Electronic Journal of Differential Equations, Vol. 2003(2003), No. 101, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

VARIATIONAL CHARACTERIZATION OF INTERIOR INTERFACES IN PHASE TRANSITION MODELS ON CONVEX PLANE DOMAINS

CLARA E. GARZA-HUME & PABLO PADILLA

ABSTRACT. We consider the singularly perturbed Allen-Cahn equation on a strictly convex plane domain. We show that when the perturbation parameter tends to zero there are solutions having a transition layer that tends to a straight line segment. This segment can be characterized as the shortest path intersecting the boundary orthogonally at two points.

1. INTRODUCTION

We consider the equation

$$\begin{aligned} -\epsilon^2\Delta u + W'(u) = 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a strictly convex subset of $\mathbb{R}^2$ with C^1 boundary and W is a double-well potential. In the case $W = (1-u^2)^2$ this corresponds to the scalar steady state Ginzburg-Landau equation. It arises in phase transition models, super conductivity, material science, etc. (see [5] for more references).

Finding solutions of (1.1) is equivalent to finding critical points of the functional

$$E_\epsilon(u) = \int_\Omega \big(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u)\big)\,dS, \tag{1.2}$$

in a suitable function space.

This problem, with and without volume constraint, has been studied by Alikakos, Bates, Chen, Fusco, Kowalczyk, Modica, Sternberg and Wei among many other authors (see [5]). Of particular interest is the characterization of solutions when ϵ tends to zero. In this situation, nontrivial solutions typically exhibit transition layers, which in the case where there is no volume constraint are expected to be staight lines. Indeed, it is well known that the value of the Lagrange multiplier corresponds to the curvature of the interface (see for instance [5].) In a recent paper, Kowalczyk ([7]) has made these assertions precise by applying the Implicit Function Theorem to construct special solutions.

2000 *Mathematics Subject Classification.* 49Q20, 35J60, 82B26.
Key words and phrases. Phase transition, singularly perturbed Allen-Cahn equation, convex plane domain, variational methods, transition layer, Gauss map, geodesic, varifold.

Submitted July 15, 2003. Published October 2, 2003.

It is the purpose of this paper to show that similar solutions can be found using variational techniques. Moreover, the variational characterization provides a natural way of describing the transition along the lines of Γ-convergence methods. We will find minima, u_ϵ, of this functional subject to a constraint which is different from the standard volume constraint and was motivated by a theorem by Poincaré and the results of [3].

Problem (1.1) was considered in [3] for the case when Ω is an oval surface embedded in $\mathbb{R}^3$ and Δu represents the Lapace-Beltrami operator with respect to the metric inherited from $\mathbb{R}^3$. In that paper it is shown that the constraint

$$G(u) = \int_{S^2} u(g^{-1}(y))\, dy = 0 \tag{1.3}$$

for ϵ small and g the Gauss map $g : \Omega \to S^2$ can be used to obtain nontrivial solutions whose interface tends to a minimal closed geodesic. Roughly speaking the idea is the following. Restriction (1.3) for functions having uniformly bounded energy as $\epsilon \to 0$ has a natural geometric interpretation. Namely, since such functions are necessarily close to ± 1 except for a small set (the transition), then the restriction guarantees that this transition divides, under the Gauss map, the unit sphere into two components of equal area. Assuming that the transition takes place on a regular curve $\gamma = u^{-1}(0)$ and using a result stated by Poincaré in [9] and proved by Berger and Bombieri in [1] it is natural to expect that minimal solutions concentrate on minimal closed geodesics, which is the content of the main result in [3].

Theorem 1.1 (Berger and Bombieri [1]). *Let Γ be the class of smooth curves γ on an oval surface such that under the Gauss map, $g(\gamma)$ divides S^2 into two components of equal area. The curve γ^* which minimizes the arc length among curves in Γ is a minimal closed geodesic.*

Recalling that the energy E_ϵ as $\epsilon \to 0$ is proportional to the length of the transition (see [12] for details) we see that minimizing E_ϵ subject to the constraint (1.3) for ϵ small is equivalent to the geometric problem of minimizing arc length in Γ and that indeed the interface should be a minimal closed geodesic. We refer to [3] for precise statements of these facts.

The case of a planar convex domain can be naturally considered as the limit of an oval surface that is gradually flattened in one direction. We prove that up to a subsequence the solutions u_ϵ have an interface that converges when $\epsilon \to 0$ to the shortest straight line that intersects the boundary orthogonally. This line would be the limit of the shortest closed geodesic on the surface. In fact, this analogy had been used in the context of dynamical systems to study the flow of billiards on convex plane domains as the limit of the geodesic flow in oval surfaces (see [8]). We point out that this line has also a minimax characterization. It is shown in [6] that the solution obtained via the Mountain Pass Theorem has a transition along this segment (in the $\epsilon \to 0$ limit).

