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MULTI-POINT BOUNDARY-VALUE PROBLEMS FOR THE P-LAPLACIAN AT RESONANCE

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Abstract. In this paper, we consider multi-point boundary-value problems for the p-laplacian at resonance. Applying an extension of Mawhin's continuation theorem, we show the existence of at least one solution.

1. Introduction

Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\alpha_j \in \mathbb{R}$ $(1 \le j \le n-2)$, $\beta_i \in \mathbb{R}$ $(1 \le i \le m-2)$ be given. In this paper, we study the differential equation

$$(\phi_p(x'))' = f(t, x(t), x'(t)) \tag{1.1}$$

subject to one of the following boundary-value conditions:

$$x'(0) = 0, \quad \phi_p(x'(1)) = \sum_{i=1}^{m-2} \beta_i \phi_p(x'(\eta_i)) \tag{1.2}$$

or

$$\phi_p(x'(0)) = \sum_{j=1}^{n-2} \alpha_j \phi_p(x'(\xi_j)), \quad x'(1) = 0 \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\sum_{i=1}^{m-2}\beta_i = \sum_{j=1}^{n-2}\alpha_j = 1$.

Mawhin [2] proved a continuation theorem for the abstract equation $Lx = Nx$ with L being a non-invertible linear operator. Since then, there has been much attention on the study of boundary-value problems at resonance (cf [1, 7, 8]). Recently, Liu [4, 5, 6] considered differential equation (1.1) with $p = 2$ and obtained some results. However, the p-laplacian operator is not linear. Our approach is different and based on an extension of Mawhin's continuation theorem in [3]. We provide sufficient conditions to guarantee the existence of solution for (1.1), (1.2) and for (1.1), (1.3). Moreover, some examples are given to demonstrate our results.

For the convenience of the reader, we first recall some notation.

Let X and Z be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. A continuous operator $M : X \cap \operatorname{dom} M \to Z$ is said to be quasi-linear if

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(i) $\operatorname{Im} M = M(X \cap \operatorname{dom} M)$ is a closed subset of Z;
(ii) $\operatorname{Ker} M = \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.

Let $P : X \to X_1$ and $Q : Z \to Z_1$ be two projectors such that $\operatorname{Im} P = \operatorname{Ker} M$, $\operatorname{Ker} Q = \operatorname{Im} M$ and $X = X_1 \oplus X_2$, $Z = Z_1 \oplus Z_2$, where $X_1 = \operatorname{Ker} M$, $Z_2 = \operatorname{Im} M$ and X_2, Z_1 are respectively the complement space of X_1 in X, Z_2 in Z. If Ω is an open and bounded subset of X such that $\operatorname{dom} M \cap \Omega \neq \Phi$, the continuous operator $N_\lambda : \overline{\Omega} \to Z, \lambda \in [0,1]$ will be called admissibly compact in $\overline{\Omega}$ with respect to M if

(iii) There is a subset Z_1 of Z with $\dim Z_1 = \dim X_1$ and an operator $K : \operatorname{Im} M \to X_2$ with $K\theta = \theta$ such that for $\lambda \in [0,1]$,

$$(I-Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Z, \tag{1.4}$$

$$(I-Q)N_0 \text{ is a zero operator, and } QN_\lambda x = 0 \Leftrightarrow QNx = 0, \lambda \in (0,1), \tag{1.5}$$

$$KM = I - P,\ K(I-Q)N_\lambda : \overline{\Omega} \to X_2 \subset X \text{ is compact}, \tag{1.6}$$

$$M[P + K(I-Q)N_\lambda] = (I-Q)N_\lambda. \tag{1.7}$$

Theorem 1.1 ([3]). *Let X and Z be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively, and $\Omega \subset X$ an open and bounded set. Suppose $M : X \cap \operatorname{dom} M \to Z$ is a quasi-linear operator and $N_\lambda : \overline{\Omega} \to Z$ is admissibly compact with respect to M. In addition, if*

(C1) $Mx \neq N_\lambda x, \lambda \in (0,1)$, $x \in \partial\Omega$
(C2) $\deg\{JQN, \Omega \cap \operatorname{Ker} M, 0\} \neq 0$, *where $N = N_1$ and $J : Z_1 \to X_1$ is a homeomorphism with $J(0) = 0$;*

then the abstract equation $Mx = Nx$ has at least one solution in $\overline{\Omega}$.

