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REGULARITY OF SOLUTIONS OF SOBOLEV TYPE SEMILINEAR INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this article, we prove the existence of mild and classical solutions of Sobolev type semilinear integrodifferential equations of the form

$$\frac{d}{dt}[Ex(t)] = A[x(t) + \int_0^t F(t-s)x(s)ds] + f(t, x(t))$$

in Banach spaces. The results are obtained by using the Banach contraction mapping principle and resolvent operator. An application is provided to illustrate the theory.

1. Introduction

Corduneanu [6] and Gripenberg et. al [10] studied the problem of existence of solutions for Volterra integral equations of various types. Grimmer [9] introduced the resolvent operators for integral equations in Banach spaces. Liu [17] studied the weak solutions of integrodifferential equations by using resolvent operators and semigroup theory. Fitzgibbon [8] investigated the existence problem for semilinear integrodifferential equations in Banach spaces. Using the method of semigroups and Banach's fixed point theorem Byszewski [5] proved the existence and uniqueness of mild, strong and classical solutions of nonlocal Cauchy problem. Lin and Liu [16] investigated the nonlocal Cauchy problem of semilinear integrodifferential equations by using resolvent operators. Brill [4] discussed the existence problem for semilinear Sobolev type equations in Banach spaces. Balachandran et. al [2] established the existence of solutions for Sobolev type integrodifferential equations in Banach spaces. Recently Balachandran et. al [1] investigated the same problem for Sobolev type delay integrodifferential equations. Several authors have studied the problem of existence of solutions of semilinear differential equations and Sobolev type equations [3, 7, 11, 12, 13, 14, 15, 18, 19, 22].

The purpose of this article is to study the regularity of solutions of Sobolev type semilinear integrodifferential equations in Banach spaces by using semigroup theory and the Banach fixed point theorem.

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2. Preliminaries

Consider the Sobolev type semilinear integrodifferential equation

$$\begin{gathered}\frac{d}{dt}[Ex(t)] = A[x(t) + \int_0^t F(t-s)x(s)ds] + f(t, x(t)),\\ x(0) = x_0, \quad t \in J = [0, b],\end{gathered} \tag{2.1}$$

where A and E are closed linear operators with domain contained in a Banach space X and ranges contained in a Banach space Y, $f : J \times X \to Y$ is a continuous function. $F(t) \in B(X)$, $0 \le t \le b$. $F(t) : W \to W$ and for $x(\cdot)$ continuous in Y, $AF(\cdot)x(\cdot) \in L^1([0, b], X)$. For $x \in X$, $F'(t)x$ is continuous in $t \in [0, b]$, where $B(X)$ is the space of all bounded linear operators on X, and W is the Banach space formed from $D(A)$, the domain of A, endowed with the graph norm.

The operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ satisfy the hypothesis:

- (A1) A and E are closed linear operators,
- (A2) $D(E) \subset D(A)$ and E is bijective,
- (A3) $E^{-1} : Y \to D(E)$ is continuous and $E^{-1}F = FE^{-1}$,
- (A4) AE^{-1} generates a strongly continuous semigroup of bonded linear operators in X.

Definition 2.1. A family of bounded linear operator $R(t) \in B(X)$ for $t \in [0, b]$ is called a resolvent operator for

$$\frac{dx}{dt} = A[x(t) + \int_0^t F(t-s)x(s)ds]$$

if

- (i) $R(0) = I$, (the identity operator on X).
- (ii) For all $x \in X$, $R(t)x$ is continuous for $t \in J$.
- (iii) $R(t) \in B(W)$, $t \in J$. For $y \in W$, $R(\cdot)y \in C^1([0, b], X) \cap C([0, b], W)$ and

$$\begin{aligned}\frac{d}{dt}R(t)y &= AE^{-1}[R(t)y + \int_0^t F(t-s)R(s)yds]\\ &= R(t)AE^{-1}y + \int_0^t R(t-s)AE^{-1}F(s)yds, \quad t \in J.\end{aligned}$$

Definition 2.2. A function $x(t) \in C([0, b], X)$ is called a mild solution of the Cauchy problem (2.1) if it satisfies the integral equation

$$x(t) = E^{-1}R(t)Ex_0 + E^{-1}\int_0^t R(t-s)f(s, x(s))ds. \tag{2.2}$$

Definition 2.3. A classical solution of (2.1) is a function $x(\cdot) \in C([0, b], W) \cap C^1([0, b], X)$ which satisfies the integrodifferential equation (2.1) on $[0, b]$.