The rest of the paper is organized as follows: in section 2 we recall some well-known facts, introduce notation and state the results on the convergence of solutions as ϵ tends to zero that we need. The setting is very similar to that of [3] but we include the essential parts for the convenience of the reader.

In section 3 we present the proof.

2. Setting

We make the following assumptions:

(A1) The function $W : \mathbb{R} \to [0,\infty)$ is $\mathcal{C}^3$ and $W(\pm 1) = 0$. For some $\gamma \in (-1,1)$, $W' < 0$ on $(\gamma, 1)$ and $W' > 0$ on $(-1,\gamma)$. For some $\alpha \in (0,1)$ and $\kappa > 0$, $W''(x) \geq \kappa$ for all $|x| \geq \alpha$.

(A2) There exist constants $2 < k \leq \frac{2n}{n-2}$ and $c > 0$ such that
$$c|x|^k \leq W(x) \leq c^{-1}|x|^k$$
$$c|x|^{k-1} \leq |W^{'}(x)| \leq c^{-1}|x|^{k-1}$$
for sufficiently large $|x|$.

(B1) The subset $U \subset \mathbb{R}^n$ is bounded, open and has Lipschitz boundary ∂U. A sequence of functions $\{u^i\}_{i=1}^{\infty}$ in $C^3(U)$ satisfies
$$\epsilon_i \Delta u^i = \epsilon_i^{-1} W^{'}(u^i) - \lambda_i \tag{2.1}$$
on U. Here, $\lim_{i\to\infty} \epsilon_i = 0$, and we assume there exist c_0, λ_0 and E_0 such that $\sup_U |u^i| \leq c_0$, $|\lambda_i| \leq \lambda_0$ and for all i,
$$\int_U \frac{\epsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\epsilon_i} \leq E_0\,.$$

Now, we recall some formalism from Geomtric Measure Theory that will be used. As in [5] let
$$\phi(s) = \int_0^s \sqrt{W(s)/2}\, ds$$
and define new functions
$$w^i = \phi \circ u^i$$
for each i and we associate to each function w^i a varifold V^i ([2, 10]) defined as
$$V^i(A) = \int_{-\infty}^{\infty} v(\{w^i = t\})(A)\, dt$$
for each Borel set $A \subset G_{n-1}(U)$, $G_{n-1}(U) = U \times G(n, n-1)$, where $G(n, n-1)$ is the Grassman manifold of unoriented $(n-1)$-dimensional planes in $\mathbb{R}^n$.

By the compactness theorem for BV functions, there exists an a.e. pointwise limit w^∞. Let ϕ^{-1} be the inverse of ϕ and define
$$u^\infty = \phi^{-1}(w^\infty).$$
$u^\infty = \pm 1$ a.e. on U and the sets $\{u^\infty = \pm 1\}$ have finite perimeter in U.

The following theorem is proved in [5].

Theorem 2.1. *Let V^i be the varifold associated with u^i (via w^i). On passing to a subsequence we can assume*
$$\lambda_i \to \lambda_\infty, \quad u^i \to u^\infty \ a.e., \quad V^i \to V \text{ in the varifold sense.}$$
Moreover,

(1) *For each $\phi \in C_c(U)$,*
$$\|V\|(\phi) = \lim_{i\to\infty} \int \phi \frac{\epsilon_i |\nabla u^i|^2}{2} = \lim_{i\to\infty} \int \phi \frac{W(u^i)}{\epsilon_i} = \lim_{i\to\infty} \int \phi |\nabla w^i|.$$

(2) $\operatorname{supp} \|\partial\{u^\infty = 1\}\| \subset \operatorname{supp} \|V\|$, *and $\{u^i\}$ converges locally uniformly to ± 1 in $U \setminus \operatorname{supp} \|V\|$, where ∂ denotes the reduced boundary.*

(3) *For each* $\tilde{U} \Subset U$ *and* $0 < b < 1$, $\{|u^i| \le 1-b\} \cap \tilde{U}$ *converges to* $\tilde{U} \cap \operatorname{supp} \|V\|$ *in the Hausdorff distance sense.*

