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ON THE INSTABILITY OF SOLITARY-WAVE SOLUTIONS FOR FIFTH-ORDER WATER WAVE MODELS

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Abstract. This work presents new results about the instability of solitary-wave solutions to a generalized fifth-order Korteweg-deVries equation of the form

$$u_t + u_{xxxxx} + bu_{xxx} = (G(u, u_x, u_{xx}))_x,$$

where $G(q,r,s) = F_q(q,r) - rF_{qr}(q,r) - sF_{rr}(q,r)$ for some $F(q,r)$ which is homogeneous of degree $p+1$ for some $p > 1$. This model arises, for example, in the mathematical description of phenomena in water waves and magneto-sound propagation in plasma. The existence of a class of solitary-wave solutions is obtained by solving a constrained minimization problem in $H^2(\mathbb{R})$ which is based in results obtained by Levandosky. The instability of this class of solitary-wave solutions is determined for $b \neq 0$, and it is obtained by making use of the variational characterization of the solitary waves and a modification of the theories of instability established by Shatah & Strauss, Bona & Souganidis & Strauss and Gonçalves Ribeiro. Moreover, our approach shows that the trajectories used to exhibit instability will be uniformly bounded in $H^2(\mathbb{R})$.

1. Introduction

In this work we study some instability properties of solitary-wave solutions to a Hamiltonian generalized fifth-order Korteweg-deVries equation of the form

$$u_t + u_{xxxxx} + bu_{xxx} = (G(u, u_x, u_{xx}))_x, \tag{1.1}$$

where we assume that the nonlinear term has the form

$$G(q,r,s) = F_q(q,r) - rF_{qr}(q,r) - sF_{rr}(q,r), \tag{1.2}$$

for some $F(q,r) \in C^3(\mathbb{R}^2)$ which is homogeneous of degree $p+1$ for some $p > 1$. That is, we assume

$$F(\lambda q, \lambda r) = \lambda^{p+1} F(q,r) \tag{1.3}$$

for all $\lambda \geq 0$ and $(q,r) \in \mathbb{R}^2$.

It is important to note that equation (1.1) arises as a model for a variety of physical phenomena. For instance by choosing $F(u,u_x) = -u^3$, we have that the corresponding model describes the evolution of gravity-capillary water waves on a shallow layer ([14], [30]), as well as a chain of coupled nonlinear oscillators ([12])

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and magneto-sound propagation in plasma ([16]). On the other hand, for the water wave problem, namely, the irrotational motion of an inviscid and incompressible fluid under the influence of gravity, where we have the interaction of small amplitude and long two-dimensional waves over a shallow horizontal bottom, we observe that Olver in [24] derived equation (1.1) as a second-order expansion for unidirectional wave propagation with a nonlinear term of the form $F(u, u_x) = uu_x^2 - au^3$. Here the parameters a and b depend on wave amplitude, wave length and surface tension (see Benney [5] and Craig & Groves [9] where equation (1.1) with this special choice of F has also been studied). Our study of instability in this paper will consider the cases (in the water wave framework) when the coefficient b in (1.1) is proportional to $\tau - \frac{1}{3}$, where τ represents the Bond number which is a nondimensionalized surface tension. So, we have that b in (1.1) may be negative, zero or positive, which occurs in the respective limits $\tau \to \frac{1}{3}^+$, $\tau = \frac{1}{3}$ (no surface tension) and $\tau \to \frac{1}{3}^-$. For a more detailed discussion about higher-order water wave models equations and specific forms of the function G, we refer the reader to Benney [5], Craig & Groves [9], Hunter & Scheurle [14], Kichenassamy & Olver [19], Olver [24], Ponce [25] and the references therein.

Our main interest in this paper is to obtain conditions which will imply that the orbit generated by a solitary-wave solution of (1.1) will be unstable. By a solitary-wave solution we mean a solution of (1.1) of the form $u(x, t) = \varphi(x + ct)$, where c represents the speed of the wave and $\varphi(\xi)$ and its derivatives tend to zero as the variable $\xi = x + ct$ approaches to $\pm\infty$. By the orbit generated by the solitary-wave φ we mean the set $\Omega_\varphi = \{\varphi(\cdot + y) \mid y \in \mathbb{R}\}$, *i.e.*, the set of all the translates of φ. So, putting this form of u in (1.1) and integrating once, we see that φ must satisfy the fourth-order ordinary differential equation

$$\varphi_{xxxx} + b\varphi_{xx} + c\varphi = G(\varphi, \varphi_x, \varphi_{xx}). \tag{1.4}$$

We observe that equation (1.4) has also been studied in other contexts, such as travelling waves equation for models of both a elastic strut (Amick & Toland [1]) and a suspension bridge (McKenna & Walter [23]). Now, the problem of existence of solutions to (1.4) with specific form of G has already been considered by several authors, by example Weinstein [29], Kichenassamy & Olver [19], Kichenassamy [18] and Champneys & Groves [8]. In fact, in [29] a variational characterization of solutions of (1.4) in the case $b < 0$ and $F(u, u_x) = |u|^{p+1}$ has been proved, while in [19] is given a classification of admissible expression for G which lead to explicit $sech^2$ solitary-wave. Finally, for a general function G such as that established in (1.2), Levandosky in [20] proved via the Concentration Compactness Method (Lions [21], [22]) (see Theorem 2.2 below) the existence of a class of solitary-wave solutions of (1.1) providing $c > b_+^2/4$, where $b_+ = max\{b, 0\}$. In this point, we note that our interest of study will be the class of solitary waves found by Levandosky above.

Now, if we consider G satisfying (1.2) and we assume the existence of such a large range of solitary-wave solutions for (1.1) found in [20], a natural issue arises: to determine whether they are stable or unstable (see Definition 3.1 below) by the flow of the fifth-order equation (1.1). This point was considered initially in [20] in which a definitive picture for the specific case $b = 0$ was obtained (see subsection 2.1 below for a review of it). In this case, the method of proving the stability (or instability) was based by making use of the variational characterization of the solitary-wave solutions and the theory of Grillakis & Shatah & Strauss established in [13]. So the use of the function $d(c) = E(\varphi) + cQ(\varphi)$, which is associated with the conserved quantities E and Q for (1.1), namely,

$$E(f) = \frac{1}{2}\int_{-\infty}^{\infty} [(f_{xx})^2 - b(f_x)^2 - 2F(f, f_x)]\, dx, \quad Q(f) = \frac{1}{2}\int_{-\infty}^{\infty} f^2\, dx, \tag{1.5}$$

was a main ingredient in the analysis of stability. In fact, as well-known the solitary waves with speed c will be stable if and only if d is convex at c. An explicit formula to $d(c)$ (see (2.5) below) has been obtained if $b = 0$ and so in this case is possible to determine the values of p for which we have stability or instability, while for $b \neq 0$ it is not easy to determine the behaviour of d. Here we will obtain a theory of instability exactly when $b \neq 0$. We note that a recent theory about the linear instability problem of solitary-wave solutions for equation (1.1), with a more general admissible expressions for G, has been established by Bridges & Derks in [7] by using the *sympletic Evans function* and the *sympletic Evans matrix* for Hamiltonian evolutions equations.

In this paper we will show a new approach to study instability of solitary-wave solutions to nonlinear Hamiltonian evolution equations. Applying this method to equation (1.1), we can obtain and improve the results established in [20] about the property of instability of the φ-orbit, Ω_φ, where the solitary-wave solution φ is obtained by solving a constrained minimization problem (see Theorem 2.2 below). Our approach makes use of the variational characterization of solutions for (1.4) as well as of modifications made in the theories established by Grillakis & Shatah & Strauss in [13], Bona & Souganidis & Strauss in [6] and Gonçalves Ribeiro in [11]. More specifically, we make an extension of some ideas from Gonçalves Ribeiro in [11] about the instability of stationary states of a semi-linear Schrödinger equation to nonlinear evolution equations which are not semi-linear. Moreover, in our analysis we avoid the use of the function d. The idea of our approach for obtaining the instability of the set Ω_φ in $H^2(\mathbb{R})$ will be to find a vector field B (see (3.3) for definition), which will be defined in a specific neighbourhood $V(\Omega_\varphi, \epsilon_0)$ of Ω_φ, such that the vector $B(\varphi)$ will produce an unstable direction if we have that the *action* defined by $S(u) = E(u) + c\, Q(u)$ has the Hessian at φ strictly negative along $B(\varphi)$, namely

$$\langle S''(\varphi)B(\varphi), B(\varphi)\rangle < 0. \tag{1.6}$$

More exactly, we will obtain that the flow created by B (see (3.13) below) through φ will produce a sequence of points $\{u_n\}$ such that $u_n \to \varphi$ in $H^2(\mathbb{R})$ as $n \to \infty$, and the flow generated by equation (1.1) starting in these points u_n, it will escape from $V(\Omega_\varphi, \epsilon_0)$ in a finite time. So, as it is shown in Theorem 4.2 below, if we choose the direction $B(\varphi) = \varphi + 2x\varphi'$ then condition (1.6) will imply that if F is homogeneous in r of degree β, $\beta \in [0, p+1]$ (and therefore homogeneous in q of degree α where $\alpha + \beta = p + 1$), then the conditions

$$\begin{gathered} b = 0 \quad \text{and} \quad \beta > \frac{9-p}{2}, \quad \text{or} \\ b < 0 \quad \text{and} \quad \beta \geq \frac{9-p}{2}, \quad \text{or} \\ \beta > \frac{9-p}{2},\ b > 0 \quad \text{and} \quad b \quad \text{small}, \end{gathered} \tag{1.7}$$

imply that Ω_φ is unstable by the flow of equation (1.1). Moreover, our approach also produces that the trajectories used to exhibit instability are uniformly bounded in

