}\sum_{k=1}^{n} f(\frac{kn}{a})(\theta), \quad \theta\in(-\infty,0],$$

which implies

$$\Big[\int_0^a f(t)dt\Big](\theta) = \int_0^a f(t)(\theta)dt, \quad \theta\in(-\infty,0].$$

□

Theorem 3.5. *Assume that (H1) and (H2) hold and $\mathcal{B}$ satisfies (C1) or (D1). Furthermore we assume that $F:\mathcal{B}\to F_0$ is continuously differentiable and F' is locally Lipschitz. Let $\varphi\in\mathcal{B}$ be continuously differentiable such that*

$$\varphi'\in\mathcal{B},\quad \varphi(0)\in F_1,\quad \varphi'(0)\in\overline{D(A)},\quad \text{and}\quad \varphi'(0)=A_{-1}\varphi(0)+F(\varphi).$$

Then the mild solution x of (1.1) belongs to $C^1(\mathbb{R}^+,X)\cap C(\mathbb{R}^+,F_1)$ and satisfies

$$\begin{gathered}\frac{d}{dt}x(t) = A_{-1}x(t)+F(x_t), \quad t\ge 0\\ x_0=\varphi.\end{gathered} \tag{3.2}$$

The proof of this theorem is based on the following Lemma.

Lemma 3.6 ([13]). *For $u_0\in F_1$ and $h\in W^{1,1}(\mathbb{R}^+,F_0)$ such that $A_{-1}u_0+h(0)\in\overline{D(A)}$, the equation*

$$\begin{gathered}\frac{d}{dt}u(t) = A_{-1}u(t)+h(t),\ t\ge 0\\ u(0)=u_0\end{gathered} \tag{3.3}$$

has a unique solution $u\in C^1(\mathbb{R}^+,X)\cap C(\mathbb{R}^+,F_1)$.

Remark 3.7. *The mild solution of (3.3) is given by*

$$u(t) = T_0(t)u_0 + \int_0^t T_{-1}(t-s)h(s)ds, \quad t\ge 0$$

Using Lemma 3.6 we can deduce that $u\in C^1(\mathbb{R}^+,X)\cap C(\mathbb{R}^+,F_1)$ and satisfies (3.3) for every $t\ge 0$.

Proof of Theorem 3.5. Set $a>0$ and let x be the mild solution of (1.1) on $[0,a]$. Consider the equation

$$\begin{gathered}y(t) = T_0(t)\varphi'(0) + \int_0^t T_{-1}(t-s)F'(x_s)y_s ds, \quad \text{for } t\in[0,a]\\ y_0=\varphi'.\end{gathered} \tag{3.4}$$

Then using the same reasoning as in the proof of Theorem 3.2 we get that Equation (3.4) has a unique solution y on $(-\infty,a]$. Let $z:(-\infty,a]\to X$ be defined by

$$z(t)=\begin{cases}\varphi(0)+\int_0^t y(s)\,ds, & \text{for } t\in[0,a]\\ \varphi(t), & \text{for } t\in(-\infty,0].\end{cases}$$

By Lemma 3.3 or Lemma 3.4, we deduce that

$$z_t = \varphi + \int_0^t y_s\, ds, \quad \text{for } t \in [0, a]. \tag{3.5}$$

We will show that $x = z$ on $[0, a]$. From (3.4),

$$\int_0^t y(s)ds = \int_0^t T_0(s)\varphi'(0)ds + \int_0^t \int_0^s T_{-1}(s-\sigma)F'(x_\sigma)y_\sigma d\sigma ds. \tag{3.6}$$

On the other hand $t \to z_t$ is differentiable from $[0, a]$ to $\mathcal{B}$, then by differentiability of F, we get that $t \to F(z_t)$ is differentiable from $[0, a]$ to F_0, since $F_0 \hookrightarrow X_{-1}$, it follows that $t \to F(z_t)$ is differentiable from $[0, a]$ to X_{-1} and then for $t \in [0, a]$, we have

$$\frac{d}{dt}\int_0^t T_{-1}(t-s)F(z_s)ds = T_{-1}(t)F(\varphi) + \int_0^t T_{-1}(t-s)F'(z_s)y_s ds,$$

which implies

$$\int_0^t T_{-1}(s)F(\varphi)ds = \int_0^t T_{-1}(t-s)F(z_s)ds + \int_0^t \int_0^s T_{-1}(s-\sigma)F'(z_\sigma)y_\sigma d\sigma ds. \tag{3.7}$$

