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SYMMETRY AND MONOTONICITY OF SOLUTIONS TO SOME VARIATIONAL PROBLEMS IN CYLINDERS AND ANNULI

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Abstract. We prove symmetry and monotonicity properties for local minimizers and stationary solutions of some variational problems related to semilinear elliptic equations in a cylinder $(-a,a)\times\omega$, where ω is a bounded smooth domain in $\mathbb{R}^{N-1}$. The admissible functions satisfy periodic boundary conditions on $\{\pm a\}\times\omega$, and some other conditions. We show also symmetry properties for related problems in annular domains. Our proofs are based on rearrangement arguments and on the Moving Plane Method.

1. Introduction

Let ω be a bounded, smooth domain in $\mathbb{R}^{N-1}$ and Ω the finite cylinder $(0,a)\times\omega$, $(a>0)$. Defining a functional J on $W^{1,2}(\Omega)$ by $J(v):=\int_\Omega((|\nabla v|^2/2)-F(v))\,dx$, we look for minimizers of J subject to the integral constraint $\int_\Omega G(v)\,dx=0$ (and possibly with respect to some further conditions), where F and G are smooth functions satisfying suitable growth conditions. Problems of this type give rise to elliptic equations in Ω subject to Neumann conditions on $\{0,a\}\times\omega$, which have been extensively studied in modelling chemical processes in tubes; see e.g. [4], and the references cited therein.

We are interested in solutions which are monotone in x_1. It is well-known that the variational problem has at least one global minimizer with this property [16]. The proof of this result is based on a rearrangement argument, which we recall below:

Let u be a minimizer, and let u^* denote its non-increasing rearrangement in the x_1-direction; for the definition see [16]. Then $u^*\in W^{1,2}(\Omega)$, $\int G(u)=\int G(u^*)$, $\int F(u)=\int F(u^*)$, and $\int|\nabla u|^2\geq\int|\nabla u^*|^2$, by the very properties of the rearrangement. Hence u^* is a minimizer, too, and since $(u^*)_{x_1}\leq 0$, the assertion follows.

But this striking argument has also its disadvantages. We first observe that is not clear from the above proof wether there exist *other, nonmonotone* minimizers. Fortunately, this difficulty is of technical nature, and it can be overcome by a careful study of the equality case in the rearrangement inequality above (see [2], compare also Remarks 3.2 and 3.4 below). On the other hand, the above argument fails if

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one tries to show the monotonicity of *local* minimizers or stationary solutions of the variational problem.

In this article we deal with global and local minimizers as well as with stationary solutions for a problem of the above type (Section 3) with *one or more* integral constraints and where the solutions satisfy a general nonlinear boundary condition on $(0, a) \times \partial\omega$. We also investigate symmetry properties for some related problems with periodicity in the infinite cylinder $\mathbb{R} \times \omega$ (Section 2) and in an annulus (Section 4), and we study similar problems with Dirichlet boundary conditions on $\mathbb{R} \times \partial\omega$ (Section 5). In the case of the stationary solutions, (Lemmata 2.5 and 4.4), we use the classical Moving Plane Method which exploits the Maximum Principle and the reflection invariance of the equation (see e.g. [13, 3]). In this method, starting from some initial stage, parallel hyperplanes are moved up to a critical position while it is proved at each intermediate stage that the difference between the solution and its reflection about the hyperplane is positive on one side of the hyperplane. However, in order to make the method work, one first needs to establish the same property at the initial stage. Thus we have to impose an additional condition on the solution which takes the form of 'two-point inequalities' with respect to reflection about some unspecified hyperplane (see the conditions (2.21), respectively (4.8) below). These inequalities cannot be deduced from the PDE, and they also *rule out entire classes of solutions*, e.g. those periodic solutions with higher order periodicity. We also emphasize that our Lemmata 2.5 and 4.4 contrast with the classical symmetry results for problems with zero Dirichlet boundary conditions (see [13]) where the role of the above mentioned two-point inequalities is played by the *positivity* of the solution which is a natural physical assumption. On the other hand, our analysis shows how far Moving Plane arguments reach in problems with periodicity.

Two further ingredients are used in the proof of symmetry (respectively monotonicity) for the local and global minimizers. The first tool is based on properties of a transformation which is often called *two-point rearrangement* in the literature. Notice that this simple type of rearrangement has been proved particularly useful in showing integral inequalities related to Steiner and cap symmetrizations (see [1, 7]). But its flexibility, and in particular its continuity properties (see Lemma 2.2) make it as well to an appropriate tool in proving qualitative properties of solutions to variational problems. Notice that the author has used two-point rearrangements to show the symmetry of local minimizers in a nonlocal model for equilibrium figures of rotating liquids (see [5]).

The second tool is the Principle of Unique Continuation. We mention that the idea to combine reflection arguments with this principle has been successfully employed by O. Lopes. He proved the symmetry of global minimizers to some variational problems in the entire space (see [20]) and in domains with rotational symmetry (see [19], and Remark 4.7 below).

In case of the periodic problem and the problem in the annulus we show that local minimizers are symmetric provided that they satisfy a 'weak version' of the above mentioned 'two-point inequality' (Lemma 2.6, respectively Lemma 4.5). Furthermore, in case of the problem in the finite cylinder we show that any local minimizer is monotone (Theorem 3.3). This result is however restricted to the case of only one constraint.

Finally we show that global minimizers of our problems are symmetric (respectively monotone), without imposing further geometrical conditions on the solution. (Theorems 2.7, 3.1 and 4.6).

2. Symmetry in periodic problems

Let us fix some notation. Let $N \in \mathbb{N}$, $(N \geq 2)$, and let ω a bounded smooth domain in $\mathbb{R}^{N-1}$. For some $a > 0$, let $\Omega = \mathbb{R} \times \omega$, $\Omega_a = (-a, a) \times \omega$ and $\Omega_a^+ = (0, a) \times \omega$. For points $x \in \Omega$ we write $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' \in \omega$. Furthermore, let $F = F(x,t), G_i = G_i(t), H = H(t)$ functions defined on $\Omega \times \mathbb{R}$, and satisfying

- F is twice differentiable with respect to t.
- $F(\cdot, t)$, $F_t(\cdot, t)$, $F_{tt}(\cdot, t)$ are in $L^\infty(\Omega)$ for all $t \in \mathbb{R}$, G_i and H are in $C^2(\mathbb{R})$
- $|F_t(x,t)|$, $|G_i'(t)| \leq c(1 + |t|^p)$, and $|H'(t)| \leq c(1 + |t|^q)$
- $f := F_t$, $g_i := G_i'$, $h := H'$, $(i = 1, \dots, n)$
- $f(x,t) = f(x_1 \pm ka, x', t)$ for all $(x,t) \in \Omega \times \mathbb{R}$

Furthermore, we assume:

$$1 \leq p < (N+2)/(N-2) \quad \text{and} \quad 1 \leq q < N/(N-2), \quad \text{if } N \geq 3; \tag{2.1}$$

and p, q are finite if $N = 2$;

$$\begin{gathered} f(x,t) \text{ is non-increasing in } x_1 \text{ for all } (x,t) \in (0,a) \times \omega \times \mathbb{R}, \\ \text{and for all } k \in \mathbb{N}. \end{gathered} \tag{2.2}$$

$$\begin{gathered} \text{If, for some constants } \alpha_1, \dots, \alpha_n, \text{ the function } \textstyle\sum_{i=1}^n \alpha_i g_i(t) \text{ vanishes} \\ \text{on some interval } c < t < d, \text{ then it vanishes everywhere on } \mathbb{R}. \end{gathered} \tag{2.3}$$

Now, let

$$X = \{v \in W^{1,2}(\Omega_a) \cap W^{1,2}_{\rm loc}(\Omega) : v \text{ is } (2a)\text{-periodic in } x_1\}$$
$$K = \{v \in X : \int_{\Omega_a} G_i(v)\, dx = 0,\ i = 1, \dots, n\}.$$

We investigate the variational problem:

$$\begin{gathered} \text{Minimize } J(v) \equiv \int_{\Omega_a} \Big(\frac{1}{2}|\nabla v|^2 - F(x,v)\Big)\, dx + \int_{(-a,a)\times\partial\omega} H(v)\, dS \\ \text{over all } v \in K\,. \end{gathered} \tag{2.4}$$

We call $u \in K$ a *global minimizer* of (2.4) if $J(v) \geq J(u)$ $\forall v \in K$, and we will say that $u \in K$ is a *local minimizer* of (2.4) if there exists a number $\varepsilon > 0$ such that $J(v) \geq J(u)$ for all $v \in K$ satisfying

$$\int_{\Omega_a} \Big(|\nabla(u-v)|^2 + (u-v)^2\Big)\, dx \leq \varepsilon.$$

In view of the non-degeneracy condition (2.3), standard arguments in the calculus of variations show (see e.g. [22]) that any local minimizer satisfies

$$-\Delta u = f(x,u) + \sum_{i=1}^n \alpha_i g_i(u) \quad \text{in } \Omega, \tag{2.5}$$

$$\partial u/\partial\nu + h(u) = 0 \quad \text{on } \partial\Omega, \tag{2.6}$$

where ν s the exterior normal, and with $\alpha_i \in \mathbb{R}$, $i = 1, \dots, n$, as Lagrange multipliers. Furthermore, we have

$$\int_{\Omega_a} \Big(\nabla u \nabla \varphi - f(x,u)\varphi \Big)\, dx + \int_{(-a,a)\times\partial\omega} h(u)\varphi \, dS = 0, \tag{2.7}$$

$$\int_{\Omega_a} \Big(|\nabla\varphi|^2 - (f_u(x,u) + \sum_{i=1}^{n} \alpha_i g_i'(u))\varphi^2 \Big)\, dx + \int_{(-a,a)\times\partial\omega} h'(u)\varphi^2 \, dS \geq 0, \tag{2.8}$$

for every $\varphi \in X$ satisfying

$$\int_{\Omega_a} g_i(u)\varphi \, dx = 0. \tag{2.9}$$

Also, the regularity assumptions on F, G_i, H, $(i = 1, \dots, n)$, imply that

$$u \in X \cap W^{2,N}(\Omega_a) \cap C(\overline{\Omega}). \tag{2.10}$$

We will say that u is a *stationary solution of* (2.4) if u satisfies (2.5),(2.6) and (2.10).