$H^2(\mathbb{R})$ (see (3.23) below). Therefore, if we have a local existence theory for equation (1.1) in $H^2(\mathbb{R})$ (see Assumption 2.1 below) then the trajectories in this case, will be defined for all time and hence the mechanics that leads to instability in our case is not produced by any singularity formation or blow-up of solutions. In this point, we would like to comment that our Assumption 2.1 related with the well-posedness problem to equation (1.1) in $H^2(\mathbb{R})$, is induced by the difficulties that appear when we work with a general non-linear term as $G(u, u_x, u_{xx})$ in (1.1) (see Section 2 to get examples of local existence theory of solutions for equation (1.1)). Finally, we do not know if in a general context the property of instability can be produced by some singularity formation of solutions of (1.1), namely, if there is a set of initial data $u_0 \in H^2(\mathbb{R})$ and $T^* = T^*(\|u_0\|_{H^2})$ with $0 < T^* < \infty$, such that the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies that $\|u(t)\|_{H^2} \to +\infty$ as $t \to T^{*-}$.

We note that similarly as occurred in the approach made in [20], we do not need in our analysis of instability to show the existence of exactly one negative eigenvalue simple and that zero is a simple eigenvalue with eigenfunction φ' associated to the linear operator

$$\begin{aligned} S''(\varphi) =& \partial_x^4 + b\, \partial_x^2 + c - F_{qq}(\varphi, \varphi') + \varphi' F_{qqr}(\varphi, \varphi') + \varphi' F_{qrr}(\varphi, \varphi')\partial_x \\ &+ \varphi'' F_{qrr}(\varphi, \varphi') + \varphi'' F_{rrr}(\varphi, \varphi')\partial_x + F_{rr}(\varphi, \varphi')\partial_x^2, \end{aligned}$$

such as is required in the instabilities theories of [6] and [13]. We would like to add that, in general for our approach, we do not know if there is a mechanics that shows that our choice of the value of the vector field B in φ, produces that the restrictions for $b \neq 0$ in (1.7) are sharp, a situation that does not occur when we use an explicit form for the function $d(c)$. In fact, our restrictions about p in (1.7) when $b = 0$ are sharp in the light of those obtained in [20] to prove stability. In this point we would like to note that there is a typo in Leavandosky's work [20] about the right formula to $d(c)$, where we must say that the correct expression for $d(c)$ is that given by (2.5) below. Finally, we think that a possible good choice for $B(\varphi)$ in cases of evolution equations in one-dimension ($(x, t) \in \mathbb{R} \times \mathbb{R}$) may be $B(\varphi) = \varphi + 2x\varphi'$, which has given sharp results in the study of instability as it has already happened in a recent work of Angulo ([2]) about the instability for solitary-wave solutions of the following generalization of the Benjamin equation (Benjamin [3], [4])

$$u_t + (u^n)_x + l\mathcal{H}u_{xx} + u_{xxx} = 0,$$

where $l \in \mathbb{R}$, $\mathcal{H}$ is the Hilbert transform and $n \in \mathbb{N}$, $n \geq 2$.

The plan of this note is as follows. Section 2 is devoted to give a review on the results known about the stability and instability of solitary-wave solutions to equation (1.1) when $b = 0$ and $F(q, r)$ is homogeneous in both q and r. In this section we also establish an assumption about the problem of local well-posedness to equation (1.1) which will be used in our instability theory, as well as, the existence of solutions for equation (1.4) which are constructed by solving a constrained minimization problem in $H^2(\mathbb{R})$. We finish this section obtaining some regularities and asymptotics properties for solitary-wave solutions of equation (1.1). Section 3 contains our criterium of instability of solitary-wave solutions for (1.1) (see Theorem 3.5 below). Finally in Section 4, we give our main result of instability of the φ-orbit Ω_φ, by the flow of the fifth-order equation (1.1).

Notation. We denote by $\widehat{f}$ the Fourier transform of f, which is defined as $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$. $|f|_{L^p}$ denotes the $L^p(\mathbb{R})$ norm of f, $1 \leq p \leq \infty$. In particular,

$|\cdot|_{L^2} = \|\cdot\|$ and $|\cdot|_{L^\infty} = |\cdot|_\infty$. The inner product of two elements $f, g \in L^2(\mathbb{R})$ will be denote by $\langle f, g\rangle$. We denote by $H^s(\mathbb{R})$, $s \in \mathbb{R}$, the Sobolev space of all f (tempered distributions) for which the norm $\|f\|^2_{H^s} = \int_{-\infty}^{\infty} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi$ is finite.

2. Review and Preliminaries

2.1. **Stability theory for equation (1.1) when $b = 0$ (review).** The problem of existence and stability of solitary-wave solutions of (1.4) with G established in (1.2), has been studied recently by Levandosky in [20]. The results about both stability and instability obtained in [20], were based essentially in the possibility of obtaining a solution to (1.4) as the Euler-Lagrange equation for a constrained minimization problem in $H^2(\mathbb{R})$ (because of this approach we have the assumptions established in (1.2)). More exactly, we have the following main results of [20]: It considers the functionals $I_{c,b}, K \in C^2(H^2(\mathbb{R}); \mathbb{R})$ defined by

$$I_{c,b}(f) = \frac{1}{2} \int_{-\infty}^{\infty} [(f_{xx})^2 - b(f_x)^2 + cf^2] \, dx \tag{2.1}$$

and

$$K(f) = \int_{-\infty}^{\infty} F(f, f_x) \, dx. \tag{2.2}$$

Define for $\lambda > 0$ and $c > b_+^2/4$, where $b_+ = \max\{b, 0\}$, the family of minimization problems

$$M_c(\lambda) = \inf\{I_{c,b}(f) : f \in H^2(\mathbb{R}) \quad \text{and} \quad K(f) = \lambda\}. \tag{2.3}$$

Then using the Concentration Compactness Method (see Lions [21,22]), it is shown in Theorem 2.3 in [20] (see also Theorem 2.2 below) that (2.3) has a solution which after a rescale (using that G is homogeneous) is a classical solution φ_c of (1.4). Now, with regard to the stability theory we have initially the following classical function $d(c)$ (see Grillakis & Shatah & Strauss [13] and Bona & Souganidis & Strauss [6]) defined for $c > b_+^2/4$, by

$$d(c) = E(\varphi_c) + c\, Q(\varphi_c) \tag{2.4}$$

where φ_c is *any* solitary-wave solution of (1.4) obtained via the minimization problem (2.3) and E, Q are defined by (1.5). We note that in terms of E, the evolution equation (1.1) has the following Hamiltonian form

$$u_t = -\partial_x E'(u),$$

and the solitary-wave equation (1.4) takes the form

$$E'(\varphi_c) + c\, Q'(\varphi_c) = 0.$$

Therefore, from the variational characterization of φ_c, (2.1) and (2.2), it is obtained the following main form for $d(c)$:

$$d(c) = \frac{p-1}{p+1} I_{c,b}(\varphi_c) = \frac{p-1}{2} K(\varphi_c) = \frac{p-1}{2} \Big(\frac{2M_c(1)}{p+1}\Big)^{(p+1)/(p-1)}.$$

So, with the condition that $d''(c) > 0$ implies stability, it is obtained that the following set of solitary-wave solutions for (1.1)

$$\mathcal{G}_c = \{\varphi \in H^2(\mathbb{R}) : 2(p-1)I_c(\varphi_c) = (p^2-1)K(\varphi_c) = 2(p+1)d(c)\},$$

is a set stable with respect to the flow generated by (1.1). We note that this criterium of stability works sharply if we have a good expression to $d(c)$. In fact, in the case when $b = 0$ in (1.1) and $F(q, r)$ is homogeneous in r of degree β, $\beta \in [0, p+1]$ (and therefore homogeneous in q of degree α where $\alpha + \beta = p + 1$), it was found in [20] that for all $c > 0$

$$d(c) = d(1)c^\gamma \quad \text{where} \quad \gamma = \frac{3p - 2\beta + 5}{4(p-1)}, \tag{2.5}$$

and therefore we have that the set $\mathcal{G}_c$ will be stable if $2 - \beta < \alpha < 10 - 3\beta$. In particular, when $F(q, r)$ depends only on q we have stability of the set $\mathcal{G}_c$ for $1 < p < 9$. In the case $b \neq 0$ in (1.1) an explicit formula for $d(c)$ is not known yet.