It follows that

$$\begin{aligned}
&z(t) \\
&= \varphi(0) + \int_0^t T_0(s)(A_{-1}\varphi(0) + F(\varphi))ds + \int_0^t \int_0^s T_{-1}(s-\sigma)F'(x_\sigma)y_\sigma d\sigma ds \\
&= \varphi(0) + \int_0^t T_{-1}(s)(A_{-1}\varphi(0) + F(\varphi))ds + \int_0^t \int_0^s T_{-1}(s-\sigma)F'(x_\sigma)y_\sigma d\sigma ds \\
&= T_{-1}(t)\varphi(0) + \int_0^t T_{-1}(s)F(\varphi)ds + \int_0^t \int_0^s T_{-1}(s-\sigma)F'(x_\sigma)y_\sigma d\sigma ds
\end{aligned}$$

Using (3.7) we obtain

$$\begin{aligned}
z(t) &= T_{-1}(t)\varphi(0) + \int_0^t T_{-1}(t-s)F(z_s)ds \\
&\quad + \int_0^t \int_0^s T_{-1}(s-\sigma)(F'(x_\sigma) - F'(z_\sigma))y_\sigma d\sigma ds.
\end{aligned}$$

On the other hand $\varphi(0) \in F_1$ which gives that $T_{-1}(t)\varphi(0) = T_0(t)\varphi(0)$, for all $t \geq 0$. Then

$$\begin{aligned}
&x(t) - z(t) \\
&= \int_0^t T_{-1}(t-s)(F(x_s) - F(z_s))ds - \int_0^t \int_0^s T_{-1}(s-\sigma)(F'(x_\sigma) - F'(z_\sigma))y_\sigma d\sigma ds.
\end{aligned}$$

Using the local Lipschitz condition on F' we get that there exists a positive constant b_0 such that

$$\sup_{0 \leq s \leq t} |x(s) - z(s)| \leq b_0 \int_0^t |x_s - z_s|_{\mathcal{B}} ds.$$

Since $x_0 = z_0 = \varphi$, axiom (A1) part (iii) implies

$$|x_t - z_t|_{\mathcal{B}} \leq K_a \sup_{0 \leq s \leq t} \sup |x(s) - z(s)|.$$

Then

$$|x_t - z_t|_{\mathcal{B}} \le K_a b_0 \int_0^t |x_s - z_s|_{\mathcal{B}}\, ds.$$

Using Gronwall's lemma, we conclude that $|x_t - z_t|_{\mathcal{B}} = 0$ for $t \in [0, a]$. Consequently, $x(t) = z(t)$ for all $t \in (-\infty, a]$. Hence, $t \to x_t$ is continuously differentiable on $[0, a]$, for every $a > 0$. We deduce that $F(x.) \in C^1(\mathbb{R}^+, F_0)$ and by Lemma 3.6 we deduce that x belongs to $C^1(\mathbb{R}^+, X) \cap C(\mathbb{R}^+, F_1)$ and satisfies (3.2). □

4. The solution semigroup and linearized stability

Let $\mathcal{H}$ be the phase space of Equation (1.1) which is given by

$$\mathcal{H} = \Big\{\varphi \in \mathcal{B} : \varphi(0) \in \overline{D(A)}\Big\}.$$

For $t \ge 0$, we define the continuous operator $U(t)$ on $\mathcal{H}$ by $U(t)\varphi = x_t(., \varphi)$, where $x(., \varphi)$ is the mild solution of (1.1). Then we can see that $(U(t))_{t\ge 0}$ is a strongly continuous semigroups on $\mathcal{H}$. We are interested in studying the behavior of solutions of Equation (1.1) near an equilibrium . We mean by an equilibrium a constant mild solution x^* of (1.1). Without loss of generality we suppose that $x^* = 0$ and $F(0) = 0$.