2. General Results

2.1. **Results for boundary-value problem (1.1), (1.2).** Let $X = \{x \in C[0,1] : x'(0) = 0,\ \phi_p(x'(1)) = \sum_{i=1}^{m-2} \beta_i \phi_p(x'(\eta_i))\}$ with norm $\|x\|_1 = \max\{\|x\|, \|x'\|\}$ and $Z = C[0,1]$ with norm $\|w\| = \max_{t\in[0,1]} |w(t)|$. Let M be the operator from $\operatorname{dom} M \subset X$ to Z with

$$\operatorname{dom} M = \{x \in C^1[0,1] : \phi_p(x') \in C^1[0,1]\}$$

and $Mx = (\phi_p(x'))'$. For any open and bounded $\Omega \subset X$, we define $N_\lambda : \overline{\Omega} \to Z$ by

$$(N_\lambda x) = \lambda f(t, x(t), x'(t)), \quad t \in [0,1]$$

Then (1.1), (1.2) can be written as

$$Mx = Nx, \quad \text{where } N = N_1.$$

Lemma 2.1. *If $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$, then there exists integer $k \in \{1, 2, \dots, m-1\}$ such that $\sum_{i=1}^{m-2} \beta_i \eta_i^k \neq 1$.*

Proof. If this were not the case, we have

$$\begin{pmatrix} \eta_1 & \eta_2 & \dots & \eta_{m-2} \\ \eta_1^2 & \eta_2^2 & \dots & \eta_{m-2}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1^{m-1} & \eta_2^{m-1} & \dots & \eta_{m-2}^{m-1} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{m-2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

This implies

$$\begin{pmatrix} \eta_1 & \eta_2 & \dots & \eta_{m-2} & 1 \\ \eta_1^2 & \eta_2^2 & \dots & \eta_{m-2}^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \eta_1^{m-1} & \eta_2^{m-1} & \dots & \eta_{m-2}^{m-1} & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{m-2} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Putting

$$A = \begin{pmatrix} \eta_1 & \eta_2 & \dots & \eta_{m-2} & 1 \\ \eta_1^2 & \eta_2^2 & \dots & \eta_{m-2}^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \eta_1^{m-1} & \eta_2^{m-1} & \dots & \eta_{m-2}^{m-1} & 1 \end{pmatrix}$$

we have

$$\det A = \eta_1\eta_2\eta_{m-2} \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ \eta_1 & \eta_2 & \dots & \eta_{m-2} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \eta_1^{m-2} & \eta_2^{m-2} & \dots & \eta_{m-2}^{m-2} & 1 \end{vmatrix} \neq 0.$$

Hence, $(\beta_1, \beta_2, \dots, \beta_{m-2}, -1)^T = (0, 0, \dots, 0, 0)^T$, which leads to a contradiction, $-1 = 0$. $\square$

Lemma 2.2. *If $\sum_{i=1}^{m-2} \beta_i = 1$, then the operator $M : \operatorname{dom} M \cap X \to Z$ is a quasi-linear and $N_\lambda : \overline{\Omega} \to Z$ is admissibly compact in $\overline{\Omega}$ with respect to M.*

Proof. Obviously, $X_1 = \operatorname{Ker} M = \{\, x = A,\ A \in \mathbb{R}\}$ and

$$Z_2 = \operatorname{Im} M = \Big\{y \in Z,\ \sum_{i=1}^{m-2} \int_{\eta_i}^1 \beta_i y(s)ds = 0\Big\}.$$

We have $\dim \operatorname{Ker} M = 1 < \infty$ and $M(X \cap \operatorname{dom} M) \subset Z$ closed. Hence, M is a quasi-linear operator. Define projectors $P : X \to X_1$ as $Px = x(0)$ for all $x \in X$ and $Q : Z \to Z_1$ as

$$Qy = \Big[\frac{k}{1 - \sum_{i=1}^{m-2} \beta_i \eta_i^k} \sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 y(s)ds\Big] t^{k-1}, \quad \forall y \in Z.$$

where Z_1 is the complement space of Z_2 in Z. (Indeed, $Z_1 \cong \mathbb{R}$). Thus $\dim X_1 = \dim Z_1 = 1$.