Assume the following conditions:

- (A5) The resolvent operator $R(t)$ is compact in X and there exists a constant $M_1 > 0$, such that $\|R(t)\| \le M_1$.
- (A6) The nonlinear operator $f : J \times X \to X$ is continuous in t on J and there exists a constant $L > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\| \le L\|x_1 - x_2\|_X, \quad t \in J,\ x_1, x_2 \in X,$$

(A7) Let $\alpha = |E^{-1}|$ and $0 < \alpha L b M_1 < 1$.

3. Main Results

Theorem 3.1. *If the hypothesis (A1) to (A7) are satisfied, then problem* (2.1) *has a mild solution on* J.

Proof. Let $Z = C(J, X)$. Then define an operator $\Phi : Z \to Z$ by

$$(\Phi x)(t) = E^{-1}R(t)Ex_0 + E^{-1}\int_0^t R(t-s)f(s, x(s))ds.$$

Now for every $x_1, x_2 \in Z$ and $t \in J$, we have

$$\begin{aligned}
\|(\Phi x_1)(t) - (\Phi x_2)(t)\| &= \|E^{-1}\int_0^t R(t-s)[f(s, x_1(s)) - f(s, x_2(s))]ds\| \\
&\le |E^{-1}|\int_0^t \|R(t-s)\|\|f(s, x_1(s)) - f(s, x_2(s))\|ds \\
&\le \alpha M_1 \int_0^t L\|x_1(s) - x_2(s)\|_X ds \\
&\le \alpha M_1 L b\|x_1(t) - x_2(t)\|_X.
\end{aligned}$$

Since $\alpha L b M_1 < 1$, the operator Φ is a contraction on E. Applying Banach's fixed point theorem we get a unique fixed point for Φ and this point is the mild solution of (2.1) on J. □

Next we prove that mild solutions are classical solutions when $f \in C^1(J \times X, Y)$.

Theorem 3.2. *Let assumptions (A1)–(A7) be satisfied and let $x(\cdot)$ be the unique mild solution of* (2.1). *Assume further that $x_0 \in D(A)$, $f \in C^1(J \times X, Y)$. Then $x(\cdot)$ is a unique classical solution of equation* (2.1).

Proof. Since (A1)–(A7) are satisfied, problem (2.1) possesses a unique mild solution which is denoted by $x(\cdot)$. We will show that $x(\cdot) \in C^1(J, X)$.

Next we shall show that the mild solution is a classical solution of (2.1) on J. To this end, let

$$B(s) = \frac{\partial}{\partial x} f(s, x(s)), \quad s \in J, \tag{3.1}$$

and

$$\begin{aligned}
k(t) = E^{-1}R(t)f(0, x_0) &+ A\Big[R(t)x_0 + \int_0^t F(t-s)R(s)x_0 ds\Big] \\
&+ E^{-1}\int_0^t R(t-s)\frac{\partial}{\partial s} f(s, x(s))ds.
\end{aligned} \tag{3.2}$$

Note that $x_0 \in W$, from Definition 2.1 and our assumptions, $k(\cdot) \in E$. Thus the method used in Pazy [20] or in the proof of Theorem 3.2 can be applied here to show that the integral equation

$$w(t) = k(t) + E^{-1}\int_0^t R(t-s)B(s)w(s)ds, \ t \in J, \tag{3.3}$$

has a unique solution $w(\cdot) \in E$. Moreover, from the assumptions we have

$$f(s, x(s+h)) - f(s, x(s)) = B(s)[x(s+h) - x(s)] + w_1(s, h),$$

$$f(s+h, x(s+h)) - f(s, x(s+h)) = \frac{\partial}{\partial s} f(s, x(s+h))h + w_2(s, h),$$

where $h^{-1}\|w_i(s,h)\| \to 0$, as $h \to 0$, uniformly on $s \in J$ for $i = 1,2$. Define