Our first result concerns the unconstrained case, $K = X$. It seems to be well-known, but I could not find a reference.

Lemma 2.1. *Let u a local minimizer of* (2.4)*, where $K = X$, and F does not depend on x_1, that is $F = F(x', t)$. Then $u_{x_1}(x) \equiv 0$ in Ω_a.*

Proof. For $\varphi \in X$, $\varphi \not\equiv 0$, we define I by

$$I(\varphi) := \Big(\int_{\Omega_a} \big(|\nabla\varphi|^2 - f_u(x',u)\varphi^2 \big)\, dx + \int_{(-a,a)\times\partial\omega} h'(u)\varphi^2 \, dS \Big) \Big(\int_{\Omega_a} \varphi^2 \, dx \Big)^{-1}.$$

In view of (2.8), we have $I(\varphi) \geq 0$. Assume that $u_{x_1} \not\equiv 0$. Then $I(u_{x_1}) = 0$, and since $u_{x_1} \in X$, we have that

$$I(u_{x_1}) = 0 = \min\{ I(\varphi) : \varphi \in X, \ \varphi \not\equiv 0 \}.$$

Then it follows from a variant of Krein-Rutman's Theorem that u_{x_1} can have one sign only. For the convenience of the reader, we give a simple proof: Assume that $M := \{ x \in \Omega_a : u_{x_1}(x) > 0 \}$ has positive measure, and set $v := \max\{u_{x_1}; 0\}$. Since $I(v) = 0$ we have then that also $-\Delta v = f_u(x', u)v$, and since u_{x_1} and v coincide on the open set M, the Principle of Unique Continuation (see Appendix) tells us that $u_{x_1} \equiv v$, which means that $u_{x_1}(x) \geq 0$, and $u_{x_1}(x) \not\equiv 0$ in Ω. But this is impossible due to the periodicity of u. Analogously we can argue in the case that $\{ x \in \Omega_a : u_{x_1}(x) < 0 \}$ has positive measure. □

In view of the above proof, it seems tempting to use (2.8) for the purpose of symmetry proofs under the presence of constraints. Let u a local minimizer of (2.4), let $F = F(x', t)$, and let for $\varphi \in X$, $\varphi \not\equiv 0$,

$$I(\varphi) := \Big(\int_{\Omega_a} \big(|\nabla\varphi|^2 - z(x)\varphi^2 \big)\, dx + \int_{(-a,a)\times\partial\omega} h'(u)\varphi^2 \, dS \Big) \Big(\int_{\Omega_a} \varphi^2 \, dx \Big)^{-1},$$

where $z := f_u(x', u) + \sum_{i=1}^{N} \alpha_i g_i'(u)$. It is easy to see that

$$\int_{\Omega_a} g_i(u) u_{x_1} \, dx = 0.$$

Proceeding analogously as above, we have then that

$$I(u_{x_1}) = 0 = \min \big\{ I(\varphi) : \varphi \not\equiv 0, \ \int_{\Omega_a} g_i(u)\varphi \, dx = 0, \ i = 1, \dots, n \big\}.$$

Then a variant of Courant-Hilbert's Theorem tells us that u_{x_1} can have at most $n+1$ nodal domains on the torus $\{(x_1,x') : x_1 \in \mathbb{R} \bmod (2a),\ x' \in \omega\}$. However, it is difficult to relate this information to symmetry properties of u, even not for only one constraint, $n = 1$.

Given any $\lambda \in \mathbb{R}$ we now define the following two transformations:

Reflexion: Let $x^\lambda := (2\lambda - x_1, x')$ and $v^\lambda(x) := v(x^\lambda)$ $\forall x \in \Omega$. For convenience, we will sometimes also write $\sigma_\lambda v$ instead of v^λ.

Two-point rearrangement: Let

$$
\begin{gathered}
T^\lambda v(x) := \begin{cases} \max\{u(x); u(x^\lambda)\} & \text{if } x \in [\lambda, \lambda + a] \times \omega \\ \min\{u(x); u(x^\lambda)\} & \text{if } x \in [\lambda - a, \lambda] \times \omega \end{cases} \\
T^\lambda v(x_1 \pm 2ka, x') := T^\lambda v(x) \quad \forall x \in [\lambda - a, \lambda + a] \times \omega,\ \forall k \in \mathbb{N}.
\end{gathered}
\tag{2.11}
$$

The transformation T^λ given above is a 'periodic variant' of the two-point rearrangements defined in the Euclidean space or on the Euclidean sphere (see [1, 7]). Below we summarize some of its properties. The proofs are easy variants of those given in [1, 7], and therefore omitted.

Lemma 2.2. (1) *If $v \in X$, $\lambda \in \mathbb{R}$, then $T^\lambda v \in X$, and*

$$
\int_{\Omega_a} |\nabla v|^2\,dx = \int_{\Omega_a} |\nabla T^\lambda v|^2\,dx. \tag{2.12}
$$

(2) *If $v \in C(\overline{\Omega})$, $\lambda \in \mathbb{R}$, then $T^\lambda v \in C(\overline{\Omega})$, and if $\psi \in C(\mathbb{R})$, then*

$$
\int_{-a}^{a} \psi(v(y,x'))\,dy = \int_{-a}^{a} \psi(T^\lambda v(y,x'))\,dy \quad \forall x' \in \overline{\omega}. \tag{2.13}
$$

(3) *Continuity of $\lambda \to T^\lambda v$: If $v \in X$ and if $\lambda_k, \lambda \in \mathbb{R}$, $k = 1, 2, \dots$, $\lim_{k\to\infty} \lambda_k = \lambda$, then $T^{\lambda_k} v \to T^\lambda v$ in $W^{1,2}(\Omega_a)$.*

Note that (2.12) and (2.13) follow from the fact that T^λ rearranges the values of v and of $|\nabla v|$ in the two corresponding points x, x^λ for a.e. $x \in [\lambda, \lambda + a] \times \omega$, and 3) follows from the continuity of the Lebesgue integral with respect to translations.

In the symmetry proof for minimizers we will need a rearrangement inequality. Notice that some variant of it for the two-point rearrangement in $\mathbb{R}^N$ has been proved in [6].

Lemma 2.3. *Let $v \in X$ and $\lambda \in [-a, a]$. Then*

$$
\int_{\Omega_a} F(x,v)\,dx \le \int_{\Omega_a} F(x, T^\lambda v)\,dx \quad \text{if } \lambda \le 0, \tag{2.14}
$$

$$
\int_{\Omega_a} F(x,v)\,dx \le \int_{\Omega_a} F(x, \sigma_\lambda(T^\lambda v))\,dx \quad \text{if } \lambda \ge 0. \tag{2.15}
$$

Proof. Let $\lambda \in [-a, 0]$. Then, if $x \in [\lambda, \lambda + a] \times \omega$, we have that $|x_1| \le |2\lambda - x_1|$. In view of (2.1) it is then easy to see that

$$
F(x, v(x)) + F(x^\lambda, v(x^\lambda)) \le F(x, T^\lambda v(x)) + F(x^\lambda, T^\lambda v(x^\lambda)) \quad \forall x \in [\lambda, \lambda + a] \times \omega.
$$

Integrating this over $[\lambda, \lambda + a] \times \omega$ gives (2.14) in view of the periodicity of v and $T^\lambda v$. The proof of (2.15) is similar and will be omitted. $\square$

Remark 2.4. **1)** Assume that u is a stationary solution of (2.4), and suppose that for some $\lambda \in [-a, 0]$,

$$u(x) \geq u(x^\lambda) \quad \forall x \in [\lambda, \lambda + a] \times \omega. \tag{2.16}$$

Setting $v_\lambda = u - u^\lambda$, we obtain from assumptions (2.2) and (2.1) that

$$\begin{gathered} -\Delta v_\lambda \geq c_\lambda(x) v_\lambda \quad \text{and} \quad v_\lambda \geq 0 \quad \text{in } (\lambda, \lambda + a) \times \omega, \\ v_\lambda(x) = 0 \quad \text{on } \{\lambda, \lambda + a\} \times \omega, \\ \frac{\partial v_\lambda}{\partial \nu} + d_\lambda(x) v_\lambda = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.17}$$

where

$$c_\lambda = \begin{cases} \big(f(x,u) - f(x,u^\lambda) + \sum_{i=1}^n \alpha_i (g_i(u) - g_i(u^\lambda))\big)/v_\lambda & \text{if } v_\lambda \neq 0 \\ 0 & \text{if } v_\lambda = 0 \end{cases} \tag{2.18}$$

and

$$d_\lambda = \begin{cases} \big(h(u) - h(u^\lambda)\big)/v_\lambda & \text{if } v_\lambda \neq 0 \\ 0 & \text{if } v_\lambda = 0 \end{cases} \tag{2.19}$$

are bounded functions. Now the Strong Maximum Principle (see Appendix) implies that either $v_\lambda \equiv 0$ in $(\lambda, \lambda + a) \times \omega$ or $v_\lambda(x) > 0$ in $(\lambda, \lambda + a) \times \overline{\omega}$. The latter also implies $(v_\lambda)_{x_1}(\lambda, x') > 0$ for all $x' \in \omega$. Thus the assumption (2.16) implies the following alternative: Either

$$u(x) = u(x^\lambda) \quad \forall x \in [\lambda, \lambda + a] \times \overline{\omega}, \quad \text{or} \tag{2.20}$$

$$u(x) > u(x^\lambda) \quad \forall x \in (\lambda, \lambda + a) \times \overline{\omega} \quad \text{and} \quad u_{x_1}(\lambda, x') > 0 \quad \forall x' \in \omega. \tag{2.21}$$

2) Next assume that F is independent of x_1, that is $F = F(x', t)$. Then we have $-\Delta v_\lambda = c_\lambda(x) v_\lambda$ in $(\lambda, \lambda + a) \times \omega$. Therefore, it is easy to see that the conclusions of 1) hold for all $\lambda \in \mathbb{R}$. Moreover, if $u_{x_1} \not\equiv 0$, if $u = u^{\lambda_0}$, and if $u_{x_1}(x) \leq 0$ in $(\lambda_0, \lambda_0 + a) \times \omega$, for some $\lambda_0 \in \mathbb{R}$, then (2.16) holds for every $\lambda \in (\lambda_0 - a, \lambda_0)$. By (2.20), (2.21) this implies that $u_{x_1}(x) < 0$ in $(\lambda_0, \lambda_0 + a) \times \overline{\omega}$.
3) Assume that

$$f(0, x'_0, t) > f(a, x'_0, t) \quad \text{for some } x'_0 \in \omega \text{ and } \ \forall t \in \mathbb{R}. \tag{2.22}$$

Then, if $u_{x_1} \not\equiv 0$, and if $u = u^{\lambda_0}$ for some $\lambda_0 \in [-a, 0]$, the elliptic equation for u shows that either $\lambda_0 = 0$ or $\lambda_0 = -a$.