(H3) F is differentiable at zero.

Then the linearized equation at zero is given by

$$\begin{gathered} \frac{d}{dt} y(t) = Ay(t) + \mathcal{L}(y_t), \quad \text{for } t \ge 0, \\ y_0 = \varphi \in \mathcal{B}, \end{gathered} \tag{4.1}$$

where $\mathcal{L} = F'(0)$. Let $(T(t))_{t\ge 0}$ be the solution semigroup associated to Equation (4.1) that is defined by $T(t)\varphi = y_t(., \varphi)$, $\varphi \in \mathcal{H}$, where $y(., \varphi)$ is the mild solution of Equation (4.1).

Theorem 4.1. *Assume that conditions (H1)–(H3) hold. Then for $t \ge 0$, the derivative at zero of $U(t)$ is $T(t)$.*

Proof. Let $x(., \varphi)$ and $y(., \varphi)$ be respectively the mild solutions of (1.1) and (4.1). By assumption (A1) part (iii), we have for all $t \ge 0$

$$\begin{aligned} |U(t)\varphi - T(t)\varphi|_{\mathcal{B}} &\le K(t) \sup_{0\le\sigma\le t} |x(\sigma, \varphi) - y(\sigma, \varphi)| \\ &\le K(t) \sup_{0\le\sigma\le t} \Big| \int_0^\sigma T_{-1}(\sigma - s)(F(U(s)\varphi) - \mathcal{L}(T(s)\varphi))ds\Big|, \\ &\le K(t)\overline{M}e^{\omega t} \int_0^t e^{-\omega s} |F(U(s)\varphi) - \mathcal{L}(T(s)\varphi)|_{F_0} ds, \\ &\le K(t)\overline{M}e^{\omega t}\Big(\int_0^t e^{-\omega s} |F(U(s)\varphi) - \mathcal{L}(U(s)\varphi)|_{F_0} ds \\ &\quad + \int_0^t e^{-\omega s} |\mathcal{L}(U(s)\varphi) - \mathcal{L}(T(s)\varphi)|_{F_0} ds\Big). \end{aligned}$$

By virtue of the differentiability of F at 0 and from (3.1) of Theorem 3.2, we deduce that for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^t e^{-\omega s} |F(U(s)\varphi) - \mathcal{L}(U(s)\varphi)|_{F_0} ds \le \varepsilon |\varphi|_{\mathcal{B}}, \quad \text{for } |\varphi|_{\mathcal{B}} \le \delta.$$

On the other hand, one has

$$\int_0^t e^{-\omega s}|\mathcal{L}(U(s)\varphi)-\mathcal{L}(T(s)\varphi)|_{F_0}ds \le |\mathcal{L}|_{\mathcal{L}(\mathcal{B},F_0)}\int_0^t e^{-\omega s}|U(s)\varphi - T(s)\varphi|_{\mathcal{B}}ds.$$

Consequently,

$$|U(t)\varphi - T(t)\varphi|_{\mathcal{B}} \le K(t)\overline{M}e^{\omega t}\Big(\varepsilon|\varphi|_{\mathcal{B}} + |\mathcal{L}|_{\mathcal{L}(\mathcal{B},F_0)}\int_0^t e^{-\omega s}|U(s)\varphi - T(s)\varphi|_{\mathcal{B}}ds\Big).$$

By Gronwall's lemma,

$$|U(t)\varphi - T(t)\varphi|_{\mathcal{B}} \le K(t)\overline{M}\varepsilon|\varphi|_{\mathcal{B}} \exp\Big((|\mathcal{L}|_{\mathcal{L}(\mathcal{B},F_0)}K(t)\overline{M}+\omega)t\Big), \quad t \ge 0.$$

Hence, we conclude that $U(t)$ is differentiable at 0 and $(D_\varphi U(t))(0) = T(t)$. □