Finally with regard to the problem of instability, it has been established in [20] that if we suppose that "*there exists a choice $\varphi(c)$ which is C^1 as a map from $(b_+^2/4, +\infty)$ to $H^2(\mathbb{R})$ such that $\varphi(c) \in \mathcal{G}_c$*" and $d''(c) < 0$, then the set $\Omega_{\varphi_c} = \{\varphi_c(\cdot + y) \mid y \in \mathbb{R}\}$ is unstable by the flow of the evolution equation (1.1). So, for $b = 0$ and $F(q, r)$ homogeneous in r of degree β, $\beta \in [0, p+1]$, it follows from (2.5) that Ω_{φ_c} will be unstable for $\alpha > 10 - 3\beta$, where $\alpha + \beta = p + 1$. In particular, when $F(q, r)$ depends only on q we have instability of Ω_{φ_c} for $p > 9$. We observe that via our approach of instability, we do not need to use the function d and hence the existence of a smooth curve of solitary-wave solutions depending on c is not necessary in our analysis.

2.2. **Cauchy problem and existence of solitary-wave solutions for equation (1.1).**

In this subsection, we establish an assumption about the initial-value problem to (1.1) and also give some results concerning to the existence of solitary-wave solutions, as well as, about regularities and asymptotic properties of these solutions.

Here we need to make an assumption more than an affirmation about the well-posedness problem to equation (1.1) in the space $H^2(\mathbb{R})$, that is, about existence, uniqueness, persistence property and continuous dependence of the solution upon the initial data. This assumption is naturally induced by the difficulties which appear when we work with a general non-linear term as $G(u, u_x, u_{xx})$ in (1.1). In fact, when $F(u, u_x)$ depends only on u is possible to obtain a local well-posedness (and global also) theory in $H^2(\mathbb{R})$ (see Kato [15], Saut [26]). For more general nonlinearities, for example $F(u, u_x) = uu_x^2 - u^3$ (so that $G(u, u_x, u_{xx}) = -3u^2 - u_x^2 - 2uu_{xx}$), Ponce in [25] has shown a well-posedness theory in $H^s(\mathbb{R})$ with $s \geq 4$. Finally in [17], Kenig & Ponce & Vega have shown that the following higher-order nonlinear dispersive equation

$$\partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \dots, \partial_x^{2j} u) = 0, \quad x, t \in \mathbb{R}, \ \ j \in \mathbb{N},$$

where $P(\cdot)$ is a polynomial having no constant or linear terms, is locally well posed in the weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^m dx)$ for some s (in general sufficiently large) and $m \in \mathbb{N}$.

Assumption 2.1. For any $u_0 \in H^2(\mathbb{R})$ there exist $T = T(\|u_0\|_{H^2}) > 0$ and a unique solution $u(t) \equiv U(t)u_0 \in C([-T, T]; H^2(\mathbb{R}))$ of (1.1) with $u(x, 0) = u_0(x)$. Moreover, for $T_1 < T$ the map $u_0 \to U(t)u_0$ is continuous from $H^2(\mathbb{R})$ to $C([-T_1, T_1]; H^2(\mathbb{R}))$.

Remark: Since our analysis of instability is based in the Sobolev space $H^2(\mathbb{R})$, Assumption 2.1 is sufficient to our purpose. But we can also suppose a local well-posedness theory to equation (1.1) in a general Hilbert space Z such that

$Z \subset H^2(\mathbb{R})$ and the embedding $Z \hookrightarrow H^2(\mathbb{R})$ is continuous, so we can deduce the existence of a continuous curve $u(t) \in C([0,T]; H^2(\mathbb{R}))$ which is solution of (1.1).

Now we establish a result of existence of solitary-wave solutions for (1.4) based in the minimization problem (2.3).

Theorem 2.2. *Suppose G has the structure established in (1.2) and $F(q,r)$ is homogeneous of degree $p+1$ for some $p>1$. Moreover, assume that there is some $f \in H^2(\mathbb{R})$ such that $K(f)>0$ and $c > b_+^2/4$, where $b_+ = \max\{b,0\}$, then:*

(i) *Any minimizing sequence $\{\psi_k\}_{k\in\mathbb{N}}$ for $M_c(\lambda)$ is relatively compact in $H^2(\mathbb{R})$ up to translations, i.e., there are sequences $\{\psi_{n_k}\}_{k\in\mathbb{N}}$ and $\{y_{n_k}\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\psi_{n_k}(\cdot - y_{n_k})$ converges strongly in $H^2(\mathbb{R})$ to some ψ, which is a minimum of $M_c(\lambda)$.*

(ii) *If the variational problem (2.3) has a solution, then there exists a μ such that every corresponding minimum point φ of $M_c(\mu)$ satisfies*

$$\varphi_{xxxx} + b\varphi_{xx} + c\varphi = G(\varphi, \varphi_x, \varphi_{xx}). \tag{2.6}$$

Moreover, $\mu = \left[\frac{M_c(1)}{p+1}\right]^{\frac{p+1}{p-1}}$.

Proof The proof of existence of minimum is based on the Concentration Compactness Method as it was shown in [20]. The value of μ is obtained via the relations

$$qF_q(q,r) + rF_r(q,r) = (p+1)F(q,r)$$

and $M_c(\lambda) = \lambda^{2/(p+1)} M_c(1)$. □

Now we will obtain some regularity and asymptotic properties of the solutions of equation (1.4) obtained via Theorem 2.2. Initially, from the homogeneity of F and (1.2) we have that

$$|G(q,r,s)| \le C\big(|q|^p + |r|^p + |r||q|^{p-1} + |s|[|q|^{p-1} + |r|^{p-1}]\big). \tag{2.7}$$

Since $\varphi \in H^2(\mathbb{R})$, we know from Sobolev's embedding that both φ and φ' are in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ (where $C_0(\mathbb{R})$ is the set of all continuous function on $\mathbb{R}$ that converges to zero at the infinity) and so from (2.7), we get $G(\varphi,\varphi',\varphi'') \in L^2(\mathbb{R})$. Since φ is a solution of (2.6) we have then that $\varphi \in H^4(\mathbb{R})$ and so

$$\varphi, \varphi', \varphi'', \varphi''' \in L^\infty(\mathbb{R}) \cap C_0(\mathbb{R}). \tag{2.8}$$

Moreover, from (1.2) it follows that $G(\varphi,\varphi',\varphi'') \in C^1(\mathbb{R})$ and from Sobolev's embedding theorem it follows that

$$|G(\varphi, \varphi_y, \varphi_{yy})|_{L^1} \le C\|\varphi\|_{H^2}.$$

Now, our attention is turned to the asymptotic properties of solutions for (2.6). Initially, we choose c such that $min_{x\in\mathbb{R}}\{c - b\xi^2 + \xi^4\} > 0$, that means, $c > b_+^2/4$ where $b_+ = \max\{b,0\}$, so we can write

$$k(\xi) \equiv \frac{1}{\xi^4 - b\xi^2 + c} = \frac{1}{\xi^4 + 2\alpha^2 cos(2\theta)\xi^2 + \alpha^4} \tag{2.9}$$

where $\alpha = c^{1/4}$ and $cos(2\theta) = -\frac{b}{2\sqrt{c}}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then from [10] we find that there exists an even function $Z \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ such that its Fourier transform satisfies $\widehat{Z}(\xi) = k(\xi)$, more precisely, we have

$$Z(x) = \frac{\pi}{2}\alpha^{-1/3} e^{-\alpha|x|\cos\theta} \sin(\theta + \alpha|x|\sin\theta)\csc(2\theta). \tag{2.10}$$

Note that $Z \in C^\infty(\mathbb{R}-\{0\})$ and $Z^{(n)} \in L^1(\mathbb{R})$ for $n = 1, 2, 3, \dots$. Now, by writing $\mathcal{M} = \partial_x^4 + b\partial_x^2$ we can rewrite equation (2.6) in the integral form

$$\varphi(x) = (c+\mathcal{M})^{-1}G(\varphi, \varphi', \varphi'')(x) = \int_{-\infty}^{\infty} Z(x-y)G(\varphi, \varphi_y, \varphi_{yy})\,dy. \tag{2.11}$$

So, from Young's inequality on convolutions we get that for all n

$$\begin{gathered} |\varphi^{(n)}|_{L^1} \le |Z^{(n)}|_{L^1}|G(\varphi, \varphi_y, \varphi_{yy})|_{L^1} < \infty, \\ \|\varphi^{(n)}\|_{L^2} \le |Z^{(n)}|_{L^1}\|G(\varphi, \varphi_y, \varphi_{yy})\|_{L^2} < \infty. \end{gathered}$$

Finally, from the following results: (2.8), $G(\cdot,\cdot,\cdot) \in C^1(\mathbb{R}^3)$, $c > b_+^2/4$, and the stable-manifold theorem (see Lemma 2.4 in [20]) we have that $\varphi, \varphi', \varphi'', \varphi'''$ decay exponentially to zero at $\pm\infty$. Then using (2.11), (2.7) and (2.8), we obtain that for all n, $\varphi^{(n)}$ also decay exponentially to zero at $\pm\infty$. Therefore, we have shown the following Lemma.