Using the classical Moving Plane Method, we now prove the following symmetry result for stationary solutions:

Lemma 2.5. *Let u be a stationary solution of* (2.4), *and assume that there exists a number $\lambda = \lambda_0 \in [-a, 0]$ for which (2.21) holds. Then there exists a number $\lambda^* \in [-a, a]$ such that*

$$u(x) = u(x^{\lambda^*}) \quad \text{and} \quad u_{x_1}(x) < 0 \quad \forall x \in (\lambda^*, \lambda^* + a) \times \overline{\omega}. \tag{2.23}$$

Furthermore, if f satisfies (2.22), then $\lambda^* = 0$.

Proof. First assume that (2.22) holds, and that $\lambda_0 \in (-a, 0)$. Setting

$$M := \{\lambda \in (-a, 0) : u^\lambda(x) < u(x) \ \forall x \in (\lambda, \lambda + a) \times \omega\}, \tag{2.24}$$

we claim that M is open. To prove this, we will need some preliminary settings. We choose $C > 0$ such that

$$C \geq |f_t(x,t) + \sum_{i=1}^{n} \alpha_i g_i'(t)| \quad \text{and} \quad C \geq |h'(t)| \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{2.25}$$

Given any $\varepsilon > 0$, let

$$\omega_\varepsilon := \{x' \in \omega : |x' - y'| > \varepsilon \ \forall y' \in \partial\omega\}, \tag{2.26}$$

and let $A_\varepsilon = A_\varepsilon(x')$, $B_\varepsilon = B_\varepsilon(x')$ defined by:

$$\begin{gathered}\Delta' A_\varepsilon \equiv \sum_{i=2}^{N} (A_\varepsilon)_{x_i x_i} = \Delta' B_\varepsilon = 0 \quad \text{in } \omega \setminus \overline{\omega_\varepsilon}, \\ A_\varepsilon = B_\varepsilon = 1 \quad \text{on } \partial\omega, \\ A_\varepsilon = 2, \quad B_\varepsilon = 1/2 \quad \text{on } \partial\omega_\varepsilon.\end{gathered}$$

Then $A_\varepsilon \in (1,2)$, $B_\varepsilon \in ((1/2),1)$ in $\omega \setminus \overline{\omega_\varepsilon}$, and since $\partial\omega$ is smooth, elliptic estimates show that

$$-\lim_{\varepsilon\to 0} \frac{\partial A_\varepsilon}{\partial\nu}(x') = \lim_{\varepsilon\to 0} \frac{\partial B_\varepsilon}{\partial\nu}(x') = +\infty$$

uniformly for all $x' \in \partial\omega$, (ν: exterior normal to $\omega \setminus \overline{\omega_\varepsilon}$). We fix $\varepsilon > 0$ small enough such that

$$-\frac{\partial A_\varepsilon}{\partial\nu} \geq C \quad \text{and} \quad \frac{\partial B_\varepsilon}{\partial\nu} \geq C \quad \text{on } \partial\omega, \tag{2.27}$$

and such that

$$\cos\sqrt{C}t \geq 1/2 \quad \text{for } 0 \leq t \leq \varepsilon. \tag{2.28}$$

Now let $\lambda \in M$. Setting $v_\lambda := u - u^\lambda$, we have that $v_\lambda > 0$ in $(\lambda, \lambda + a) \times \overline{\omega}$ and then $(v_\lambda)_{x_1}(\lambda, x') > 0 > (v_\lambda)_{x_1}(\lambda + a, x')$ for all $x' \in \omega$, by Remark 2.4. We claim that this implies

$$(v_\lambda)_{x_1}(\lambda, x') > 0 > (v_\lambda)_{x_1}(\lambda + a, x') \quad \forall x' \in \overline{\omega}. \tag{2.29}$$

Let $U_\varepsilon := (\lambda, \lambda+\varepsilon) \times (\omega \setminus \overline{\omega_\varepsilon})$. If $\kappa > 0$ is sufficiently small, we have that

$$v_\lambda(x) \geq \kappa A_\varepsilon(x')(e^{\sqrt{C}x_1} - 1) \quad \text{on } \partial U_\varepsilon \setminus \partial\Omega. \tag{2.30}$$

Setting $w(x) := v_\lambda(x) - \kappa A_\varepsilon(x')(e^{\sqrt{C}x_1} - 1)$, and using (2.18), (2.19), (2.25), (2.27) and (2.30) we find that

$$\begin{gathered}-\Delta w \geq c_\lambda w + \kappa A_\varepsilon(x') e^{\sqrt{C}x_1}\Big(C + c_\lambda(1 - e^{-\sqrt{C}x_1})\Big) \geq c_\lambda w \quad \text{in } U_\varepsilon, \\ w \geq 0 \quad \text{on } \partial U_\varepsilon \setminus \partial\Omega, \\ \frac{\partial w}{\partial\nu} + d_\lambda(x) w = -\kappa(e^{\sqrt{C}x_1} - 1)\big((A_\varepsilon)_\nu + d_\lambda\big) \geq 0 \quad \text{on } \partial U_\varepsilon \cap \partial\Omega.\end{gathered} \tag{2.31}$$

Then we define

$$V(x) := \frac{w(x)}{B_\varepsilon(x')\cos\sqrt{C}(x_1 - \lambda)}, \quad (x \in U_\varepsilon). \tag{2.32}$$

Note that $B_\varepsilon(x')\cos\sqrt{C}(x_1-\lambda)\in[(1/4),1]$ in U_ε, in view of (2.28). Using (2.31) and (2.32) we obtain

$$-\Delta V+2\sqrt{C}V_{x_1}\tan\sqrt{C}(x_1-\lambda)-2\sum_{i=2}^{N}V_{x_i}\frac{(B_\varepsilon)_{x_i}}{B_\varepsilon}\geq(c_\lambda-C)V\quad\text{in }\;U_\varepsilon,$$
$$V\geq 0\quad\text{on}\partial U_\varepsilon\setminus\partial\Omega,$$
$$\frac{\partial V}{\partial\nu}+\big(d_\lambda+\frac{\partial B_\varepsilon}{\partial\nu}\big)V\geq 0\quad\text{on }\partial U_\varepsilon\cap\partial\Omega.$$

From (2.18) and (2.25) we have that $c_\lambda-C\leq 0$ in U_ε, and by (2.19) and (2.27) we have that $d_\lambda+(B_\varepsilon)_\nu\geq 0$ on $(\lambda,\lambda+\varepsilon)\times\partial\omega$. The Weak Maximum Principle (see Appendix) tells us that $V\geq 0$ in U_ε. This proves the first inequality in (2.29). Similarly one shows the second inequality in (2.29). Having proved (2.29), we find by continuity some $\delta>0$ such that $v_\mu>0$ in $(\mu,\mu+a)\times\omega$ $\forall\mu\in(\lambda-\delta,\lambda+\delta)$. This shows that M is open.

Let M_1 be the connected component of M containing λ_0, and let $\lambda^*:=\sup\{\lambda:\lambda\in M_1\}$. By continuity, we have $u^{\lambda^*}(x)\leq u(x)$ in $(\lambda^*,\lambda^*+a)\times\omega$. Assume that $\lambda^*<0$. The Strong Maximum Principle tells us that either $u^{\lambda^*}(x)<u(x)$ or $u^{\lambda^*}(x)=u(x)$ throughout $(\lambda^*,\lambda^*+a)\times\omega$. The first of these two possibilities is excluded due to the maximality of λ^* and since M_1 is open, and the second possibility is impossible in view of Remark 2.4, part 3. Hence $\lambda^*=0$. Similarly we show that $\inf\{\lambda:\lambda\in M_1\}=-a$. It follows that $u(x)>u^\lambda(x)$, and then $u_{x_1}(\lambda,x')>0$ $\forall x\in(\lambda,\lambda+a)\times\overline{\omega}$ and $\forall\lambda\in(-a,0)$. This proves (2.23) in the case under consideration.

Next assume that (2.22) holds and that $\lambda_0=0$, or $\lambda_0=-a$, respectively. Then we can prove similarly as above that $u^\lambda(x)<u(x)$ in $(\lambda,\lambda+a)\times\omega$ for $\lambda\in(-\delta,0)$, (respectively $\lambda\in(-a,-a+\delta)$), provided that $\delta>0$ is small enough. Hence $M\neq\emptyset$, and we may obtain (2.23) as in the previous case.