Theorem 4.2. *Assume that conditions (H1)–(H3) hold. If the zero equilibrium of $(T(t))_{t\ge0}$ is exponentially stable, then the zero equilibrium of $(U(t))_{t\ge0}$ is locally exponentially stable in the sense that there exist $\delta > 0$ $\mu > 0$, $k \ge 1$ such that*

$$|U(t)\varphi| \le ke^{-\mu t}|\varphi|, \quad \text{for } |\varphi| \le \delta\ t \ge 0.$$

Moreover if $\mathcal{H}$ can be decomposed as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_i$ are T-invariant subspaces of $\mathcal{H}$ and $\mathcal{H}_1$ is finite-dimensional and with $\omega_0 = \lim_{h\to\infty}\frac{1}{h}\log|T(h)/\mathcal{H}_2|$ we have

$$\inf\big\{|\lambda| : \lambda \in \sigma(T(t)/\mathcal{H}_1)\big\} > e^{\omega_0 t}.$$

Then zero is not stable in the sense that there exist $\varepsilon > 0$ and sequence $(\varphi_n)_n$ converging to 0 and $(t_n)_n$ of positive reals such that $|U(t_n)\varphi_n| > \varepsilon$.

The proof of this theorem is based on Theorem 4.1 and on the following result.

Theorem 4.3 ([6])**.** *Let $(V(t))_{t\ge0}$ be a nonlinear strongly continuous semigroup on a subset Ω of a Banach space Z and assume that $x_0 \in \Omega$ is an equilibrium of $(V(t))_{t\ge0}$ such that $V(t)$ is differentiable at x_0 for each $t \ge 0$, with $W(t)$ the derivative at x_0 of $V(t)$, $t \ge 0$. Then $(W(t))_{t\ge0}$ is a strongly continuous semigroup of bounded linear operators on Z. If the zero equilibrium of $(W(t))_{t\ge0}$ is exponentially stable, then x_0 is locally exponentially stable of $(V(t))_{t\ge0}$. Moreover if Z can be decomposed as $Z = Z_1 \oplus Z_2$ where Z_i are W-invariant subspaces of Z and Z_1 is finite-dimensional and with $\omega_0 = \lim_{h\to\infty}\frac{1}{h}\log|W(h)/Z_2|$ we have*

$$\inf\big\{|\lambda| : \lambda \in \sigma(W(t)/Z_1)\big\} > e^{\omega_0 t}.$$

Then the equilibrium x_0 is not stable in the sense that there exist $\varepsilon > 0$ and sequence $(x_n)_n$ converging to x_0 and $(t_n)_n$ of positive reals such that $|V(t_n)x_n - x_0| > \varepsilon$.

5. Applications

To apply our abstract result, we consider the partial functional differential equations with infinite delay,

$$\begin{gathered}\frac{\partial}{\partial t}v(t,\xi) = -\frac{\partial}{\partial \xi}v(t,\xi) + m(\xi)\int_{-\infty}^0 K(\theta, v(t+\theta,\xi))d\theta, \quad \text{for } \xi \in [0,1],\ t \ge 0,\\ v(t,0) = 0, \quad t \ge 0,\\ v(\theta,\xi) = v_0(\theta,\xi), \quad \text{for } \xi \in [0,1]\ \theta \in (-\infty,0],\end{gathered} \tag{5.1}$$

where K is a continuous function from $(-\infty,0]\times\mathbb{R}$ into $\mathbb{R}$ and $v_0 : (-\infty,0]\times[0,1] \to \mathbb{R}$ is an appropriate function. Here and hereafter we suppose that m is not necessarily continuous on $[0,1]$ and $m \in L^\infty(0,1)$. Let A be the operator defined on $X = C([0,1];\mathbb{R})$ by

$$D(A) = \{g \in C^1([0,1];\mathbb{R}) : g(0) = 0\}, \quad Ag = -g'.$$

Then $\overline{D(A)} = C_0([0,1];\mathbb{R}) = \{g \in C([0,1];\mathbb{R}) : g(0) = 0\}$.

Lemma 5.1 ([13]). *The operator A is a Hille-Yosida operator on X. The part A_0 of A in $C_0([0,1];\mathbb{R})$ generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $C_0([0,1];\mathbb{R})$ which is given for $g \in C_0([0,1];\mathbb{R})$ by*

$$(T_0(t)g)(\xi) = \begin{cases} g(\xi - t), & \text{if } t \leq \xi \\ 0, & \text{if } t > \xi. \end{cases}$$

Let $\mathrm{Lip}_0[0,1]$ be the space of Lipschitz continuous function on $[0,1]$ vanishing at zero.