(4) $\sigma^{-1}V$ *is an integral varifold. Moreover, the density* $\theta(x) = \sigma N(x)$ *of* V *satisfies*

$$N(x) = \{\begin{cases} \text{odd} & \mathcal{H}^{n-1} \ a.e. \ x \ in \ M^\infty, \\ \text{even} & \mathcal{H}^{n-1} \ a.e. \ x \ in \ \operatorname{supp} \ \|V\| \setminus M^\infty, \end{cases}$$

where M^∞ *is the reduced boundary of* $\{u^\infty = 1\}$ *and*

$$\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\, ds.$$

(5) *The generalized mean curvature* H *of* V *is*

$$H(x) = \begin{cases} \frac{\lambda_\infty}{\theta(x)} \nu^\infty(x) & \mathcal{H}^{n-1} \ a.e. \ in \ M^\infty \\ 0 & \mathcal{H}^{n-1} \ a.e. \ x \in \operatorname{supp} \ \|V\| \setminus M^\infty, \end{cases}$$

where ν^∞ *is the inward normal for* M^∞.

The next theorem is also proved in [5].

Theorem 2.2. *In addition to assumptions* (A1), (A2), (B1) *suppose* $\{u^i\}$ *are locally energy minimizing on* $\tilde{U} \Subset U$ *for* E_{ϵ_i} *(with or without volume constraint). Then* $N(x) = 1$, $\mathcal{H}^{n-1}$ *a.e. on* $\tilde{U} \cap \operatorname{supp}\|V\|$. *The set* $\partial\{u^\infty = 1\}$ *on* $\tilde{U}$ *has constant mean curvature* $\frac{\lambda_\infty}{\sigma}\nu^\infty$ *and no energy loss occurs on* $\tilde{U}$.

Motivated by Theorem 1.1 we consider the Gauss map g from the *boundary* of the domain to S^1 and introduce the restriction

$$G(u) = \int_{S^1} u(g^{-1}(\theta))\, d\theta = 0. \tag{2.2}$$

Let L be the length of the shortest straight line that intersects $\partial\Omega$ orthogonally at two points. For simplicity, we will refer to such a line as a minimal orthogonal line. The fact that such a straight line exists can be shown directly using calculus or as follows. Choose a direction and find two tangents to the domain in that direction, called L_1 and L_2 in figure 1. Draw the lines orthogonal to L_1 and L_2 and find the (signed) distance between them, a. If we now rotate the chosen direction together with the corresponding parallel lines, the distance between the orthogonal lines varies continuously and after a half turn the distance is $-a$ so it has to go through zero. This shows the existence of a line orthogonal to the boundary at two points. By compactness there exists at least one which is shortest.

As we mentioned in the introduction, in [3] it is proved that the supports of the varifolds associated with minimal solutions of equation (1.1) on an oval surface tend to a minimal geodesic. It is natural to expect that if we make the oval tend to a convex set on the plane, the interface of the solution will tend to a straight line orthogonal to the boundary.

3. Main Result

Theorem 3.1. *Let* u_{ϵ_n}, $\epsilon_n \to 0$ *be a sequence of minimizers of* (1.2) *(with* W *satisfying (A1) and (A2)) in* $V = \{u \in H^1(\Omega) : \frac{\partial u}{\partial \nu} = 0\}$ *subject to the constraint* (2.2)*). Then up to a subsequence, the support of the varifold associated with* u_{ϵ_n} *converges locally in the Hausdorff distance sense to a minimal orthogonal line.*

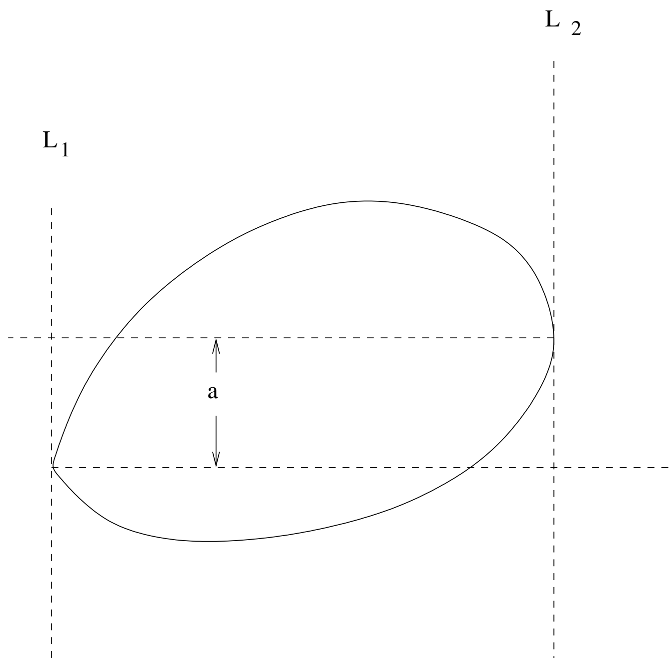

Figure 1. Construction of the minimal orthogonal line

Proof. We will first show that the Lagrange multiplier $\lambda_\epsilon = 0$. The functional to be minimized is given by (1.2). Variations of this functional subject to the constraint satisfy