Assume finally that (2.22) is not satisfied. Then F is independent of x_1, that is $F=F(x',t)$. Setting now $M:=\{\lambda\in\mathbb{R}:u^\lambda(x)<u(x)\ \forall x\in(\lambda,\lambda+a)\}$, we first show as in the above cases that M is open. Letting M_1 the connected component of M containing λ_0, we notice that M_1 is bounded, since, if $\lambda\in M$, then $\lambda+a\notin M$, by periodicity. As before, this means that $u=u^{\lambda^*}$, where $\lambda^*=\sup\{\lambda:\lambda\in M_1\}$. Hence we have $u(2\lambda^*-2\lambda+x_1,x')=u(2\lambda-x_1,x')<u(x)\ \ \forall x\in(\lambda,\lambda+a)\times\omega$ and $\forall\lambda\in(\lambda_0,\lambda^*)$, which means that $u_{x_1}(x)\leq 0$ in $[\lambda^*,\lambda^*+a]\times\omega$. As we already know, the latter also implies $u_{x_1}(x)<0$ in $(\lambda^*,\lambda^*+a)\times\overline{\omega}$. This completes the proof. $\square$

Lemma 2.6. *Let u be a local minimizer of* (2.4)*, and let $u_{x_1}\not\equiv 0$. Furthermore, assume that there exists a number $\lambda=\lambda_0\in[-a,0]$ such that (2.16) holds. Then (2.23) holds with $\lambda^*=\lambda_0$ or with $\lambda^*=\lambda_0+a$. Moreover, if f satisfies (2.22), then $\lambda^*=\lambda_0=0$.*

Proof. In view of Remark 2.4, either $u^{\lambda_0}=u$ or (2.21) holds with $\lambda=\lambda_0$. In the second case the assertions follow by Lemma 2.5. Thus it remains to consider the case that $u=u^{\lambda_0}$.

Since $J(T^\lambda u)\leq J(u)$ for $\lambda\in(-a,0)$, by Lemma 2.3, and since $T^\lambda\to u$ in $W^{1,2}(\Omega_a)$, as $\lambda\to\lambda_0$, this implies that $T^\lambda u$ is a local minimizer for $-\delta<\lambda<0$, provided that $\delta>0$ is small enough. Hence $T^\lambda u$ satisfies equation (2.5) for any of these λ, with the α_i's replaced by (possibly different) Lagrangean multipliers

α_i', $(i = 1, \dots, n)$. Now let $\lambda \in (-\delta, 0)$, and assume that $u \not\equiv T^\lambda u$. Then $U := \{x \in (\lambda, \lambda + a) \times \omega : u(x) < u(x^\lambda)\}$ is open and nonempty. Assume then that $f(x, u^\lambda(x)) \not\equiv f(x^\lambda, u^\lambda(x))$ in U. Since $f(x, t) \ge f(x^\lambda, t)$ on $(\lambda, \lambda + a) \times \omega \times \mathbb{R}$, the set $U_1 := \{x \in U : f(x, u^\lambda(x)) > f(x^\lambda, u^\lambda(x))\}$ is open and nonempty. Hence,

$$\begin{aligned} F(x, u(x)) + F(x^\lambda, u(x^\lambda)) &< F(x, u^\lambda(x)) + F(x^\lambda, u^\lambda(x^\lambda)) \\ &= F(x, T^\lambda u(x)) + F(x^\lambda, T^\lambda u(x^\lambda)) \quad \forall x \in U_1, \end{aligned}$$

by (2.1). Integrating over $(\lambda, \lambda + a) \times \omega$ shows that

$$\int_{\Omega_a} F(x, u)\, dx < \int_{\Omega_a} F(x, T^\lambda u)\, dx. \tag{2.33}$$

But this implies that $J(T^\lambda u) < J(u)$, contradicting the minimality of u. Hence

$$f(x, u^\lambda(x)) = f(x^\lambda, u^\lambda(x)) \quad \forall x \in U.$$

Since $u^\lambda = T^\lambda u$ in U, this implies

$$\sum_{i=1}^{n} (\alpha_i' - \alpha_i) g_i(T^\lambda u(x)) = 0 \quad \forall x \in U.$$

Since $u = T^\lambda u$ in $\Big((\lambda, \lambda + a) \times \omega\Big) \setminus U$, we furthermore obtain from this that

$$\sum_{i=1}^{n} (\alpha_i' - \alpha_i) g_i(T^\lambda u(x)) = 0 \quad \forall x \in (\lambda, \lambda + a) \times \omega.$$

In view of the non-degeneracy condition (2.3), this implies that $\alpha_i' = \alpha_i$, $(i = 1, \dots, n)$. Setting $w := u^\lambda - T^\lambda u$, we have then that $-\Delta w = c(x) w$ in Ω, where c is a bounded function. Since $w \equiv 0$ on U, the Principle of Unique Continuation tells us that we must have $u^\lambda = T^\lambda u$ throughout Ω. In other words, we have shown that, if $\lambda \in (-\delta, 0)$, then $u = T^\lambda u$ or $u^\lambda = T^\lambda u$.

Now assume that there exists a strictly increasing sequence $\lambda_k \to \lambda_0$ such that $u = T^{\lambda_k} u$ $\forall k \in \mathbb{N}$. Then we have that $u(2\lambda_0 - 2\lambda_k + x_1, x') = u(x^{\lambda_k}) \le u(x)$ $\forall x \in [\lambda_k, \lambda_k + a] \times \omega$ and $\forall k \in \mathbb{N}$. Hence, $u_{x_1}(x) \le 0$ $\forall x \in [\lambda_0, \lambda_0 + a] \times \omega$.
If we furthermore assume, that F is independent of x_1, then Remark 2.4 tells us that we must have $u(x) > u^\lambda(x)$ in $(\lambda, \lambda + a) \times \overline{\omega}$ for $\lambda \in (\lambda_0 - a, \lambda_0)$, in view of $u_{x_1} \not\equiv 0$. This also means $u_{x_1}(x) < 0$ in $(\lambda_0, \lambda_0 + a) \times \overline{\omega}$.

If, on the other hand, f satisfies (2.22), then we obtain from the elliptic equation for u and from the assumption $u_{x_1} \not\equiv 0$ that we must have $\lambda_0 = 0$, which also means $u_{x_1}(x) < 0$ in $(0, a) \times \overline{\omega}$. Next assume that there exists a strictly increasing sequence $\lambda_k \to \lambda_0$ such that $u^{\lambda_k} = T^{\lambda_k} u$ $\forall k \in \mathbb{N}$. Arguing analogously as above we then find that $u_{x_1}(x) > 0$ in $(\lambda_0, \lambda_0 + a) \times \overline{\omega}$, and $\lambda_0 = 0$ if f satisfies (2.22). $\square$

Theorem 2.7. *Let u a global minimizer of* (2.4)*, and let $u_{x_1} \not\equiv 0$. Then there exists a number $\lambda^* \in [-a, a]$ such that (2.23) holds. Moreover, if f satisfies (2.22) then $\lambda^* = 0$.*

Proof. If $v \in X$, then Lemma 2.3 shows that $J(v) \ge J(T^\lambda v)$ $\forall \lambda \in [-a, 0]$. Proceeding as in the previous proof we find then that $u = T^\lambda u$ or $u^\lambda = T^\lambda u$, and $T^\lambda u$ is a global minimizer of (2.4) for any of these λ. The periodicity of u shows that, if $\lambda \in \mathbb{R}$, then $u = T^\lambda u$ if and only if $u^{\lambda + a} = T^{\lambda + a} u$. By the continuity of the mapping $\lambda \to T^\lambda u$ in $W^{1,2}(\Omega_a)$, we then find a value $\lambda_0 \in [-a, 0]$ such that $u = u^{\lambda_0}$, and the assertions follow as in the previous proof. $\square$

3. Monotonicity in Neumann problems

Let $\Omega_a^+ = (0,a) \times \omega$ and

$$K_+ = \big\{ v \in W^{1,2}(\Omega_a^+) : \int_{\Omega_a^+} G_i(v)\,dx = 0,\ i = 1,\dots,n \big\}.$$

then we study the problem:

$$\begin{gathered} \text{Minimize } J_+(v) \equiv \int_{\Omega_a^+} \Big(\frac{1}{2}|\nabla v|^2 - F(x,v) \Big)\,dx + \int_{(0,a)\times\partial\omega} H(v)\,dS \\ \text{over all } v \in K_+\,. \end{gathered} \tag{3.1}$$

Defining global and local minimizers of (3.1) analogously as those of (2.4), we obtain that any local minimizer u of (3.1) satisfies

$$u \in W^{2,N}(\Omega_a^+) \cap C(\overline{\Omega_a^+}), \tag{3.2}$$

$$-\Delta u = f(x,u) + \sum_{i=1}^n \alpha_i g_i(u) \quad \text{in } \Omega_a^+, \tag{3.3}$$

$$\partial u/\partial\nu + h(u) = 0 \quad \text{on } (0,a)\times\partial\omega, \tag{3.4}$$

$$u_{x_1}(0,x') = u_{x_1}(a,x') = 0 \quad \forall x' \in \omega, \tag{3.5}$$

with $\alpha_i \in \mathbb{R}$, $i = 1,\dots,n$, as Lagrange multipliers. More generally, we call a function $u \in W^{1,2}(\Omega_a^+)$ satisfying (3.2)–(3.5) a stationary solution of (3.1).

Extending a function $v \in K_+$ by reflection and periodic continuation to Ω,

$$v(x_1 \pm 2ka, x') = v(-x_1 \pm 2ka, x') = v(x_1,x') \quad \forall x \in \Omega_a^+, \tag{3.6}$$

we see that $v \in K$. Then it is easy to see that (3.1) can be equivalently formulated as:

$$\text{minimize } J(v) \text{ over all } v \text{ in } K_e, \tag{3.7}$$

where $K_e := \{v \in K : v(-x_1,x') = v(x)\ \forall x \in \Omega\}$. Therefore, any local minimizer u of (3.7) satisfies (2.5)–(2.10), where now the functions φ in (2.7)–(2.9) are in K_e. Note that, using Lemma 2.1, we recover from the above equivalence the well-known result that in case of the unconstrained problem, ($n = 0$), any local minimizer u of (3.1) satisfies $u_{x_1}(x) \equiv 0$ in Ω_a^+ (see [8]).