Taking $K : \operatorname{Im} M \to X_2$ as follows:

$$K(I - Q)h(t) = \int_0^t \phi_p^{-1}\Big(\int_0^s ((I - Q)h(\tau)d\tau\Big)ds, \quad \forall h(t) \in Z.$$

here X_2 is the complement space of X_1 in X. It is clear that $K\theta = \theta$ and (1.4), (1.5) hold. For for all $x \in X \cap \operatorname{dom} M$, we have

$$M[P + K(I - Q)N_\lambda]x = (I - Q)N_\lambda x$$

and

$$KMx = K(I - Q)Mx = \int_0^t \phi_p^{-1}(\int_0^s (\phi_p(x'(\tau)))' d\tau)ds = x(t) - x(0) = (I - P)x.$$

For any open and bounded set $\Omega \subset X, \lambda \in [0,1]$, by using the Ascoli-Arzela Theorem, we can prove that $[K(I-Q)N_\lambda](\overline{\Omega})$ is relatively compact and continuous. This implies that (1.6), (1.7) are valid. Hence, N_λ is admissibly compact in $\overline{\Omega}$ with respect to M. $\square$

For the next theorem we have the assumptions:

(H1) There exist functions $a, b, r \in C([0,1],[0,+\infty))$ such that for all $(x,y) \in \mathbb{R}^2$, $t \in [0,1]$,
$$|f(t,x,y)| \le \phi_p[a(t)|x| + b(t)|y| + r(t)].$$

(H2) There exists a constant $C > 0$ such that for any $x \in X$, if $|x(t)| > c$, for all $t \in [0,1]$, one has
$$\operatorname{sgn}(x(t)) \sum_{i=2}^{m-2} \beta_i \int_{\eta_i}^{1} f(t,x(t),x'(t))dt < 0 \tag{2.1}$$
or
$$\operatorname{sgn}(x(t)) \sum_{i=2}^{m-2} \beta_i \int_{\eta_i}^{1} f(t,x(t),x'(t))dt > 0 \tag{2.2}$$

Theorem 2.3. *Suppose $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\beta_i \in \mathbb{R}$ ($1 \le i \le m-2$) are given. Under Assumptions (H1) and (H2), the boundary-value problem (1.1), (1.2) with $\sum_{i=1}^{m-2} \beta_i = 1$ has at least one solution provided that $\|a\| + \|b\| < 1$.*

Proof. Set $\Omega_1 = \{x \in X \cap \operatorname{dom} M,\ Mx = N_\lambda x,\ \lambda \in (0,1)\}$, then for $x \in \Omega_1$, $Mx = N_\lambda x$. Thus, we have $Nx \in \operatorname{Im} M = \operatorname{Ker} Q$ and
$$\sum_{i=2}^{m-2} \beta_i \int_{\eta_i}^{1} f(t,x(t),x'(t))dt = 0.$$
By (H2), there exists $t_0 \in [0,1]$ such that $|x(t_0)| \le C$. Hence, we have $\|x\| \le C + \|x'\|$, in view of the relation $x(t) = x(t_0) + \int_{t_0}^{t} x'(s)ds$. At the same time,
$$|x'(t)| \le \phi_p^{-1}(\|Nx\|) \le \|a\|\|x\| + \|b\|\|x'\| + \|r\|.$$
Noticing $\|a\| + \|b\| < 1$, we know the set Ω_1 is bounded. Let $\|x\|_1 \le C^*$ for all $x \in \Omega_1$.