$$\begin{aligned} DE_\epsilon(u)\phi &= \int_\Omega \epsilon \nabla u \nabla \phi + \frac{1}{\epsilon} W'(u)\phi + \lambda_\epsilon \int_{\partial\Omega} \phi f \\ &= \int [-\epsilon \Delta u + \frac{1}{\epsilon} W'(u)]\phi + \int_{\partial\Omega} [\epsilon \frac{\partial u}{\partial \nu}\phi + \lambda_\epsilon \phi f] = 0 \end{aligned}$$

for all $\phi \in C^\infty$. Here f is the derivative of the transformation from S^1 to $\partial\Omega$, i.e. the curvature of the boundary and therefore positive because of the strict convexity of the domain. This implies that

$$-\epsilon \Delta u + \frac{1}{\epsilon} W'(u) = 0 \text{ on } \Omega\,, \tag{3.1}$$

$$\epsilon \frac{\partial u}{\partial \nu} + \lambda_\epsilon f(x) = 0 \text{ on } \partial\Omega \tag{3.2}$$

The first of these is simply the fact that a critical point of the functional satisfies equation (1.1). The second one, using the fact that $\frac{\partial u}{\partial \nu} = 0$, implies that

$$\int_\Omega \lambda_\varepsilon f \phi = 0.$$

Since f is strictly positive by strict convexity of the boundary this implies that $\lambda_\epsilon = 0$.

Again from [5], Theorem 2.1 above, we know that V^i, the varifold associated with u_{ϵ_i}, converges (in the sense of varifolds) to a rectifiable varifold V. By [11] we also know that V has to divide the domain into two components by intersecting the boundary in two points. Moreover, by the boundary conditions, the intersection has to be orthogonal. If it did not coincide with a minimal orthogonal line we would be able to construct a trial function with less energy by making the transition layer of such a function coincide with that line using a standard procedure (see [11, 3]). The assertion about the convergence in the Hausdorff sense follows also immediately from Theorem 2.1. $\square$

Acknowledgements. This work was supported in part by Conacyt Group Project G25427-E and project 34203-E and completed while the authors were visiting the Mathematical Institute, Oxford University. We thank Stanley Alama and Manuel del Pino for many useful discussions and Ana Pérez Arteaga for her computational support.

References

[1] Berger M. S., Bombieri E., *On Poincaré's Isoperimetric Problem for Simple Closed Geodesics*, J. Functional An., **42** (1981), 274-298.

[2] Federer, Geometric Measure Theory, *Die Grundlehren der mathematischen Wissenschaften*, Band 153, Springer-Verlag, New York (1969).

[3] Garza-Hume, C. E., Padilla, P. *Closed geodesics on oval surfaces and pattern formation*, Comm. in Analysis and Geom., Vol. 11, no. 2 (2003), pp. 223-233.

[4] Gamkrelidze, R. V (Ed.), *Encylopaedia of Mathematical Sciences* vol. 28, Geometry I, Springer-Verlag Berlin, Heidelberg 1991.

[5] J. E. Hutchinson, Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. **10** (2000), pp.49-84.

[6] B. Ko, P. Padilla, *Geometric properties of interfaces of the mountain pass solution in a singularly perturbed equation*, preprint.

[7] Kowalczyk, M.,*On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions*, To appear in Ann. Mat. Pura Aplic.

[8] Moser, J., *Dynamical Systems, past and present*, Proc. Int. Congress of Math, Vol I. Berlin 1998, Doc. Math. 1998. Extra Vol. I, 381-402.

[9] Poincaré, H., *Sur les lignes géodésiques des surfaces convexes*, Trans. Amer. Math. Soc. **6** (1905), 237-274.

[10] Simon, L. *Lectures on Geometric Measure Theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3** (1983).

[11] Sternberg, P., Zumbrum, K., *Connectivity of phase boundaries in strictly convex domains*, Arch. Rational Mech. Anal. **141** (1998), no. 4, 375-400.

[12] Tonegawa, Y., *Phase filed with a Variable Chemical Potential*, Hokkaido Univ. Preprint Series in Math, series 509, Dec. 2000, Japan.

Clara E. Garza-Hume
Department of Applied Mathematics, UNAM, Mexico City, Mexico
E-mail address: `clara@mym.iimas.unam.mx`

Pablo Padilla
Department of Applied Mathematics, UNAM, Mexico City, Mexico
E-mail address: `pablo@mym.iimas.unam.mx`