Lemma 2.3. *Suppose that φ is a solution of (2.6) obtained via Theorem 2.2-(ii). Then $\varphi \in H^n(\mathbb{R})$, $\varphi^{(n)} \in L^1(\mathbb{R})$, $n = 0, 1, 2, \dots$, and there exist $\sigma > 0$ and constants $C_n > 0$ such that*

$$|\varphi^{(n)}(x)| \le C_n e^{-\sigma|x|}, \quad \textit{for all } x \in \mathbb{R}.$$

Moreover, $x\varphi^{(n)} \in L^1(\mathbb{R})$ for $n \in \mathbb{N}$.

We finish this Section giving for $u(t)$, solution of (1.1), an estimate of how fast its tail near infinity grows with t. This estimate will be a key step in the proof of instability. We note that this type of information for solutions of dispersive evolution equations has already used as an essential ingredient in studies about instability (see [6] and [28]).

Theorem 2.4. *Let $p \ge 2$ and suppose Assumption 2.1. Assume $u_0 \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\mathcal{M}u_0 \equiv (\partial_x^4 + b\partial_x^2)u_0 \in L^1(\mathbb{R})$. If $u(t)$ is the solution of (1.1) with initial data $u(0) = u_0$, then*

$$\sup_{-\infty<x<\infty}\Big|\int_{-\infty}^{x} u(y,t)\,dy\Big| \le C\Big(1 + t^{\theta/(1+\theta)}\Big)$$

for $0 \le t < T_0$, where T_0 denotes the existence time for u. Moreover, $\theta = 2$ if $b \ne 0$, $\theta = 4$ if $b = 0$ and the constant C depends only on sup $\{\|u(t)\|_{H^2} \;:\; 0 \le t < T_0\}$, $|u_0|_{L^1}$ and $|\mathcal{M}u_0|_{L^1}$.

Proof The proof follows the same lines of the proof of Theorem 4.3 in [28] or Lemma 5.12 in [20], and is based essentially on the following estimate for the free group $U_0(t)$ associated to the linear evolution equation

$$u_t + u_{xxxxx} + bu_{xxx} = 0, \qquad u(0) = u_0,$$

namely, for all $t > 0$ we have

$$|U_0(t)u_0|_\infty \le C_0(1+t)^{-1/(1+\theta)}|u_0|_{L^1}$$

where $\theta = 2$ if $b \ne 0$ and $\theta = 4$ if $b = 0$. $\square$

3. Instability Theory for Equation (1.1)

In this section the attention is turned to establish a criterium of instability for solitary-wave solutions associated to equation (1.1), which are obtained via the variational problem (2.3). By using the notations from [11], for any $X \subset H^2(\mathbb{R})$ and $\delta > 0$ we define $\mathcal{V}(X,\delta)$, the δ-neighbourhood of X in $H^2(\mathbb{R})$, by

$$\mathcal{V}(X,\delta) = \cup_{v\in X} B_\delta(v) = \{g \in H^2(\mathbb{R}) : \inf_{v\in X} \|v-g\|_{H^2} < \delta\}$$

where $B_\delta(v) = \{u \in H^2(\mathbb{R}) : \|u-v\|_{H^2} < \delta\}$. So, based in Assumption 2.1 we have the following definition of stability.

Definition 3.1. We say that $X \subset H^2(\mathbb{R})$ is stable by the flow of (1.1) if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $u_0 \in \mathcal{V}(X,\delta)$, the solution $u(t)$ of (1.1) with initial data $u(0) = u_0$ satisfies $u(t) \in \mathcal{V}(X,\epsilon)$ for all $t \in \mathbb{R}$. Otherwise, we say X is unstable by the flow of (1.1).

For each $y \in \mathbb{R}$, let τ_y be the translation operator defined by $\tau_y v = v(\cdot + y)$. If $Y \subset L^p(\mathbb{R})$, we denote $\Omega_Y = \{\tau_y v \mid y \in \mathbb{R}, v \in Y\}$. With this notation we introduce for φ solution of (1.4), the set $\Omega_\varphi \equiv \{\tau_y\varphi \mid y \in \mathbb{R}\}$, called the φ-orbit.

In the next, we give some basic Lemmas which will be used in the proof of the criterium of instability given in Theorem 3.5.

Lemma 3.2. *There exist $\epsilon_0 > 0$ and a unique function $\Lambda : \mathcal{V}(\Omega_\varphi,\epsilon_0) \to \mathbb{R}$, which is a C^2-functional, such that $\Lambda(\varphi) = 0$ and such that for all $v \in \mathcal{V}(\Omega_\varphi,\epsilon_0)$ and all $w \in \Omega_\varphi$, $\|v - \tau_{\Lambda(v)}\varphi\| \le \|v-w\|$. Moreover, for $y \in \mathbb{R}$*

$$\begin{gathered}\Lambda(\tau_y v) = \Lambda(v) + y\\ \Lambda'(v) = -\frac{1}{\langle v, \varphi''(\cdot + \Lambda(v))\rangle}\varphi'(\cdot + \Lambda(v)).\end{gathered}\tag{3.1}$$

In particular, for $v \in \Omega_\varphi$ we have

$$\begin{gathered}\langle \Lambda'(v), v\rangle = 0,\\ \Lambda'(v) = \frac{1}{\|\varphi'\|^2}v'.\end{gathered}\tag{3.2}$$

For the proof of this lemma see Theorem 3.1 in [11] or Lemma 3.3 in [2].

Remark: We note that the value $\Lambda(v)$, obtained in Lemma 3.2, can be seen as the "optimal" translation for φ to be the best approximation of v to the φ-orbit, Ω_φ, in the $L^2(\mathbb{R})$-norm .

Now, we define our main vector field in the study of instability. We consider initially a function $\psi \in L^\infty(\mathbb{R})$ such that $\psi' \in H^2(\mathbb{R})$ and φ a nonzero-solution of (1.4), then for $v \in V(\Omega_\varphi,\epsilon_0)$ we define $B_\psi(v)$ by the formula

$$B_\psi(v) \equiv \tau_{\Lambda(v)}\psi' - \frac{\langle v, \tau_{\Lambda(v)}\psi'\rangle}{\langle v, \tau_{\Lambda(v)}\varphi''\rangle}\,\tau_{\Lambda(v)}\varphi''.\tag{3.3}$$

We note that the vector field B_ψ is an extension of the formula (4.2) in [6] as well as of the formula (5.9) in [20] (a similar vector field was also used in [2] and [11]). Now, for $v \in \Omega_\varphi$ we can obtain an easy geometric interpretation of the vector $B_\psi(v)$ as being the derivative of the " orthogonal component of $\tau_{\Lambda(v)}\psi$ with regard to $\tau_{\Lambda(v)}\varphi'$ ". In fact, let $y \in \mathbb{R}$ such that $v = \tau_y\varphi$. Then from Lemma 3.2 it follows that

$\Lambda(v) = y$. Now, even that $\psi \notin L^2(\mathbb{R})$ we have from Lemma 2.3 that

$$\frac{\langle v, \tau_{\Lambda(v)}\psi' \rangle}{\langle v, \tau_{\Lambda(v)}\varphi'' \rangle}\,\tau_y\varphi' = \frac{\langle \tau_y\varphi, \tau_y\psi' \rangle}{\langle \tau_y\varphi, \tau_y\varphi'' \rangle}\,\tau_y\varphi' = \frac{\langle \tau_y\varphi', \tau_y\psi \rangle}{\|\tau_y\varphi'\|^2}\,\tau_y\varphi' \equiv P_{\shortparallel}.$$

So, defining $Q_\perp \equiv \tau_y\psi - P_{\shortparallel}$ we obtain that $\langle Q_\perp, \tau_y\varphi' \rangle = 0$ and $B_\psi(v) = \partial_x Q_\perp$.