Lemma 5.2 ([13]). *The following properties hold:*

(i) *The Favard class F_0 of the extrapolated semigroup $(T_{-1}(t))_{t\geq 0}$ is given by $F_0 = L^\infty(0,1)$.*
(ii) *The Favard class F_1 of semigroup $(T_0(t))_{t\geq 0}$ is given by $F_1 = \mathrm{Lip}_0[0,1]$, where the norm is given by*

$$|g|_{\mathrm{Lip}} = \sup_{0\leq x_1 < x_2 \leq 1} \frac{|g(x_1) - g(x_2)|}{x_1 - x_2}.$$

(iii) *The extrapolated operator A_{-1} coincides on F_1 with the a.e. derivative.*

Set $\gamma > 0$. For the phase space, we choose $\mathcal{B}$ to be defined by

$$\mathcal{B} = C_\gamma = \left\{\phi \in C((-\infty,0];X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } X\right\}$$

with the norm $|\phi|_\gamma = \sup_{\theta\leq 0} e^{\gamma\theta}|\phi(\theta)|$, $\phi \in C_\gamma$.

Lemma 5.3 ([12]). *The space C_γ satisfies Assumptions (A1), (A2), (B1), (C1), and (D1).*

We assume the following:

(a) K is measurable in (θ, z), $K(.,0)$ is integrable on $(-\infty,0]$ and there exists a positive function G such that

$$|K(\theta,z_1) - K(\theta,z_2)| \leq G(\theta)|z_1 - z_2|, \quad \text{for } \theta \in (-\infty,0]\ z_1,z_2 \in \mathbb{R}.$$

(b) $G(.)e^{-\gamma\cdot}$ is integrable on $(-\infty,0]$,
(c) $v_0 \in C((-\infty,0]\times[0,1];\mathbb{R}$, $v_0(0,0) = 0$ and $\lim_{\theta\to-\infty} e^{\gamma\theta}v_0(\theta,.)$ exists in X.

By making the following change of variables:

$$x(t)(\xi) = v(t,\xi), t \geq 0, \quad \xi \in [0,1],$$
$$\varphi(\theta)(\xi) = v_0(\theta,\xi), \theta \leq 0, \quad \xi \in [0,1],$$
$$F(\phi)(\xi) = m(\xi)\int_{-\infty}^0 K(\theta,\phi(\theta)(\xi))d\theta, \quad \xi \in [0,1],\ \phi \in C_\gamma,$$

Equation (5.1) takes the abstract form

$$\begin{gathered} \frac{dx}{dt}(t) = Ax(t) + F(x_t), \quad t \geq 0, \\ x_0 = \varphi \in C_\gamma. \end{gathered} \tag{5.2}$$

Proposition 5.4. *Assume that (a)–(c) above hold. Then (5.2) has a unique mild solution on* $(-\infty,\infty)$.

Proof. Since $m \in L^\infty(0,1)$, it follows that F doesn't take its values in X. However F takes its values in $L^\infty(0,1)$. In fact if $\phi_1 \in C_\gamma$, then

$$\begin{aligned}|F(\phi_1)|_{L^\infty(0,1)} &\le \operatorname*{ess\,sup}_{x\in[0,1]} |m(x)| \sup_{0\le\xi\le1}\int_{-\infty}^0 G(\theta)|\phi_1(\theta)(\xi)|d\theta \\ &\quad + \operatorname*{ess\,sup}_{x\in[0,1]} |m(x)| \int_{-\infty}^0 |K(\theta,0)|d\theta.\end{aligned}$$

Moreover,

$$\sup_{0\le\xi\le1}\int_{-\infty}^0 G(\theta)|\phi_1(\theta)(\xi)|d\theta \le \int_{-\infty}^0 e^{-\gamma\theta}G(\theta)d\theta \sup_{-\infty<\theta\le0,\ 0\le\xi\le1} e^{\gamma\theta}|\phi_1(\theta)(\xi)|.$$