$$w_h(t) = \frac{x(t+h) - x(t)}{h} - w(t). \tag{3.4}$$

Then from (3.1)–(3.4) and the fact that $x(\cdot)$ is a mild solution, we obtain

$$\begin{aligned}
&w_h(t)\\
&= h^{-1}E^{-1}[R(t+h)Ex_0 - R(t)Ex_0] - A\Big[R(t)x_0 + \int_0^t F(t-s)R(s)x_0(s)ds\Big]\\
&\quad + h^{-1}E^{-1}\Big[\int_0^{t+h} R(t+h-s)f(s,x(s))ds - \int_0^t R(t-s)f(s,x(s))ds\Big]\\
&\quad - E^{-1}\Big[R(t)f(0,x_0) + \int_0^t R(t-s)\frac{\partial}{\partial s}f(s,x(s))ds\Big]\\
&\quad - E^{-1}\int_0^t R(t-s)\frac{\partial}{\partial x}f(s,x(s))w(s)ds\\
&= h^{-1}E^{-1}[R(t+h)Ex_0 - R(t)Ex_0] - A\Big[R(t)x_0 + \int_0^t F(t-s)R(s)x_0(s)ds\Big]\\
&\quad + h^{-1}\Big[E^{-1}\int_0^h R(t+h-s)f(s,x(s))ds - E^{-1}R(t)f(0,x_0)\Big]\\
&\quad + h^{-1}E^{-1}\int_0^t R(t-s)[w_1(s,h) + w_2(s,h)]ds\\
&\quad + E^{-1}\int_0^t R(t-s)\frac{\partial}{\partial s}f(s,x(s+h))ds - E^{-1}\int_0^t R(t-s)\frac{\partial}{\partial s}f(s,x(s))ds\\
&\quad + E^{-1}\int_0^t R(t-s)\frac{\partial}{\partial x}f(s,x(s))\Big(\frac{x(s+h)-x(s)}{h}\Big)ds\\
&\quad - E^{-1}\int_0^t R(t-s)\frac{\partial}{\partial x}f(s,x(s))w(s)ds\\
&= h^{-1}E^{-1}[R(t+h)Ex_0 - R(t)Ex_0] - A\Big[R(t)x_0 + \int_0^t F(t-s)R(s)x_0(s)ds\Big]\\
&\quad + h^{-1}E^{-1}\int_0^h R(t+h-s)f(s,x(s))ds - E^{-1}R(t)f(0,x_0))\\
&\quad + h^{-1}E^{-1}\int_0^t R(t-s)[w_1(s,h) + w_2(s,h)]ds\\
&\quad + E^{-1}\int_0^t R(t-s)\frac{\partial}{\partial s}[f(s,x(s+h)) - f(s,x(s))]ds\\
&\quad + E^{-1}\int_0^t R(t-s)\frac{\partial}{\partial x}f(s,x(s))\Big[\frac{x(s+h)-x(s)}{h} - w(s)\Big]ds
\end{aligned}$$

and

$$\begin{aligned}
&\|w_h(t)\|\\
&\le |E^{-1}|\,\Big\|h^{-1}[R(t+h)Ex_0 - R(t)Ex_0] - A\Big[R(t)x_0 + \int_0^t F(t-s)R(s)x_0(s)ds\Big]\Big\|
\end{aligned}$$

$$
\begin{aligned}
&+ |E^{-1}| \big\| h^{-1} \int_0^h R(t+h-s) f(s,x(s))ds - E^{-1}R(t)f(0,x_0)) \big\| \\
&+ |E^{-1}| \big\| h^{-1} \int_0^t R(t-s)[w_1(s,h)+w_2(s,h)]ds \big\| \\
&+ |E^{-1}| \big\| \int_0^t R(t-s) \frac{\partial}{\partial s}[f(s,x(s+h)) - f(s,x(s))]ds \big\| \\
&+ |E^{-1}| \big\| \int_0^t R(t-s) \frac{\partial}{\partial x} f(s,x(s)) \big[\frac{x(s+h)-x(s)}{h} - w(s) \big] ds \big\| \\
\le\; & |E^{-1}| \big\| h^{-1}[R(t+h)Ex_0 - R(t)Ex_0] - A[R(t)x_0 + \int_0^t F(t-s)R(s)x_0(s)ds] \big\| \\
&+ |E^{-1}| \big\| h^{-1} \int_0^h R(t+h-s) f(s,x(s))ds - E^{-1}R(t)f(0,x_0) \big\| \\
&+ |E^{-1}| \int_0^t \|R(t-s)\| \|w_1(s,h)+w_2(s,h)\| ds \\
&+ |E^{-1}| \int_0^t \|R(t-s)\| \frac{\partial}{\partial s} \|f(s,x(s+h)) - f(s,x(s))\| ds + N \int_0^t \|w_h(s)\| ds,
\end{aligned}
$$

where

$$
N = \alpha \max_{t>0} \|R(t-s) \frac{\partial}{\partial x} f(s,x(s))\|_{B(X)}.
$$