On the other hand, under the presence of constraints, $n \geq 1$, it seems difficult to use (2.7), (2.8) for symmetric proofs of u, since $u_{x_1} \notin K_e$! Nevertheless, there holds the following

Theorem 3.1. *Let u be a global minimizer for* (3.1) *and $u_{x_1}(x) \not\equiv 0$ in Ω_a^+. Then either $u_{x_1}(x) > 0$ or $u_{x_1}(x) < 0$ for all $x \in (0,a)\times\overline{\omega}$. Furthermore, if f satisfies (2.22) then $u_{x_1}(x) < 0$ in Ω_a^+.*

Proof. Since $K_e \subset K$, $\min\{J(v) : v \in K\} \leq \min\{J(v) : v \in K_e\}$. On the other hand, if $\widetilde{u}$ is a minimizer of (2.4), then $\widetilde{u}^{\lambda^*} = \widetilde{u}$ for some $\lambda^* \in [-a,0]$. Hence $\widetilde{u}^{(\lambda^*/2)} \in K_e$, $J(\widetilde{u}^{(\lambda^*/2)}) \leq J(u)$, which means that $\min\{J(v) : v \in K\} = \min\{J(v) : v \in K_e\}$. Hence any global minimizer of (3.1) is a global minimizer of (2.4). In view of Theorem 2.7, this proves the assertion. □

Remark 3.2. An alternative proof of Theorem 3.1 can be performed by using the results of [2] mentioned in the introduction: Let u^* denote the non-increasing rearrangement of u in the variable x_1. By the very properties of the rearrangement we have then $u^* \in K_+$, $\int_{\Omega_a^+} |\nabla u^*|^2\,dx \leq \int_{\Omega_a^+} |\nabla u|^2\,dx$, and $\int_{\Omega_a^+} F(x,u^*)\,dx \geq$

$\int_{\Omega_a^+} F(x,u)\,dx$. (The latter inequality is well-known, too. It also follows easily from an analogous inequality for Steiner symmetrization given in [6].) Hence u^* is a minimizer, too. But this means that we must have equality in both inequalities above. By a result, given in [2], it follows that, if $x' \in \omega$, then either $u_{x_1}(x_1, x') \ge 0$ for all $x_1 \in (0,a)$, or $u_{x_1}(x_1,x') \le 0$ for all $x_1 \in (0,a)$. It is easy to see that this implies $u^* = T^{-a/2}u$. We may now argue analogously as in the proof of Theorem 2.7: If $F = F(x',t)$, then this implies that $u = T^{-a/2}u$ or $u^{-a/2} = T^{-a/2}u$, which means that either $u = u^*$ or $-u = (-u)^*$ in this case. If f satisfies (2.22) then $\int F(x, T^{-a/2}u) \ge \int F(x,u)$, with equality sign if and only if $u = T^{-a/2}u$. This implies $u = u^*$.

Theorem 3.3. *Let F be independent of x_1, that is $F = F(x',t)$. Let u be a local minimizer for* (3.1)*, where K_+ has only one constraint, ($n = 1$), and let $u_{x_1} \not\equiv 0$ in Ω_a^+. Then either $u_{x_1}(x) > 0$ or $u_{x_1}(x) < 0$ for all $x \in (0,a) \times \overline{\omega}$.*

Proof. We first claim that there exists a $\delta \in (0,a)$ such that the following alternative takes place:

$$\begin{aligned} &\text{Either} \quad \int_{\Omega_a^+} G_1(u^y)\,dx > 0 \quad \forall y \in (0,\delta], \\ &\text{or} \quad \int_{\Omega_a^+} G_1(u^y)\,dx < 0 \quad \forall y \in (0,\delta]. \end{aligned} \tag{3.8}$$

Suppose (3.8) is not true. Then there exist strictly decreasing sequences $y_k' \to 0$, $y_k'' \to 0$, such that

$$\int_{\Omega_a^+} G_1(u^{y_k'})\,dx \ge 0 \ge \int_{\Omega_a^+} G_1(u^{y_k''})\,dx.$$

By continuity and since u is even in x_1, we find then another strictly decreasing sequence $y_k \to 0$, such that

$$\int_{\Omega_a^+} G_1(u^{y_k})\,dx = 0 = \int_{\Omega_a^+} G_1(u^{-y_k})\,dx.$$

By Lemma 2.2, and since $u \in K_e$, we have that

$$2J_+(u) = J(u) = J(u^{-y_k}) = J_+(u^{-y_k}) + J_+(u^{y_k}).$$

Hence both u^{y_k} and u^{-y_k} are local minimizers, provided that k is large enough - say $k \ge k_0$. This means that $u_{x_1}(\pm y_k, x') = 0$ $\forall x' \in \omega$, for these k. We claim that this actually implies

$$u_{x_1}(x) = 0 \quad \text{in } (0, y_{k_1}) \times \omega, \tag{3.9}$$

for some (large) number $k_1 \in \mathbb{N}$.

First note that $u_{x_1} \in W^{2,N}(\Omega_a^+)$ and

$$-\Delta u_{x_1} = \Big(f_u(x',u) + \alpha_1 g_1'(u)\Big) u_{x_1} \quad \text{in } \Omega_a^+. \tag{3.10}$$

Let $C > 0$, $\varepsilon > 0$ and B_ε be chosen in such a way that (2.25)–(2.27) are satisfied. Choose then $k_1 \in \mathbb{N}$ such that $\cos\sqrt{C}x_1 \ge \frac{1}{2}$ for all $x_1 \in [0, y_{k_1}]$. Setting

$$V(x) := \frac{u_{x_1}(x)}{B_\varepsilon(x')\cos\sqrt{C}x_1}, \quad (x \in (0, y_{k_1}) \times \omega),$$

we find that

$$
\begin{gathered}
-\Delta V+2\sqrt{C}V_{x_1}\tan\sqrt{C}x_1-2\sum_{i=2}^{N}V_{x_i}\frac{(B_\varepsilon)_{x_i}}{h_\varepsilon}\\
=\big(f_u(x',u)+\alpha_1 g_1'(u)-C\big)V \quad \text{in } (0,y_{k_1})\times\omega,\\
V(0,x')=V(y_{k_1},x')=0 \quad \forall x'\in\omega,\\
\frac{\partial V}{\partial\nu}+\Big(h'(u)+\frac{\partial B_\varepsilon}{\partial\nu}\Big)V=0 \quad \text{on } (0,y_{k_1})\times\partial\omega.
\end{gathered}
$$

From (2.25) we obtain $f'(u)+\alpha_1 g_1'(u)-C\le 0$ in $(0,y_{k_1})\times\omega$, and $h'(u)+(B_\varepsilon)_\nu\ge 0$ on $(0,y_{k_1})\times\partial\omega$. The Weak Maximum Principle then gives (3.9). In view of (3.10), the Principle of Unique Continuation then shows that $u_{x_1}\equiv 0$ in Ω_a^+, a contradiction. Having proved (3.8), we will assume from now on that

$$
\int_{\Omega_a^+} G_1(u^y)\,dx>0 \quad \forall y\in(0,\delta].
$$

(The proof for the other case, $\int_{\Omega_a^+} G_1(u^y)\,dx<0$ $\forall y\in(0,\delta]$, is similar and will be omitted.) We have $\int_{\Omega_a^+} G_1(u^{-y})\,dx=-\int_{\Omega_a^+} G_1(u^y)\,dx<0$, and since $T^\lambda u\to u$ in $W^{1,2}(\Omega)$, as $\lambda\to 0$, we find numbers $\Lambda(y)>0$ such that

$$
\int_{\Omega_a^+} G_1(\sigma_y(T^\lambda u))\,dx>0>\int_{\Omega_a^+} G_1(\sigma_{-y}(T^\lambda u))\,dx
$$

for all $\lambda\in[0,\Lambda(y)]$, and for all $y\in(0,\delta]$. But then we find decreasing sequences $\lambda_k\to 0$, $y_k\to 0$, such that $\lambda_k\ne 0$ and

$$
\int_{\Omega_a^+} G_1(\sigma_{y_k}(T^{\lambda_k}u)\,dx=0 \quad \forall k\in\mathbb{N}. \tag{3.11}
$$

Hence, for large enough k - say $k\ge k_2$ - $\sigma_{y_k}(T^{\lambda_k}u)$ is a local minimizer, too. Now we may argue as in the proof of Lemma 2.6: First one shows that both $\sigma_{y_k}(T^{\lambda_k}u)$ and $\sigma_{y_k}u(\equiv u^{y_k})$ satisfy the same elliptic equation as u. Using the Principle of Unique Continuation, this means that for any $k\ge k_2$, either $u=T^{\lambda_k}u$ or $u^{\lambda_k}=T^{\lambda_k}u$. Since $u=u^0$, this finally implies that either $u_{x_1}(x)>0$ or $u_{x_1}(x)<0$ for all $x\in(0,a)\times\omega$. □

Remark 3.4. 1) Using the equivalence of problem (3.1) with (3.7), Lemma 2.5 immediately yields the following result: Let u be a stationary solution of (3.1), and let u be extended to Ω by (3.6). Furthermore, assume that there exists a number $\lambda=\lambda_1\in(-a,0)$ such that (2.21) is satisfied. Then the conclusions of Theorem 3.1 hold.

2) Lachand-Robert [17, 18] studied (3.1) under the Dirichlet boundary conditions $u=0$ on $(0,a)\times\partial\omega$ for one constraint $G_1(v)=|v|^{p+1}$, and for $F(v)\equiv 0$. He showed among other things that local minimizers u are monotone in x_1. He also proved that $u_{x_1}\equiv 0$ if Ω_a^+ has small width, that is if a is small, and that $u_{x_1}\not\equiv 0$ if a is large. However, the monotonicity proof in [18] essentially depends on the special form of the nonlinearity G_1.

Also results analogous to [17, 18] have been obtained in [2] for certain critical points of functionals J_+ satisfying the conditions of the Mountain Pass Theorem.