Next, we have some basic properties of the vector field B_ψ.

Lemma 3.3. *For $\psi \in L^\infty(\mathbb{R})$ such that $\psi', \psi'' \in H^2(\mathbb{R})$ and φ nonzero-solution of (1.4), we have*

$$B_\psi : V(\Omega_\varphi, \epsilon_0) \to H^2(\mathbb{R}) \quad \text{is } C^1 \quad \text{with bounded derivative}, \tag{3.4}$$

$$B_\psi \quad \text{commutes with translations}, \tag{3.5}$$

$$\langle B_\psi(v), v \rangle = 0 \quad \text{for all } v \in V(\Omega_\varphi, \epsilon_0), \tag{3.6}$$

$$\text{if } \langle \varphi, \psi' \rangle = 0 \quad \text{then} \quad B_\psi(\varphi) = \psi', \tag{3.7}$$

$$\text{if } \langle \varphi', \psi' \rangle = 0 \quad \text{then} \quad \langle B_\psi(v), v' \rangle = 0, \quad \text{for all } v \in \Omega_\varphi. \tag{3.8}$$

Proof The proof of statements (3.4), (3.5), (3.6) follows the same lines from the proof of Lemma 3.5 in [2]. Statements (3.7) and (3.8) follow from equality $\Lambda(\varphi) = 0$ and from the formula

$$B_\psi(\varphi) = \psi' + \frac{\langle \varphi, \psi' \rangle}{\|\varphi'\|^2}\varphi''. \tag{3.9}$$

□

Before establishing our instability criterium, we need to assure the convergence of an integral which will be used in our analysis.

Lemma 3.4. *Let $p \geq 2$ and φ a solution of (2.6) obtained via Theorem 2.2-(ii) . Assume that $u_0 \in V(\Omega_\varphi, \epsilon_0)$ such that $u_0, \mathcal{M}u_0 \equiv (\partial_x^4 + b\partial_x^2)u_0 \in L^1(\mathbb{R})$. If $u(t)$ is a solution of (1.1) which corresponds to the initial data u_0 and $u(t) \in V(\Omega_\varphi, \epsilon_0)$ for $t \in [0, T_1]$, then we have that for $\psi(x) = \int_{-\infty}^x [\varphi(y) + 2y\varphi'(y)]dy$ it follows that*

$$\mathcal{A}_\psi(u(t)) \equiv \int_{-\infty}^{\infty} \psi(x - \Lambda(u(t)))u(x,t)\,dx < \infty, \qquad \text{for all } t \in [0, T_1]. \tag{3.10}$$

Proof Let H be the Heaviside function and $\gamma = \int_{\mathbb{R}}[\varphi(y) + 2y\varphi'(y)]dy$ (note that $\gamma < \infty$ from Lemma 2.3) then

$$\mathcal{A}_\psi(u(t)) = \int_{-\infty}^{\infty} [\psi(x - \Lambda(u(t))) - \gamma H(x - \Lambda(u(t)))]u(x,t)dx + \gamma \int_{\Lambda(u(t))}^{\infty} u(x,t)dx.$$

It follows then from Cauchy-Schwarz inequality, (1.5) and Theorem 2.4 that

$$|\mathcal{A}_\psi(u(t))| \leq \|\psi - \gamma H\|\|u_0\| + |\gamma|C_0\Big(1 + t^{\theta/(1+\theta)}\Big),$$

and therefore

$$|\mathcal{A}_\psi(u(t))| \leq K_1\Big(1 + t^{\theta/(1+\theta)}\Big) \tag{3.11}$$

where θ is given by Theorem 2.4 and K_1 is a constant. To show that $\psi - \gamma H \in L^2(\mathbb{R})$ it is sufficient to know that $\int_{\mathbb{R}} |y|^{1/2}|\varphi(y)| + |y|^{3/2}|\varphi'(y)|dy < \infty$, which is true by Lemma 2.3. This proves the Lemma. □

Now we establish a sufficient condition of instability for solitary-wave solutions of equation (1.1) which are obtained via the variational problem (2.3).

Theorem 3.5 (Criterium of Instability). *Let $p \geq 2$ and φ be a solution from (1.4) obtained via Theorem 2.2-(ii) with $\lambda = \mu$. If there is $\psi \in L^\infty(\mathbb{R})$ such that $\psi', \psi'' \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $\psi''', \psi^{(v)} \in L^1(\mathbb{R})$, and it is chosen such that (3.11) is true with $0 < \theta/(1+\theta) < 1$ and*

$$\langle S''(\varphi)B_\psi(\varphi), B_\psi(\varphi)\rangle < 0 \tag{3.12}$$

where $S(u) = E(u) + c\,Q(u)$, then there exist $\epsilon > 0$ and a sequence $\{u_0^j\}$ in $V(\Omega_\varphi, \epsilon)$ satisfying:

(i) *$u_0^j \to \varphi$ in $H^2(\mathbb{R})$ as $j \to \infty$.*
(ii) *For all $j \in \mathbb{N}$, $u(t) = U(t)u_0^j$ is uniformly bounded in $H^2(\mathbb{R})$, but escapes from $V(\Omega_\varphi, \epsilon)$ in a finite time.*

The proof of this theorem follows the same spirit of ideas as those established in [11] and [2] (see also [27]), therefore we will explain only the main points of the argument. We will divide the proof in a few Lemmas.

Lemma 3.6. *Let φ be a solution from (1.4) obtained via Theorem 2.2-(ii) with $\lambda = \mu$. Suppose that there is $\psi \in L^\infty(\mathbb{R})$ such that $\psi', \psi'' \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $\psi''', \psi^{(v)} \in L^1(\mathbb{R})$ and the action S satisfies (3.12). Then there are positive numbers ϵ_3 and σ_3 such that*

$$\text{for} \quad v_0 \in V(\Omega_\varphi, \epsilon_3), \quad \text{there exists } s \in (-\sigma_3, \sigma_3): \quad S(\varphi) \leq S(v_0) + P(v_0)s$$

where $P(u) = \langle S'(u), B_\psi(u)\rangle$.

Proof We consider for $v_0 \in V(\Omega_\varphi, \epsilon_0)$, where ϵ_0 is given by Lemma 3.2, the initial-value problem

$$\begin{gathered} \frac{d}{ds}v(s) = B_\psi(v(s)) \\ v(0) = v_0. \end{gathered} \tag{3.13}$$

So, from (3.4) we have that (3.13) admits for each $v_0 \in V(\Omega_\varphi, \epsilon_0)$ a unique maximal solution $v \in C^2((-\sigma, \sigma); V(\Omega_\varphi, \epsilon_0))$, where $v(0) = v_0$ and $\sigma = \sigma(v_0) \in (0, \infty]$. Moreover, for each $\epsilon_1 < \epsilon_0$, there exists $\sigma_1 > 0$ such that $\sigma(v_0) \geq \sigma_1$, for all $v_0 \in V(\Omega_\varphi, \epsilon_1)$. Hence, we can define for fixed ϵ_1, σ_1 the following dynamical system

$$\begin{gathered} \mathcal{W} : (-\sigma_1, \sigma_1) \times V(\Omega_\varphi, \epsilon_1) \to V(\Omega_\varphi, \epsilon_0) \\ (s, v_0) \to \mathcal{W}(s)v_0, \end{gathered}$$

where $s \to \mathcal{W}(s)v_0$ is the maximal solution of (3.13) with initial data v_0. Now we establish some basic properties of the flow $s \to \mathcal{W}(s)v_0$. In fact, from Lemma 3.3 one has that $\mathcal{W}$ is a C^1-function, also we have that for each $v_0 \in V(\Omega_\varphi, \epsilon_1)$ the function $s \in (-\sigma_1, \sigma_1) \to \mathcal{W}(s)v_0$ is C^2 and the flow $s \to \mathcal{W}(s)v_0$ *commutes with translations*. Finally, from the relation

$$\mathcal{W}(s)\varphi = \varphi + \int_0^s \tau_{\Lambda(\mathcal{W}(t)\varphi)}\psi'\,dt - \int_0^s D(t)\tau_{\Lambda(\mathcal{W}(t)\varphi)}\varphi''\,dt,$$

(where the map $s \in (-\sigma_1, \sigma_1) \to D(s)$ is a continuous function), Lemma 2.3 and the hypothesis on ψ, we have the following consequences

$$\mathcal{W}(s)\varphi,\ \partial_x^2\mathcal{W}(s)\varphi,\ \partial_x^4\mathcal{W}(s)\varphi \in L^1(\mathbb{R}) \quad \text{for all } s \in (-\sigma_1, \sigma_1). \tag{3.14}$$