It follows that for every $\phi_1 \in C_\gamma$, $F(\phi_1) \in L^\infty(0,1)$. Moreover for $\phi_1,\phi_2 \in C_\gamma$, we have

$$\begin{aligned}&|F(\phi_1)-F(\phi_2)|_{L^\infty(0,1)} \\ &= \operatorname*{ess\,sup}_{x\in[0,1]} |m(x)| \sup_{0\le\xi\le1}\int_{-\infty}^0 G(\theta)|\phi_1(\theta)(\xi)-\phi_2(\theta)(\xi)|d\theta \\ &\le |m|_{L^\infty(0,1)} \sup_{0\le\xi\le1}\int_{-\infty}^0 e^{-\gamma\theta}G(\theta)(e^{\gamma\theta}|\phi_1(\theta)(\xi)-\phi_2(\theta)(\xi)|)d\theta, \\ &\le |m|_{L^\infty(0,1)}\Big(\int_{-\infty}^0 e^{-\gamma\theta}G(\theta)d\theta\Big) \sup_{-\infty<\theta\le0,\ 0\le\xi\le1} e^{\gamma\theta}|\phi_1(\theta)(\xi)-\phi_2(\theta)(\xi)|.\end{aligned}$$

Then F is Lipschitzian from C_γ to $L^\infty(0,1)$. Assumption (**c**) implies that $\varphi \in C_\gamma$ and $\varphi(0) \in \overline{D(A)}$. As a consequence, (5.2) has a unique mild solution v on $(-\infty,\infty)$. □

For the regularity, we assume that F is continuously differentiable from C_γ to $L^\infty(0,1)$ and F' is locally Lipschitz. Let $v_0 \in C((-\infty,0]\times[0,1];\mathbb{R})$ be such that:

(d) $\frac{\partial v_0}{\partial\theta}$ exists and continuous on $(-\infty,0]\times[0,1]$, $\lim_{\theta\to-\infty}\big(e^{\gamma\theta}\big(\frac{\partial}{\partial\theta}v_0(\theta,.)\big)\big)$ exists in X and $v_0(0,.) \in \mathrm{Lip}_0[0,1]$,

(e) $\frac{\partial}{\partial\theta}v_0(0,\xi) = -\frac{\partial}{\partial\xi}v_0(0,\xi) + m(\xi)\int_{-\infty}^0 K(\theta,v_0(\theta,\xi))d\theta$, for a.e. $\xi\in[0,1]$,

(f) $\frac{\partial}{\partial\theta}v_0(0,0) = 0$.

Proposition 5.5. *Assume that (a)–(f) above hold. Let v be the mild solution of (5.2). Then v belongs to* $C^1(\mathbb{R}^+, C([0,1],\mathbb{R})) \cap C(\mathbb{R}^+, \mathrm{Lip}_0[0,1])$ *and satisfies*

$$\begin{gathered}\frac{\partial}{\partial t}v(t,\xi) = -\frac{\partial}{\partial\xi}v(t,\xi) + m(\xi)\int_{-\infty}^0 K(\theta, v(t+\theta,\xi))d\theta, \\ \text{for a.e. } \xi\in[0,1],\ t\ge0 \\ v(t,0)=0, \quad t\ge0 \\ v(\theta,\xi) = v_0(\theta,\xi), \quad \text{for } (\theta,\xi)\in(-\infty,0]\times[0,1].\end{gathered} \tag{5.3}$$

Proof. Conditions (d), (e), and (f) imply that φ is continuously differentiable and

$$\varphi' \in C_\gamma, \quad \varphi(0)\in F_1, \quad \varphi'(0)\in\overline{D(A)}, \quad \varphi'(0) = A_{-1}\varphi(0) + F(\varphi).$$

Since all assumptions of Theorem 3.5 are satisfied, we deduce that the mild solution v belongs to $C^1(\mathbb{R}^+, C_0([0,1],\mathbb{R})) \cap C(\mathbb{R}^+, \mathrm{Lip}_0[0,1])$ and satisfies (5.3). $\square$

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