Kawohl [16] proved that the second eigenfunction of the Laplacian subject to Neumann boundary conditions in Ω_a^+ is monotone in x_1. This corresponds to problem (3.1) with $F \equiv 0$, $H \equiv 0$, $n = 2$, $G_1(t) = t$, and $G_2(t) = t^2 - 1$.
3) The author was not able to generalize Theorem 3.3 for potentials F that depend on x_1, or to cases with more than one constraint.
4) The problems (2.4) and (3.1) may also be formulated in a more general setting: Let the gradient term $|\nabla v|^2$ in the functionals J and J_+ be replaced by

$$\sum_{i,j=2}^{N} a_{ij}(x')v_{x_i}v_{x_j} + a_{11}(x')v_{x_1}^2,$$

where $a_{11}, a_{ij} \equiv a_{ji} \in C^1(\overline{\omega})$, $(i, j = 2, \dots, N)$, $\sum_{i,j=2}^{N} a_{ij}(x')\xi_i\xi_j + a_{11}\xi_1^2 \ge c_0|\xi|^2$, $(c_0 > 0)$, and let the nonlinearities G_i, H, $(i = 1, \dots, n)$, depend additionally on x', and satisfying suitable smoothness and growth conditions. The changes in the formulation of the problems (2.4) and (3.1) are then obvious, and the results of Section 2 and 3 remain valid. We leave the details to the reader.
5) We finally mention that elliptic problems in Ω_a^+ have been extensively investigated under Dirichlet conditions on $\{0, a\} \times \omega$, where the monotonicity of the solutions often follows from moving plane and sliding arguments (see [3, 4] and the references cited therein). A typical situation is the following one: Let $g \in C(\overline{\Omega_a^+})$ and strictly decreasing in the variable x_1, and let $u \in W^{2,N}(\Omega_a^+) \cap C(\overline{\Omega_a^+})$ satisfy $-\Delta u = f(u)$ and $g(0, x') > u(x) > g(a, x')$ in Ω_a^+, and $u = g$ on $\partial\Omega_a^+$, where f is smooth. Then $u_{x_1} < 0$ in Ω_a^+. The same result holds if one replaces the Dirichlet conditions on $(0, a) \times \partial\omega$ by Neumann conditions.

4. Codimension one symmetry in an annulus

In this section, let Ω_1 be the annulus $B_{R_2} \setminus \overline{B_{R_1}}$ in $\mathbb{R}^N$, $(R_2 > R_1 > 0)$. Let F, G_i, H be functions satisfying (2.2), (2.3), $F = F(t)$, and

$$K_1 = \big\{v \in W^{1,2}(\Omega_1) : \int_{\Omega_1} G_i(v)\,dx = 0,\ i = 1, \dots, n\big\}.$$

We consider the variational problem:

$$\begin{gathered}\text{Minimize } J_1(v) \equiv \int_{\Omega_1} \Big(\frac{1}{2}|\nabla v|^2 - F(v)\Big)\,dx + \int_{\partial\Omega_1} H(v)\,dS \\ \text{over all } v \in K_1.\end{gathered} \tag{4.1}$$

Defining global and local minimizers of (4.1) analogously as those for (2.4), we have that any local minimizer of (4.1) satisfies

$$-\Delta u = f(u) + \sum_{i=1}^{n} \alpha_i g_i(u) \quad \text{in } \Omega_1, \tag{4.2}$$

$$\partial u/\partial\nu + h(u) = 0 \quad \text{on } \partial\Omega_1, \quad (\nu: \text{ is the exterior normal}),$$

$$\int_{\Omega_1} \Big(\nabla u \nabla\varphi - f(u)\varphi\Big)\,dx + \int_{\partial\Omega_1} h(u)\varphi\,dS = 0, \tag{4.3}$$

$$\int_{\Omega_1} \Big(|\nabla\varphi|^2 - (f'(u) + \sum_{i=1}^{n} \alpha_i g_i'(u))\varphi^2\Big)\,dx + \int_{\partial\Omega_1} h'(u)\varphi^2\,dS \ge 0 \tag{4.4}$$

for every $\varphi \in W^{1,2}(\Omega_1)$ satisfying $\int_{\Omega_1} g_i(u)\varphi\,dx = 0$, and $u \in W^{2,2}(\Omega_1) \cap C^1(\overline{\Omega_1})$.

with $\alpha_i \in \mathbb{R}$, $i = 1, \dots, n$, as Lagrange multipliers. As before, we say that u is a stationary solution of (P_1) if u satisfies the conditions (4.2).

We first recall the well-known result that in case of the unconstrained problem, $n = 0$, local minimizers u of (4.1) are radial, that is $u = u(|x|)$. This statement has been shown in [21] for the case of Dirichlet boundary conditions. But the argument used in [21] is applicable in our case, too. For the convenience of the reader we give a proof: We proceed similarly as in the proof of Lemma 2.1. Setting

$$I(\varphi) := \Big(\int_{\Omega_1} \big(|\nabla \varphi|^2 - f'(u)\varphi^2 \big)\, dx + \int_{\partial\Omega_1} h'(u)\varphi^2\, dS \Big) \Big(\int_{\Omega_1} \varphi^2\, dx \Big)^{-1},$$

we find that $I(\varphi) \geq 0$ for all $\varphi \in W^{1,2}(\Omega_1)$ with $\varphi \not\equiv 0$, by (4.4). Now fix any coordinate system $x = (x_1, x_2, x'')$, $(x \in \mathbb{R}^N,\ x'' \in \mathbb{R}^{N-2})$, and let $v := x_1 u_{x_2} - x_2 u_{x_1}$ the angular derivative of u in the (x_1, x_2)-plane. We have that $-\Delta v = f'(u)v$ in Ω_1, and $(\partial v/\partial \nu) + h'(u)v = 0$ on $\partial\Omega_1$. Assuming $v \not\equiv 0$ we then find $I(v) = 0$. As in the proof of Lemma 2.1, this means that v can have one sign only, which is impossible since $\int_{\Omega_1} v\, dx = 0$. Hence u is radial.

In this connection we also mention the well-known result of Casten and Holland [8] that in case of Neumann boundary conditions, that is $H(v) \equiv 0$, any local minimizer of the unconstrained problem is constant. This result has been generalized by O. Lopes [19] to a related elliptic system.

Our aim now is to show that global minimizers - and, under some additional assumption also local minimizers and stationary solutions - of (4.1) have some partial symmetry (see the Definition below).

Definition 4.1. We say that $v \in W^{1,2}(\Omega_1)$ has codimension one symmetry if there is a unit vector $e \in \mathbb{R}^N$ such that v depends on $r := |x|$ and $\theta := \arccos\Big((x, e)/|x|\Big)$ only, ($(\cdot, \cdot)$: Euclidean scalar product), that is, $v(x) = \widetilde{v}(r, \theta)$.

For the symmetry proofs below we will use an 'angular variant' of the reflection method of Section 2. If $e \in \mathbb{R}^N$, $|e| = 1$, then let $H \equiv H_e$ the (open) half-space $\{x \in \mathbb{R}^N : (x, e) > 0\}$. We write σ_H for reflection in ∂H, that is $\sigma_H x := x - 2(x, e)e$ $\forall x \in \mathbb{R}^N$.

We first establish a simple geometrical criterion.

Lemma 4.2. *Let $u \in W^{1,2}(\Omega_1) \cap C(\Omega_1)$ and assume that for any half-space H with $0 \in H$ we have either $u(x) \geq u(\sigma_H x)$ $\forall x \in H \cap \Omega_1$ or $u(x) \leq u(\sigma_H x)$ $\forall x \in H \cap \Omega_1$. Then u has codimension one symmetry. Moreover, with the notations of Definition 4.1 we have $(\partial \widetilde{u}/\partial \theta)(r, \theta) \leq 0$ $\forall (r, \theta) \in (R_1, R_2) \times [0, \pi]$.*

Proof. Let l_1, l_2 be mutually orthogonal unit vectors. We fix a coordinate system $x = (x_1, x_2, x'')$, $(x'' \in \mathbb{R}^{N-2})$, in which $l_1 = (1, 0, \dots, 0)$ and $l_2 = (0, 1, 0, \dots, 0)$. Introducing polar coordinates in the (x_1, x_2)- plane, $x_1 = z\cos\varphi$, $x_2 = z\sin\varphi$, $(\varphi \in [-\pi, \pi],\ z \geq 0)$, the function $\widehat{u}(z, \varphi, x'') := u(x)$ is (2π)- periodic in φ, on the set $\{(z, \varphi, x'') : R_1^2 \leq z^2 + |x''|^2 \leq R_2^2,\ z \geq 0,\ \varphi \in \mathbb{R}\}$. Proceeding as in the proof of Lemma 2.6 we find some value $\lambda^* \in [-\pi, \pi]$ such that $\widehat{u}$ is either non-increasing or nondecreasing in φ for $\varphi \in (\lambda^*, \lambda^* + \pi)$, and writing $e_1 = (\cos\lambda^*, \sin\lambda^*, 0, \dots, 0)$ and $H_1 = \{x : (x, e_1) > 0\}$, we also have that $u(x) = u(\sigma_{H_1} x)$ $\forall x \in \Omega_1$. Let S denote the subspace of $\mathbb{R}^N$ orthogonal to $\operatorname{span}\{e_1\}$. Fixing two mutually orthogonal unit vectors l_3, l_4 in S we find another unit vector $e_2 \in \operatorname{span}\{l_3, l_4\}$ such that $u(x) = u(\sigma_{H_2} x)$ $\forall x \in \Omega_1$ where $H_2 = \{x : (x, e_2) > 0\}$. Letting S' the orthogonal complement of $\operatorname{span}\{e_1, e_2\}$, we then find a further unit vector $e_3 \in S'$, such that $u(x) = u(\sigma_{H_3} x)$ in Ω_1, where $H_3 := \{x : (x, e_3)\}$. Continuing in this manner we find $N-1$ mutually orthogonal unit vectors $e_1, \dots, e_{N-1}$ such that $u(x) = u(\sigma_{H_i} x)$ in Ω_1, where $H_i = \{x : (x, e_i) > 0\}$, $(i = 1, \dots, N-1)$. Now let e a unit vector orthogonal to e_i, $(i = 1, \dots, N-1)$, and let H any halfspace with $e \in \partial H$. Then H may be written as $\{x : (x, l) > 0\}$, where l is a unit vector in $\operatorname{span}\{e_1, \dots, e_{N-1}\}$. We claim that

$$u(x) = u(\sigma_H x) \quad \forall x \in \Omega_1. \tag{4.5}$$

Indeed, since $-tl = \sigma_H(tl) = \sigma_{H_{N-1}} \cdots \sigma_{H_1}(tl)$ for all $t \in \mathbb{R}$, we have

$$u(tl) = u(\sigma_H(tl)) \quad \forall t \in \mathbb{R}.$$

In view of our assumption, this implies (4.5). With the notation of Definition 4.1 this implies that $u = \widetilde{u}(r, \theta)$. After replacing e by $(-e)$ if necessary - the latter also implies $(\partial \widetilde{u}/\partial \theta)(r, \theta) \le 0$ $\forall (r, \theta) \in (R_1, R_2) \times [0, \pi]$, due to our considerations in the (x_1, x_2)-plane. □