Now, given $v_0 \in V(\Omega_\varphi, \epsilon_1)$ we get from Taylor's Theorem that there is $\theta \in (0, 1)$ such that $S(\mathcal{W}(s)v_0) = S(v_0) + P(v_0)s + \frac{1}{2}R(\mathcal{W}(\theta s)v_0)s^2$, where P and

R are functionals defined on $V(\Omega_\varphi, \epsilon_1)$ by $P(v) = \langle S'(v), B_\psi(v)\rangle$ and $R(v) = \langle S''(v)B_\psi(v), B_\psi(v)\rangle + \langle S'(v), B'_\psi(v)(B_\psi(v))\rangle$. Since R, $\mathcal{W}$ are continuous, $S'(\varphi) = 0$ and $R(\varphi) < 0$, then there exist $\epsilon_2 \in (0, \epsilon_1]$, $\sigma_2 \in (0, \sigma_1]$ such that

$$S(\mathcal{W}(s)v_0) \leq S(v_0) + P(v_0)s \quad \text{for } v_0 \in B_{\epsilon_2}(\varphi), \quad s \in (-\sigma_2, \sigma_2). \tag{3.15}$$

Since $\mathcal{W}(s)v_0$ *commutes with translations*, we obtain from the relation $V(\Omega_\varphi, \epsilon_2) = \Omega_{B_{\epsilon_2}(\varphi)}$, the extension

$$\begin{gathered} \text{for } v_0 \in V(\Omega_\varphi, \epsilon_2), \quad s \in (-\sigma_2, \sigma_2) \\ S(\mathcal{W}(s)v_0) \leq S(v_0) + P(v_0)s. \end{gathered} \tag{3.16}$$

So that, for $v_0 = \mathcal{W}(\tau)\varphi$ with $\tau \neq 0$ small enough, we get

$$S(\varphi) \leq S(\mathcal{W}(\tau)\varphi) - P(\mathcal{W}(\tau)\varphi)\tau. \tag{3.17}$$

Moreover, from (3.12) the function $\tau \to S(\mathcal{W}(\tau)\varphi)$ has a strict local maximum in 0 and therefore

$$S(\mathcal{W}(\tau)\varphi) < S(\varphi) \quad \text{for } \tau \in (-\sigma_2, \sigma_2),\ \tau \neq 0, \tag{3.18}$$

and so from (3.17) and (3.18) we have that for some $\sigma_3 \leq \sigma_2$

$$P(\mathcal{W}(\tau)\varphi) < 0, \quad \tau \in (0, \sigma_3). \tag{3.19}$$

On the other hand, we have that for $K(v) = \int_{-\infty}^{\infty} F(v, v_x)\,dx$

$$\langle K'(\varphi), B_\psi(\varphi)\rangle \neq 0, \tag{3.20}$$

otherwise, $B_\psi(\varphi)$ would be tangent to $\mathcal{F} = \{v \in H^2(\mathbb{R})|\ K(v) = \mu\}$ and then one would have $\langle S''(\varphi)B_\psi(\varphi), B_\psi(\varphi)\rangle \geq 0$ (since φ minimizes S on $\mathcal{F}$ by Theorem 2.2), but this is a contradiction with (3.12). So, if we consider the function

$$(v_0, s) \in V(\Omega_\varphi, \epsilon_1) \times (-\sigma_1, \sigma_1) \longrightarrow K(\mathcal{W}(s)v_0) = \int_{-\infty}^{\infty} F(\mathcal{W}(s)v_0, \partial_x \mathcal{W}(s)v_0)\,dx,$$

which is a C^1- mapping with $(\varphi, 0) \to K(\varphi) = \mu$, it follows from (3.20) that

$$\frac{d}{ds} K(\mathcal{W}(s)v_0)\Big|_{(\varphi,0)} = \langle K'(\varphi), B_\psi(\varphi)\rangle \neq 0,$$

and therefore from the Implicit Function Theorem it follows that, there exist $\epsilon_3 \in (0, \epsilon_2)$ and $\sigma_3 \in (0, \sigma_2]$ such that for all $v_0 \in B_{\epsilon_3}(\varphi)$, there exists a unique $s \equiv s(v_0) \in (-\sigma_3, \sigma_3)$ such that $K(\mathcal{W}(s)v_0) = \mu$, *i.e.* $\mathcal{W}(s)v_0 \in \mathcal{F}$. Then, applying (3.15) to $(v_0, s(v_0)) \in B_{\epsilon_3}(\varphi) \times (-\sigma_3, \sigma_3)$ and using that φ minimizes S on $\mathcal{F}$, we have that for $v_0 \in B_{\epsilon_3}(\varphi)$ there exists $s \in (-\sigma_3, \sigma_3)$ such that $S(\varphi) \leq S(v_0) + P(v_0)s$. Therefore, using again that $\mathcal{W}(s)v_0$ *commutes with translations* and $V(\Omega_\varphi, \epsilon_2) = \Omega_{B_{\epsilon_2}(\varphi)}$ we obtain the extension

$$\text{for each } v_0 \in V(\Omega_\varphi, \epsilon_3), \quad \text{there exists } s \in (-\sigma_3, \sigma_3): \quad S(\varphi) \leq S(v_0) + P(v_0)s.$$

This shows the Lemma. □

Since $B_{-\psi}(\varphi) = -B_\psi(\varphi)$ we can assume from (3.20) that $\langle K'(\varphi), B_\psi(\varphi)\rangle < 0$. So, if we consider the continuous flow $\tau \to \mathcal{W}(\tau)\varphi$ which is solution of (3.13) with initial data φ, we will have for $\tau > 0$ and small enough that we can get some δ small such that

$$K(\mathcal{W}(\tau)\varphi) = K(\varphi) + \int_0^\tau \langle K'(\mathcal{W}(\xi)\varphi), B_\psi(\mathcal{W}(\xi)\varphi)\rangle\,d\xi = \mu - \delta < \mu\,. \tag{3.21}$$

Lemma 3.7. *Suppose φ, ψ and S satisfy the same hypotheses as those established in Lemma 3.6 and ψ is chosen such that (3.11) is true with $0 < \theta/(1+\theta) < 1$. Define*

$$\mathbb{D} = \{v \in H^2(\mathbb{R})|\ S(v) < S(\varphi),\ K(v) < \mu\} \cap \{v \in H^2(\mathbb{R})|\ P(v) < 0\} \equiv \mathbb{B} \cap \mathcal{P} \tag{3.22}$$

where $P(v) = \langle S'(v), B_\psi(v)\rangle$. Then we have:

(i) *$\mathcal{W}(\tau)\varphi \in \mathbb{D}$ for all $\tau \in (0, \sigma_3)$.*
(ii) *$\mathbb{B}$ is invariant by the flow $u(t)$ of equation (1.1).*
(iii) *The flow $u(t)$ of (1.1) with initial data in $\mathbb{B}$ has a uniformly bounded trajectory in $H^2(\mathbb{R})$*
(iv) *For $\mathcal{A}_\psi$ defined in (3.10) we have $\partial_t \mathcal{A}_\psi(u(t)) = -P(u(t))$.*

Proof (*i*) From (3.18), (3.19) and (3.21) we get that the flow generated by (3.13) through φ satisfies $\mathcal{W}(\tau)\varphi \in \mathbb{D}$ for all $\tau \in (0, \sigma_3)$.

(*ii*) Let $u_0 \in \mathbb{B} \subset H^2(\mathbb{R})$. Then from Assumption 2.1 (local existence theory), there exists $T_0 > 0$ such that $u(t) = U(t)u_0 \in H^2(\mathbb{R})$ and satisfies (1.1) for all $t \in [0, T_0)$. So from (1.5) we have $S(U(t)u_0) = S(u_0) < S(\varphi)$. Therefore from this last relation and the property of minimization of S on $\mathcal{F} = \{v \in H^2(\mathbb{R})|\ K(v) = \mu\}$ by φ, we have that for all $t \in [0, T_0)$, $K(U(t)u_0) \neq \mu$. Finally, since $t \to K(U(t)u_0)$ is continuous on $[0, T_0)$ we obtain that $K(U(t)u_0) < \mu$ for all $t \in [0, T_0)$. This shows property (ii).