If (2.1) holds, taking $H(x,\lambda) = -\lambda Ix + (1-\lambda)JQNx$, $\lambda \in [0,1]$ and
$$\Omega_2 = \{x \in \operatorname{Ker} M,\ H(x,\lambda) = 0\},$$
where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homomorphism such that $J(At^{k-1}) = A$, for all $A \in \mathbb{R}$. We claim that for any $x \in \Omega_2$, $\|x\|_1 \le C$. In fact, if this were not the case, there exist $\lambda_0 \in [0,1]$ and $|x_0| > C$ such that $H(x_0,\lambda_0) = 0$. Without loss of generality, suppose $x_0 > C$ and $\sum_{i=1}^{m-2} \beta_i \eta_i^k < 1$. If $\lambda_0 = 1$, then $x_0 = 0$ otherwise, it implies that
$$0 \le \lambda_0 x_0 = (1-\lambda_0) \frac{k}{1 - \sum_{i=1}^{m-2} \beta_i \eta_i^k} \sum_{i=2}^{m-2} \beta_i \int_{\eta_i}^{1} f(t,x_0,0)dt < 0,$$
which leads to a contradiction, $0 < 0$.

When (2.2) holds, putting $H(x,\lambda) = \lambda Ix + (1-\lambda)JQNx$. By a similar argument, we can obtain the same conclusion.

Next, we shall prove that all conditions of Theorem 1.1 are satisfied. Let Ω be an open bounded subset of X such that $\bigcup_{i=1}^{2} \overline{\Omega}_i \subset \Omega$. Clearly, we have

$$Mx \neq N_\lambda x, \quad \lambda \in (0,1),\ x \in \partial\Omega$$

and

$$H(x,\lambda) \neq 0, \quad \lambda \in [0,1],\ x \in \partial\Omega \cap \operatorname{Ker} M.$$

In view of the homotopy property of degree,

$$\deg\{JQN, \Omega \cap \operatorname{Ker} M, 0\} = \deg\{-I, \Omega \cap \operatorname{Ker} M, 0\} = -1.$$

Theorem 1.1 can be generalized to obtain the existence of at least one solution for (1.1), (1.2) with $\sum_{i=2}^{m-2} \beta_i = 1$. $\square$

2.2. **Results for Problem (1.1), (1.3).** Let $X = \{x \in C[0,1] : \phi_p(x'(0)) = \sum_{j=1}^{n-2} \alpha_j \phi_p(x'(\xi_j)),\ x'(1) = 0\}$. $Z, M, \operatorname{dom} M$ and N_λ be defined as above.

Lemma 2.4. *If $\sum_{j=1}^{n-2} \alpha_j = 1$ and $\sum_{j=1}^{n-2} \alpha_j \xi_j \neq 0$ then operator $M : \operatorname{dom} M \cap X \to Z$ is a quasi-linear and $N_\lambda : \overline{\Omega} \to Z$ is admissibly compact in $\overline{\Omega}$ with respect to M.*

Note that $\operatorname{Ker} M = \{x = A, A \in \mathbb{R}\}$ and $\operatorname{Im} M = \{y \in Z,\ \sum_{j=1}^{n-2} \alpha_j \int_0^{\xi_j} y(s)ds = 0\}$. We define $P : X \to X_1$, and $Q : Z \to Z_1$ as follows:

$$(Px)(t) = x(0), \quad (Qy)(t) = \frac{1}{\sum_{j=1}^{n-2} \alpha_j \xi_j} \sum_{j=1}^{n-2} \alpha_j \int_0^{\xi_j} y(s)ds.$$

Let $K(I-Q)h(t) = -\int_0^t \phi_p^{-1}\big[\int_s^1 ((I-Q)h(\tau)d\tau\big]ds$, for $h(t) \in Z$. Using the method in the proof of Lemma 2.2, we can show the theorem bellow. However, we need the assumption:

- (H3) There exists a constant $D > 0$ such that for all $x \in X$, if $|x(t)| > D$ for some $t \in [0,1]$, one has
$$\operatorname{sgn}(x(t)) \sum_{j=1}^{n-2} \alpha_j \int_0^{\xi_j} f(t, x(t), x'(t))dt < 0 \tag{2.3}$$
or
$$\operatorname{sgn}(x(t)) \sum_{j=1}^{n-2} \alpha_j \int_0^{\xi_j} f(t, x(t), x'(t))dt > 0 \tag{2.4}$$

Theorem 2.5. *Suppose that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$, and $\alpha_j \in \mathbb{R}$ $(1 \leq j \leq n-2)$ are given. Under Assumptions (H1), (H3), the boundary-value problem (1.1), (1.3) with $\sum_{j=1}^{n-2} \alpha_j = 1$, $\sum_{j=1}^{n-2} \alpha_j \xi_j \neq 0$ has at least one solution provided that $\|a\| + \|b\| < 1$.*

The proof of this theorem is similar to that of Theorem 2.3; therefore, we omit it here.

3. Examples

Example 3.1. Consider the boundary-value problem, at resonance,

$$\begin{gathered}(\phi_p(x'))' = \frac{1}{3}x + \frac{(\cos(x'))^2}{2(t+1)} - 3t^2, \quad t \in [0,1] \\ x'(0) = 0, \quad \phi_p(x'(1)) = \sum_{i=1}^{m-2} \beta_i \phi_p(x'(\eta_i))\end{gathered} \tag{3.1}$$

where $p > 2$, $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$, $\sum_{i=1}^{m-2} \beta_i = 1$, $\sum_{i=1}^{m-2} \beta_i \eta_i \neq 1$ and $\beta_i (1 \leq i \leq m-2) \geq 0$.

Let $f(t,x,y) = \frac{1}{3}x + \frac{(\cos(x'))^2}{2(t+1)} - 3t^2$, then $|f(t,x,y)| \leq \frac{1}{3}|x| + \frac{1}{2}|y| + 3t^2 \leq \phi_p(\frac{1}{3}|x| + \frac{1}{2}|y| + 3)$. Clearly, there exists a constant $C > 0$ such that if $|x(t)| > C$, for any $t \in [0,1]$, one has

$$\operatorname{sgn}(\mathrm{x(t)}) \sum_{i=2}^{m-2} \beta_i \int_{\eta_i}^{1} f(t, x(t), x'(t)) dt > 0.$$

Applying Theorem 2.3, we obtain that (3.1) has at least one solution.

Example 3.2. Consider the boundary-value problem

$$\begin{gathered}(\phi_p(x'))' = \frac{1}{4}x + \frac{1}{12}x\sin(x') + t \\ \phi_p(x'(0)) = 3\phi_p(x'(\frac{1}{4})) - 2\phi_p(x'(\frac{1}{2})), \; x'(1) = 0,\end{gathered} \tag{3.2}$$

where $p > 2$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{2}$, $\alpha_1 = 3$, $\alpha_2 = -2$. Taking $f(t,x,y) = \frac{1}{4}x + \frac{1}{12}x\sin(x') + t$, then $|f(t,x,y)| \leq \frac{1}{12}|x| + \frac{1}{4}|x| + 1 \leq \phi_p(\frac{1}{3}|x| + 1)$ and for any $A \in \mathbb{R}$,

$$3\int_0^{\frac{1}{4}} f(t,A,0)dt - 2\int_0^{\frac{1}{2}} f(t,A,0)dt = -\frac{A^3}{4} - \frac{5}{32}.$$

Hence, there exists a constant $D > 0$ such that for $t \in [0,1]$

$$\sum_{j=1}^{n-2} \alpha_j \int_0^{\xi_j} f(t,D,0)dt < 0 < \sum_{j=1}^{n-2} \alpha_j \int_0^{\xi_j} f(t,-D,0)dt.$$

From Theorem 2.5, Problem (3.2) has at least one solution.

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