Remark 4.3. Assume that u is a stationary solution of (4.1), and suppose that there is a half-space H with $0 \in \partial H$ such that

$$u(x) \ge u(\sigma_H x) \quad \forall x \in H \cap \Omega_1. \tag{4.6}$$

Using (4.2), we obtain, analogously as in Remark 2.4, that the following alternative takes place: Either

$$u(x) = u(\sigma_H x) \quad \forall x \in \overline{H} \cap \overline{\Omega_1}, \quad \text{or} \tag{4.7}$$

$$u(x) > u(\sigma_H x) \quad \forall x \in H \cap \overline{\Omega_1} \quad \text{and} \tag{4.8}$$

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on } \partial H \cap \Omega_1, \quad (\nu \text{ is the exterior normal to } H).$$

Lemma 4.4. *Let u a stationary solution of* (4.1)*, and assume that there is a half-space H with $0 \in \partial H$ such that* (4.8) *holds. Then u has codimension one symmetry. Moreover, if $\widetilde{u}$ is as in Definition 4.1, then*

$$\begin{gathered} \text{either} \quad \frac{\partial \widetilde{u}}{\partial \theta}(r, \theta) \equiv 0, \\ \text{or} \quad \frac{\partial \widetilde{u}}{\partial \theta}(r, \theta) < 0 \quad \text{in } [R_1, R_2] \times (0, \pi). \end{gathered} \tag{4.9}$$

Proof. Fix N mutually orthogonal unit vectors $l_1, \dots, l_N$ such that $H = H_{l_1}$. Choose a coordinate system $x = (x_1, x_2, x'')$, $(x'' \in \mathbb{R}^{N-2})$, so that $l_1 = (1, 0, \dots, 0)$ and $l_2 = (0, 1, 0, \dots, 0)$. Now we proceed similarly as in the proof of Lemma 2.5: For $\lambda \in \mathbb{R}$, let H_λ denote the half-space $\{x = (z\cos\varphi, z\sin\varphi, x'') : \varphi \in (\lambda, \lambda + \pi)\} = \{x : (x, e_\lambda) > 0\}$, where $e_\lambda := (\cos\lambda, \sin\lambda, 0, \dots, 0)$, and let u^λ the reflection of u in H_λ, that is $u^\lambda(x) := u(\sigma_{H_\lambda} x)$ $\forall x \in \Omega_1$. Defining $M := \{\lambda \in \mathbb{R} : u^\lambda(x) < u(x)\ \forall x \in H_\lambda \cap \Omega_1\}$, we first claim that M is open.

Let C be defined by (2.25). Given any $\varepsilon > 0$, let $\Omega_1^\varepsilon := B_{R_2 - \varepsilon} \setminus \overline{B_{R_1 + \varepsilon}}$, and let A_ε, B_ε the solutions of the following boundary value problems:

$$\begin{gathered} \Delta A_\varepsilon = \Delta B_\varepsilon = 0 \quad \text{in } \Omega_1 \setminus \overline{\Omega_1^\varepsilon}, \\ A_\varepsilon = B_\varepsilon = 1 \quad \text{on } \partial B_{R_2} \cup \partial B_{R_1}, \\ A_\varepsilon = 2, \quad B_\varepsilon = 1/2 \quad \text{on } \partial B_{R_2 - \varepsilon} \cup \partial B_{R_1 + \varepsilon}. \end{gathered}$$

Then $1 \le A_\varepsilon \le 2$, $1/2 \le B_\varepsilon \le 1$, $A_\varepsilon = A_\varepsilon(|x|)$, $B_\varepsilon = B_\varepsilon(|x|)$, and

$$-\lim_{\varepsilon \to 0} \frac{A_\varepsilon}{\partial \nu}(x) = \lim_{\varepsilon \to 0} \frac{\partial B_\varepsilon}{\partial \nu}(x) = +\infty \quad \text{on } \partial\Omega_1.$$

We fix $\varepsilon > 0$ small enough such that

$$-\frac{\partial A_\varepsilon}{\partial \nu}(x) \ge C, \quad \frac{\partial B_\varepsilon}{\partial \nu}(x) \ge C \quad \text{on } \partial\Omega_1, \tag{4.10}$$

and we choose $\varepsilon' > 0$ such that

$$\varepsilon' \le \frac{\sqrt{C}}{3 \sup\{|\nabla A_\varepsilon(x)| : x \in \Omega_1 \setminus \overline{\Omega_1^\varepsilon}\}} \quad \text{and} \quad \varepsilon' \le \frac{\pi}{4\sqrt{C}}. \tag{4.11}$$

Now let $\lambda \in M$. Setting $v_\lambda := u - u^\lambda$, we have that $v_\lambda > 0$ in $H_\lambda \cap \overline{\Omega_1}$, and then $(\nabla v_\lambda, e_\lambda) > 0$ on $\partial H_\lambda \cap \Omega_1$, by Remark 4.3. We claim that this actually implies

$$(\nabla v_\lambda, e_\lambda) > 0 \quad \text{on } \partial H_\lambda \cap \overline{\Omega_1}. \tag{4.12}$$

Let $U_\varepsilon := \{x \in \Omega_1 \setminus \overline{\Omega_1^\varepsilon} : 0 < (x, e_\lambda) < \varepsilon'\}$. If then $\kappa > 0$ is sufficiently small, we have that

$$v_\lambda(x) \ge \kappa A_\varepsilon(x)\big(e^{\sqrt{2C}(x,e_\lambda)} - 1\big) \quad \text{on } \partial U_\varepsilon \setminus \partial\Omega_1. \tag{4.13}$$

Setting $w(x) := v_\lambda(x) - \kappa A_\varepsilon(x)(e^{\sqrt{2C}(x,e_\lambda)} - 1)$, and using (2.17)-(2.19), (2.25), (4.10), and (4.13), we find that

$$\begin{gathered}
-\Delta w \ge c_\lambda w + \kappa A_\varepsilon e^{\sqrt{2C}(x,e_\lambda)} \Big(2\sqrt{2C}\frac{(\nabla A_\varepsilon, e_\lambda)}{A_\varepsilon} + 2C + c_\lambda(1 - e^{-\sqrt{2C}(x,e_\lambda)})\Big) \\
\ge c_\lambda w \quad \text{in } U_\varepsilon \\
w \ge 0 \quad \text{on } \partial U_\varepsilon \setminus \partial\Omega, \\
\frac{\partial w}{\partial \nu} + d_\lambda(x) w = -\kappa(e^{\sqrt{2C}x_1} - 1)\big((A_\varepsilon)_\nu + d_\lambda\big) \ge 0 \quad \text{on } \partial U_\varepsilon \cap \partial\Omega.
\end{gathered} \tag{4.14}$$

We then define

$$V(x) := \frac{w(x)}{B_\varepsilon(x) \cos \sqrt{C}(x, e_\lambda)}, \quad (x \in U_\varepsilon). \tag{4.15}$$

Note that

$$B_\varepsilon(x) \cos \sqrt{C}(x, e_\lambda) \in [(1/4), 1] \quad \text{in } U_\varepsilon,$$

in view of (4.11). Using (4.14) and (4.15) we obtain

$$\begin{gathered}
-\Delta V + 2\sqrt{2C}(\nabla V, e_\lambda) \tan \sqrt{2C}(x, e_\lambda) - 2\frac{(\nabla V, \nabla B_\varepsilon)}{B_\varepsilon} \\
\ge (c_\lambda - C)V \quad \text{in } U_\varepsilon, \\
V \ge 0 \quad \text{on } \partial U_\varepsilon \setminus \partial\Omega_1, \\
\frac{\partial V}{\partial \nu} + \Big(d_\lambda + \frac{\partial B_\varepsilon}{\partial \nu}\Big)V \ge 0 \quad \text{on } \partial U_\varepsilon \cap \partial\Omega.
\end{gathered}$$

As in the proof of Lemma 2.5, this implies $V \ge 0$ in U_ε, which shows (4.12). Hence by continuity, we find some $\delta > 0$ such that

$$v_\mu(x) > 0 \quad \text{in } H_\mu \cap \overline{\Omega}_1 \quad \forall \mu \in (\lambda - \delta, \lambda + \delta). \tag{4.16}$$

This implies that M is open.

Recalling that $0 \in M$, let M_1 the connected component of M containing 0. Since then $\pm\pi \notin M_1$, we have that $u = u^{\lambda^*}$, where $\lambda^* := \sup_{M_1} \lambda$. Hence $u_\nu(x) > 0$ on $\partial H_\lambda \cap \Omega_1$ $\forall \lambda \in (\lambda^* - \pi, \lambda^*)$. With the notations of the previous proof this implies

$$\begin{gathered} \widehat{u}(z, \varphi, x'') = \widehat{u}(z, 2\lambda^* - \varphi, x''), \\ \frac{\partial \widehat{u}}{\partial \varphi}(z, \varphi, x'') < 0 \quad \text{for } \varphi \in [\lambda^*, \lambda^* + \pi],\ R_1^2 \le z^2 + |x''|^2 \le R_1^2,\ z \ge 0. \end{gathered} \tag{4.17}$$

We may repeat the above considerations in the (l_1, l_i)-plane, $(i = 3, \dots, N)$, to obtain that for every half-space H with $0 \in \partial H$ we have either $u(x) \ge u(\sigma_H x)$ for all $x \in H \cap \Omega_1$ or $u(x) \le u(\sigma_H x)$ for all $x \in H \cap \Omega_1$. By Lemma 4.2 this implies that u has codimension one symmetry. Finally, the inequality in (4.9) follows from (4.17). □

The following statement is an analogue of Lemma 2.6.