(*iii*) Let $u_0 \in \mathbb{B}$. Then from the conservation quantities in (1.5) by the flow of equation (1.1), we have that

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty} [\partial_x^2 U(t)u_0]^2 - b[\partial_x U(t)u_0]^2 + c[U(t)u_0]^2 dx &= S(U(t)u_0) + K(U(t)u_0)\\ &< S(\varphi) + \mu \end{aligned}\tag{3.23}$$

for all $t \in [0, T_0)$. So, if we have $b = 0$ in (3.23) then a simple interpolation argument shows that there is a constant $M(\varphi, \mu) > 0$ such that $\|\partial_x U(t)u_0\| \le M(\varphi, \mu)$ and therefore $\|U(t)u_0\|^2_{H^2} \le M(\varphi, \mu)$ for all $t \in [0, T_0)$. If $b > 0$ then from restriction $c > b^2/4$ we get

$$\begin{aligned}&\frac{1}{2}\Big(1 - \frac{b}{2\sqrt{c}}\Big) \int_{-\infty}^{\infty} [\partial_x^2 U(t)u_0]^2 + c[U(t)u_0]^2 dx\\ &\le \frac{1}{2}\int_{-\infty}^{\infty} [\partial_x^2 U(t)u_0]^2 - b[\partial_x U(t)u_0]^2 + c[U(t)u_0]^2 dx < S(\varphi) + \mu,\end{aligned}$$

and again an interpolation argument shows that $\|U(t)u_0\|^2_{H^2} \le M(\varphi, \mu)$ for all $t \in [0, T_0)$. Note that this last fact shows that the flow $t \to U(t)u_0$ will be a global solution for equation (1.1) if we have in fact a local existence theory of solutions in $H^2(\mathbb{R})$.

(*iv*) Let $u(t)$ be a solution of (1.1). Then as long as this flow remains in $V(\Omega_\varphi, \epsilon)$ (note that from hypothesis $\mathcal{A}_\psi(u(t)) < \infty$) we have that

$$\begin{aligned}\partial_t \mathcal{A}_\psi(u(t)) &= \int_{-\infty}^{\infty} \Big(\langle \tau_{\Lambda(u(t))}\psi'(x), u(t)\rangle \Lambda'(u(t)) + \tau_{\Lambda(u(t))}\psi(x)\Big)\frac{\partial u}{\partial t}(x,t)\, dx\\ &= -\langle\langle \tau_{\Lambda(u(t))}\psi'(x), u(t)\rangle \frac{d}{dx}\Lambda'(u(t)) + \tau_{\Lambda(u(t))}\psi'(x), E'(u(t))\rangle\\ &= -\langle B_\psi(u(t)), S'(u(t))\rangle + c\langle B_\psi(u(t)), u(t)\rangle = -P(u(t)),\end{aligned}\tag{3.24}$$

where in the last equality we have used the fact that $\langle B_\psi(v), v\rangle = 0$ for all $v \in V(\Omega_\varphi, \epsilon)$. This shows the Lemma. □

Proof of Theorem 3.5 By (i) in Lemma 3.7, let $\tau_j \in (0, \sigma_3)$ such that $\tau_j \to 0$ as $j \to \infty$ and define $u_0^j \equiv \mathcal{W}(\tau_j)\varphi$. Then $u_0^j \to \varphi$ in $H^2(\mathbb{R})$ as $j \to \infty$. Since $u_0^j \in \mathbb{D} \subset \mathbb{B}$, it follows from Lemma 3.7 that $t \to u(t) \equiv U(t)u_0^j$ is uniformly bounded for all j. So, to conclude the proof of our criterium of instability we need only to verify that $t \to U(t)u_0^j$ escapes from $V(\Omega_\varphi, \epsilon_3)$ for some $\epsilon_3 > 0$ and for all $j \in \mathbb{N}$ in a finite time. In fact, let $\epsilon_3 > 0$ determined in Lemma 3.6 and define

$$T_j = \sup\{\tau > 0 : U(t)u_0^j \in V(\Omega_\varphi, \epsilon_3), \quad \text{for all } t \in (0, \tau)\}.$$

Then, it follows from Lemma 3.6 that for all $j \in \mathbb{N}$ and $t \in (0, T_j)$ there exists $s = s_j(t) \in (-\sigma_3, \sigma_3)$ satisfying $S(\varphi) \le S(U(t)u_0^j) + P(U(t)u_0^j)s = S(u_0^j) + P(U(t)u_0^j)s$. Now, since $u_0^j \in \mathbb{D}$ then $t \to U(t)u_0^j \in \mathcal{P}$ for $t \in (0, T_j)$, so we have that

$$-P(U(t)u_0^j) \ge \frac{S(\varphi) - S(u_0^j)}{\sigma_3} = \eta_j > 0, \quad \text{for all} \quad t \in (0, T_j). \tag{3.25}$$

Now suppose that for some j, $T_j = +\infty$. Then from the properties obtained by the flow $\tau \to \mathcal{W}(\tau)\varphi$ in (3.14) and considering that (3.11) is true, we obtain from (3.24) and (3.25) that $\mathcal{A}_\psi(U(t)u_0^j)) \ge t\eta_j + \mathcal{A}_\psi(u_0^j)$ for all $t \in (0, +\infty)$. Then from (3.11) it follows

$$K_1 \ge \frac{t\eta_j + \mathcal{A}_\psi(u_0^j)}{1 + t^{\theta/(1+\theta)}} \quad \text{for all } t \in (0, +\infty),$$

which is a contradiction . Therefore $T_j < +\infty$, which means that $u(t) = U(t)u_0^j$ eventually leaves $V(\Omega_\varphi, \epsilon_3)$. This proves the Theorem. □

4. Instability of the Orbit Ω_φ

In this section we give conditions to assure inequality (3.12) and so to obtain the instability of the φ-orbit, Ω_φ, with respect to equation (1.1). For that, we start showing the following Lemma.

Lemma 4.1. *Let $p \ge 2$ and suppose that $F(q, r)$, homogeneous of degree $p + 1$, satisfies relation (1.2). Consider φ a solution of (1.4) obtained via Theorem 2.2-(ii) with $\lambda = \mu$. Then for $\psi(x) = \int_{-\infty}^x [\varphi(y) + 2y\varphi'(y)]dy$ we get that*

$$\begin{aligned} &\langle S''(\varphi)B_\psi(\varphi), B_\psi(\varphi)\rangle \\ &= 8b \int_{-\infty}^{\infty} (\varphi')^2\,dx + (9-p)(p-1)\int_{-\infty}^{\infty} F(\varphi, \varphi')\,dx \\ &\quad + 4\int_{-\infty}^{\infty} [4\varphi' F_r(\varphi, \varphi') - \varphi\varphi' F_{qr}(\varphi, \varphi') - 2(\varphi')^2 F_{rr}(\varphi, \varphi')]\,dx. \end{aligned} \tag{4.1}$$

Proof Since $\langle \psi', \varphi\rangle = 0$ it follows immediately from (3.9) that $B_\psi(\varphi) = \psi'$ and therefore we need only to estimate the quantity $\langle S''(\varphi)\psi', \psi'\rangle$. In fact, it denotes by $\mathcal{L} = S''(\varphi)$ the linear operator

$$\begin{aligned} \mathcal{L} =& \partial_x^4 + b\,\partial_x^2 + c - F_{qq}(\varphi, \varphi') + \varphi' F_{qqr}(\varphi, \varphi') + \varphi' F_{qrr}(\varphi, \varphi')\partial_x \\ &+ \varphi'' F_{qrr}(\varphi, \varphi') + \varphi'' F_{rrr}(\varphi, \varphi')\partial_x + F_{rr}(\varphi, \varphi')\partial_x^2. \end{aligned} \tag{4.2}$$

Initially, from the homogeneity of F we obtain the relations

$$\begin{gathered}\varphi F_q(\varphi,\varphi') + \varphi' F_r(\varphi,\varphi') = (p+1)F(\varphi,\varphi'),\\ \varphi^2 F_{qq}(\varphi,\varphi') + 2\varphi\varphi' F_{qr}(\varphi,\varphi') + (\varphi')^2 F_{rr}(\varphi,\varphi') = p(p+1)F(\varphi,\varphi').\end{gathered} \tag{4.3}$$

Now, using (1.2) and (1.4), we have from (4.2)

$$\begin{gathered}\mathcal{L}\varphi = F_q - \varphi' F_{qr} - \varphi F_{qq} + \varphi(F_{qr})_x + \varphi'(F_{rr})_x\\ \mathcal{L}(x\varphi') = -2b\varphi'' - 4c\varphi + 4F_q - 4\varphi' F_{qr} - 2\varphi'' F_{rr} + (\varphi')^2 F_{qrr} + \varphi'\varphi'' F_{rrr}.\end{gathered} \tag{4.4}$$