From the definition of resolvent operator and our assumptions, it is clear that the norm of each one of the first four terms on the right hand side of the above equation tends to zero as $h \to 0$. Therefore, we have

$$
\|w_h(t)\|_X \le \epsilon(h) + N \int_0^t \|w_h(s)\|_X ds, \tag{3.5}
$$

and $\epsilon(h) \to 0$ as $h \to 0$. From (3.5) it follows by Gronwall's inequality that

$$
\|w_h(t)\|_X \le \epsilon(h) e^{TN},
$$

and, therefore, $\|w_h(t)\|_X \to 0$ as $h \to 0$, $t \in J$. This implies that $x(t)$ is differentiable on J and that $w(t)$ is the derivative of $x(t)$. Since $w \in E$, x is continuously differentiable on J.

Finally, to show that x is the classical solution of problem (2.1). Observe that, from the continuous differentiability of x and $f \in C^1(J \times X, X)$, $t \to f(t, x(t))$ is continuously differentiable on J. As shown in [16], the linear Cauchy problem

$$
\begin{gathered}
v'(t) = AE^{-1}\Big[v(t) + \int_o^t F(t-s)v(s)ds\Big] + f(t,x(t)), \quad 0 \le t \le b, \\
v(0) = x_0,
\end{gathered}
$$

has a unique classical solution $v(\cdot)$ given by

$$
v(t) = R(t) + \int_0^t R(t-s) f(s,x(s))ds. \tag{3.6}
$$

The right hand side of (3.6) is $x(t)$ since $x(\cdot)$ is the mild solution. So we have $v(t) = x(t)$, $t \in J$, and hence, $x(\cdot)$ is the classical solutions of (2.1). Hence the theorem is proved. $\square$

4. Application

Consider the semilinear integrodifferential system

$$\begin{gathered}\frac{d}{dt}[Ex(t)] = A[x(t) + \int_0^t F(t-s)x(s)ds] + (Bu)(t) + f(t, x(t)),\\ x(0) = x_0, \quad t \in J = [0, b],\end{gathered} \tag{4.1}$$

where A and E are closed linear operators with domain contained in a Banach space X and the ranges are contained in a Banach space Y, the state $x(\cdot)$ takes values in the Banach space X and the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control function with U as a Banach space and B is a bounded function from U into X. Then for the system (4.1) there exists a mild solution of the form

$$x(t) = E^{-1}R(t)Ex_0 + E^{-1}\int_0^t R(t-s)[(Bu)(s) + f(s, x(s))ds],$$

and $Ex(t) \in C([0, b], Y) \cap C^1([0, b], Y)$.

Definition 4.1. The system (4.1) is said to be controllable on the interval J if, for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(t)$ of (4.1) satisfies $x(b) = x_1$.

We assume the following hypothesis:

(A8) The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^b E^{-1}R(b-s)Bu(s)ds$$

has induces an inverse operator $\tilde{W}^{-1}$ defined on $L^2(J, U)/\ker W$ and there exist positive constants M_2, M_3 such that $|B| \le M_2$ and $|\tilde{W}^{-1}| \le M_3$ (see [21]).

(A9) $0 < \alpha M_1 L b[\alpha M_1 M_2 M_3 b + 1] < 1$.

Theorem 4.2. *If the hypothesis (A1)–(A9) are satisfied then the system* (4.1) *is controllable on J.*

Proof. Using the hypothesis (A8) for an arbitrary function $x(\cdot)$ define the control

$$u(t) = \tilde{W}^{-1}\Big[x_1 - E^{-1}R(b)Ex_0 - E^{-1}\int_0^b R(b-s)f(s, x(s))ds\Big](t).$$

Now we show that, when using this control, the operator $\Psi : Z_b^0 \to Z_b^0$ defined by

$$\begin{aligned}(\Psi x)(t) = E^{-1}R(t)Ex_0 + E^{-1}\int_0^t R(t-\eta)B\tilde{W}^{-1}\Big[x_1 - E^{-1}R(b)Ex_0\\ - E^{-1}\int_0^b R(b-s)f(s, x(s))ds\Big](\eta)d\eta + E^{-1}\int_0^t R(t-s)f(s, x(s))ds,\end{aligned}$$

has a fixed point. This fixed point is then a solution of (4.1).

Clearly $x(b) = x_1$ which means that the control u steers that the semilinear integrodifferential system from the initial state x_0 to x in time b, provided we can obtain a fixed point of the nonlinear operator Ψ. The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted. □

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