Lemma 4.5. *Let u a local minimizer of* (4.1). *Suppose that u is not radial, and that there exists a half-space $H \subset \mathbb{R}^N$, with $0 \in \partial H$, such that (4.6) holds. Then the conclusions of Lemma 4.4 hold.*

Proof. We use the notation of the previous proof. We also define two-point rearrangements $T^\lambda u$ with respect to the half-space $H_\lambda = \{x : (x, e_\lambda) > 0)\}$, $(\lambda \in [-\pi, \pi])$, by

$$T^\lambda u(x) := \begin{cases} \max\{u(x); u(\sigma_{H_\lambda} x)\} & \text{if } x \in H_\lambda \cap \Omega_1, \\ \min\{u(x); u(\sigma_{H_\lambda} x)\} & \text{if } x \in \Omega_1 \setminus H_\lambda. \end{cases} \tag{4.18}$$

The properties of Lemma 2.2 hold for this type of rearrangement, too. Hence we may argue analogously as in the proof of Theorem 3.1 to obtain that either $u = T^\lambda u$ $\forall \lambda \in [\lambda_0, \lambda_0 + \pi]$ or $u^\lambda = T^\lambda u$ $\forall \lambda \in [\lambda_0, \lambda_0 + \pi]$. Since we may repeat these considerations in any (l_1, l_i)- plane, $(i = 3, \dots, N)$, we then find as in the previous proof, that u has codimension one symmetry. Finally the inequality in (4.9) follows easily from the fact that u is non-radial and from Remark 4.3. □

Theorem 4.6. *Let u be a global minimizer of* (4.1)*, and assume that u is not radial. Then the conclusions of Lemma 4.4 hold.*

Proof. For any half-space H with $0 \in H$, let $\sigma_H u$ and $T^H u$ denote reflection and two-point rearrangement with respect to H, respectively, that is $\sigma_H u(x) := u(\sigma_H x)$ $\forall x \in \Omega_1$, and

$$T^H u(x) := \begin{cases} \max\{u(x); u(\sigma_H x)\} & \text{if } x \in H \cap \Omega_1, \\ \min\{u(x); u(\sigma_H x)\} & \text{if } x \in \Omega_1 \setminus H\,. \end{cases} \tag{4.19}$$

Arguing as in proof of Theorem 2.7 we find that either $u = T^H u$ or $\sigma_H u = T^H u$. Hence u has codimension one symmetry, by Lemma 4.2. The inequality in (4.9) then follows as in the previous proof. □

Remark 4.7. 1) Homogeneous variational problems with one constraint in domains with rotational symmetry under Neumann boundary conditions, that is, $F(v) = -(1/2)v^2$, $n = 1$, $G_1(v) = |v|^{p+1}$ and $H \equiv 0$, have been extensively studied by Esteban and other authors (see [9, 10, 11, 12]). It turned out that the global minimizers are not radially symmetric, but have the symmetry property of Lemma 4.4.

2) O. Lopes [19] studied problem (4.1) for one constraint under Neumann conditions. He also investigated a variational problem that is related to an elliptic system,

$$E(v_1,\dots,v_K) := \int_{\Omega_1} \Big((1/2)\sum_{i=1}^{K} |\nabla v_i|^2 - F(|x|, v_1,\dots,v_K) \Big)\, dx \to \mathrm{Inf}\ !,$$

subject to

$$v_i \in W^{1,2}(\Omega_1),\ (i=1,\dots,K), \quad \text{and} \quad \int_{\Omega_1} G(|x|, v_1,\dots,v_K)\, dx = 1,$$

where the functions F and G satisfy appropriate smoothness and growth conditions. He proved both in the scalar and in the vector case, that local minimizers are not radial. Furthermore, he showed in the system case that any component of the (vector-valued) global minimizer has codimension one symmetry (he did not prove the inequality (4.9)). This result is surprising, since no cooperativity condition is imposed on F.

We mention that the symmetry proofs in [19, 20] are based on a reflection device in which the Principle of Unique Continuation plays a crucial role, too. However, the transformations used in that method are different from ours: in particular, they are not rearrangements. This fact restricts the applicability of the method to problems with one constraint.

3) Lemmata 4.2, 4.4 and Theorem 4.6 remain valid if the nonlinearities F, G_i, H, $(i=1,\dots,n)$, depend also on $|x|$, provided that they satisfy suitable smoothness and growth conditions and/or if Ω_1 is a ball B_R, $(R>0)$.

5. Dirichlet boundary conditions

In this section we reconsider the problems (2.4), (3.1) and (4.1) when the solutions u satisfy Dirichlet boundary conditions on $\partial\Omega$, respectively on $\partial\Omega_1$.

1) Letting

$$\begin{aligned} X := \big\{ & v \in W^{1,2}(\Omega_a) \cap W^{1,2}_{\mathrm{loc}}(\Omega) : v \text{ is } (2a)\text{-periodic in } x_1 \\ & \text{and } v = g \text{ on } \mathbb{R} \times \partial\omega \big\}, \end{aligned}$$

where $g \in W^{1,2}(\Omega_a) \cap C(\overline{\Omega_a})$, $g = g(x')$, and omitting the boundary term in J and J_+, the results of Section 2 and 3 remain valid, except the fact that the derivative u_{x_1} is now positive only on $(\lambda^* - a, \lambda^*) \times \omega$.

The proofs require only some minor modifications which we summarize below. First observe that, if u satisfies (2.16), then we again obtain the alternative (2.20), (2.21), except that the first condition in (2.21) is satisfied on the set $(\lambda, \lambda+a)\times\omega$ only. Then, defining M by (2.24) as in the proof of Lemma 2.5, the proof of its openness is modified as follows: Let $W_\varepsilon := \{x \in (\lambda+\varepsilon, \lambda+a-\varepsilon)\times\omega : |x-y| > \varepsilon\ \forall y \in \partial\Omega\}$, and let $\lambda \in M$. Since $v_\lambda = u - u^\lambda > 0$ in $(\lambda, \lambda+a)\times\omega$, we find by continuity some $\delta(=\delta(\varepsilon)) > 0$ such that $v_\mu > 0$ in W_ε for any $\mu \in (\lambda-\delta, \lambda+\delta)$. Then

$$\begin{aligned} -\Delta v_\mu \geq c_\mu v_\mu \quad & \text{in } \big((\mu, \mu+a)\times\omega\big) \setminus \overline{W_\varepsilon}, \\ v_\mu \geq 0 \quad & \text{on } \partial\big((\mu,\mu+a)\times\omega \setminus \overline{W}_\varepsilon\big), \end{aligned} \tag{5.1}$$

where the coefficients c_μ and d_μ are defined by (2.18),(2.19) and satisfy the (uniform!) inequalities (2.25). Now the Maximum Principle in Small Domains (see

Appendix) tells us that if ε is small enough then the solution v_μ of (5.1) is nonnegative $\forall\mu \in (\lambda-\delta, \lambda+\delta)$. Applying the Strong Maximum Principle then gives that $v_\mu > 0$ in $(\mu, \mu+a)\times\omega$. This proves that M is open.
2) Replacing $W^{1,2}(\Omega_1)$ by $W_0^{1,2}(\Omega_1)$ and omitting the boundary term in J_1 in the definition of problem (4.1), the results of Section 4 remain valid, except that the angular derivative $(\partial\widetilde{u}/\partial\theta)(r,\theta)$ is now positive only on $(R_1, R_2)\times(0,\pi)$.
Observe first that if u satisfies (4.6) then we obtain the alternative (4.7),(4.8), except that the first condition in (4.8) is satisfied on the set $H\cap\Omega_1$ only. Then, defining the set M as in the proof of Lemma 4.4, the proof of its openness can be accomplished by using the Maximum Principle in Small Domains similarly as above. We leave the details to the reader.

6. Appendix

In this Appendix let Ω be a domain in $\mathbb{R}^N$, and let $a_{ij} \in C^1(\overline{\Omega})$, $d \in C(\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$, with $a_{ij} = a_{ji}$, $a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2$ for all $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, $c_0 > 0$, $(i,j = 1,\dots,N)$.

Weak Maximum principle. (see [14]) Let $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_1 open and smooth, $c \geq 0$, $d \geq 0$. Then, if $u \in W^{2,N}(\Omega)\cap C(\overline{\Omega})$ satisfies

$$-a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu \geq 0 \quad \text{in } \Omega, \tag{6.1}$$

$$u \geq 0 \quad \text{on } \Gamma_0, \quad \frac{\partial u}{\partial \nu} + du \geq 0 \quad \text{on } \Gamma_1, \tag{6.2}$$

it follows that $u \geq 0$ in Ω.

Strong Maximum Principle. (see [14]) Let $\partial\Omega$ smooth in a neighborhood of $x_0 \in \partial\Omega$, and let $u \in W^{2,N}(\Omega)\cap C^1(\Omega\cup\{x_0\})$ satisfy (6.1), $u \geq 0$ in Ω, and $u(x_0) = 0$. Then

$$\limsup_{t\to+0} \frac{1}{t}\big(u(x_0 - t\nu) - u(x_0)\big) \geq 0, \quad (\nu \text{is the exterior normal}),$$

where the equality sign is attained only if $u \equiv 0$ in Ω.

Maximum Principle in Small Domains. (see [3]) There exists a number $\delta > 0$ depending only on $\operatorname{diam}\Omega$, N, c_0 and on the L^∞-bounds for the functions a_{ij}, b_i and c, such that if $|\Omega| < \delta$, and if $u \in W^{2,N}(\Omega)\cap C(\overline{\Omega})$ satisfies (6.1) and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.

Principle of Unique Continuation. (see [15]) Let $u \in W^{2,2}(\Omega)$ satisfy

$$-a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu = 0 \quad \text{in } \Omega, \tag{6.3}$$

and suppose that there is a nonempty open subset U of Ω such that $u \equiv 0$ in U. Then $u \equiv 0$ in Ω.

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