So, using the first equation in (4.4), integration by parts and (4.3) we get that

$$\begin{aligned}\langle \mathcal{L}\varphi, \varphi\rangle &= \int_{-\infty}^{\infty} [\varphi F_q - \varphi\varphi' F_{qr} - \varphi^2 F_{qq} + \varphi^2 (F_{qr})_x + \varphi\varphi'(F_{rr})_x]\,dx\\ &= \int_{-\infty}^{\infty} [\varphi F_q - \varphi\varphi' F_{qr} - \varphi\varphi'' F_{rr} - p(p+1)F]\,dx\\ &= \int_{-\infty}^{\infty} [\varphi F_q + \varphi' F_r - p(p+1)F]\,dx = (p+1)(1-p)\int_{-\infty}^{\infty} F\,dx.\end{aligned} \tag{4.5}$$

Now, from the first equation in (4.4), some integrations by parts and the first equation of (4.3), we estimate the following term

$$\begin{aligned}\langle \mathcal{L}\varphi, x\varphi'\rangle &= \int_{-\infty}^{\infty} [-\varphi F_q - 2x\varphi(F_r)_x - 2x\varphi'(F_q)_x - \varphi\varphi' F_{qr} - (\varphi')^2 F_{rr}]\,dx\\ &= \int_{-\infty}^{\infty} [-\varphi F_q + 2x(F)_x + 2(\varphi F_q + \varphi' F_r) - \varphi\varphi' F_{qr} - (\varphi')^2 F_{rr}]\,dx\\ &= (p-1)\int_{-\infty}^{\infty} F\,dx + \int_{-\infty}^{\infty} [\varphi' F_r - \varphi\varphi' F_{qr} - (\varphi')^2 F_{rr}]\,dx.\end{aligned} \tag{4.6}$$

We are going to estimate the term $\langle \mathcal{L}(x\varphi'), x\varphi'\rangle$. Initially from the second equation in (4.4) and integration by parts we get

$$\begin{aligned}&\langle \mathcal{L}(x\varphi'), x\varphi'\rangle\\ &= \int_{-\infty}^{\infty} [b(\varphi')^2 + 4x\varphi' F_q - 4x(\varphi')^2 F_{qr} - 2x\varphi'\varphi'' F_{rr} + x(\varphi')^2 (F_{rr})_x + 2c\varphi^2]\,dx\\ &= \int_{-\infty}^{\infty} [b(\varphi')^2 + 4x\varphi' F_q - 4x\varphi'(F_r)_x - (\varphi')^2 F_{rr} + 2c\varphi^2]\,dx\\ &= \int_{-\infty}^{\infty} [b(\varphi')^2 - 4F + 4\varphi' F_r - (\varphi')^2 F_{rr} + 2c\varphi^2]\,dx.\end{aligned} \tag{4.7}$$

We need now an expression for $\int_{-\infty}^{\infty} 2c\varphi^2\,dx$. In fact, multiplying (1.4) by $x\varphi'$, integrating by parts several times and using the relation

$$\int_{-\infty}^{\infty} x\varphi' \partial_x^4 \varphi\,dx = \frac{3}{2}\int_{-\infty}^{\infty} (\varphi'')^2\,dx$$

we have

$$\int_{-\infty}^{\infty} \Big[\frac{3}{2}(\varphi'')^2 - \frac{b}{2}(\varphi')^2 + F - \varphi' F_r - \frac{c}{2}\varphi^2\Big]\,dx = 0. \tag{4.8}$$

Moreover, from (1.4) we obtain immediately that

$$\int_{-\infty}^{\infty} (\varphi'')^2\,dx = \int_{-\infty}^{\infty} [b(\varphi')^2 + (p+1)F - c\varphi^2]\,dx. \tag{4.9}$$

So, from (4.8) and (4.9) we get the main relation

$$\int_{-\infty}^{\infty}\Big[b(\varphi')^2+\frac{3}{2}\varphi F_q+\frac{1}{2}\varphi' F_r+F\Big]\,dx=\int_{-\infty}^{\infty}2c\varphi^2\,dx. \tag{4.10}$$

Then, from (4.7) and (4.10) we obtain

$$\langle\mathcal{L}(x\varphi'),x\varphi'\rangle=\int_{-\infty}^{\infty}\Big[2b(\varphi')^2+\frac{3(p-1)}{2}F\Big]\,dx+\int_{-\infty}^{\infty}[3\varphi' F_r-(\varphi')^2F_{rr}]\,dx. \tag{4.11}$$

Finally, from (4.5), (4.6) and (4.11) we get (4.1). This completes the Proof. $\square$

With Theorem 3.5 and Lemma 4.1 we are ready to establish our Theorem of instability for solitary-wave solutions associated to the fifth-order equation (1.1).

Theorem 4.2 (Instability for Ω_φ)**.** *Let $p\geq 2$ and suppose that $F(q,r)$, homogeneous of degree $p+1$, satisfies relation (1.2). Consider φ a solution of (1.4) obtained via Theorem 2.2-(ii) with $\lambda=\mu$. Then if $F(\varphi,\varphi')$ is homogeneous in φ' of degree β, $\beta\in[0,p+1]$, then the conditions*

$$b=0 \quad \textit{and} \quad \beta>\frac{9-p}{2}, \quad \textit{or}$$
$$b<0 \quad \textit{and} \quad \beta\geq\frac{9-p}{2}, \quad \textit{or}$$
$$\beta>\frac{9-p}{2},\ b>0 \quad \textit{and} \quad b \quad \textit{small},$$

imply that the φ-orbit, Ω_φ, is unstable by the flow of (1.1).

Proof From Lemma 3.4 we have that (3.11) is true since we have $0<\theta/(1+\theta)<1$, and from Lemma 2.3 we obtain the properties of regularity on $\psi(x)=\int_{-\infty}^{x}[\varphi(y)+2y\varphi'(y)]dy$ required by Theorem 3.5. So, we only need to verify condition (3.12) and therefore from Lemma 4.1 we need to know when expression (4.1) is negative. In fact, let $F(\varphi,\varphi')$ be homogeneous in φ' of degree β, $\beta\in[0,\ p+1]$. Since F satisfies the relations $\varphi' F_r=\beta F$, $(\varphi')^2F_{rr}=\beta(\beta-1)F$, $\varphi F_q=\alpha F$ and $\varphi^2F_{qq}=\alpha(\alpha-1)F$, where $\alpha+\beta=p+1$, we have initially from (4.1) and the second equation in (4.3) that

$$4\int_{-\infty}^{\infty}[4\varphi' F_r-\varphi\varphi' F_{qr}-2(\varphi')^2F_{rr}]\,dx=4\beta(5-\beta-p)\int_{-\infty}^{\infty}F\,dx$$

so from (4.1) we get

$$\begin{aligned}&\langle S''(\varphi)B_\psi(\varphi),B_\psi(\varphi)\rangle\\&=\Big[(1-p)(p-9)+4(5-p)\beta-4\beta^2\Big]\int_{-\infty}^{\infty}F(\varphi,\varphi')\,dx+8b\int_{-\infty}^{\infty}(\varphi')^2\,dx,\end{aligned} \tag{4.12}$$

which is negative if either $b=0$ and $\beta>\frac{9-p}{2}$ or $b<0$ and $\beta\geq\frac{9-p}{2}$.

Now if we consider $\beta>\frac{9-p}{2}$ and $b>0$ then it follows from condition $c>b^2/4$, properties obtained for φ and Theorem 2.2, that

$$\begin{aligned}b\int_{-\infty}^{\infty}(\varphi')^2\,dx&\leq b\int_{-\infty}^{\infty}[(\varphi'')^2+\varphi^2]\,dx\leq\frac{b\quad\max\{1,c\}}{c}\int_{-\infty}^{\infty}[(\varphi'')^2+c\varphi^2]\,dx\\&\leq\frac{4b\quad\max\{1,c\}}{\sqrt{c}(2\sqrt{c}-b)}\,I_{c,b}(\varphi)=\frac{4b\quad\max\{1,c\}}{\sqrt{c}(2\sqrt{c}-b)}\,\mu^{\frac{2}{p+1}}M_c(1)\\&=\frac{4b\quad\max\{1,c\}}{\sqrt{c}(2\sqrt{c}-b)}\Big(\frac{1}{p+1}\Big)^{\frac{2}{p-1}}[M_c(1)]^{\frac{p+1}{p-1}}.\end{aligned} \tag{4.13}$$

So, we get from (4.12) and (4.13) that

$$\begin{aligned}\langle S''(\varphi)B_\psi(\varphi), B_\psi(\varphi)\rangle &\le \Big[[(9-p)(p-1)+4(5-p)\beta-4\beta^2]\Big(\frac{1}{p+1}\Big)^{\frac{p+1}{p-1}} \\ &\quad + \frac{32b \quad \max\{1,c\}}{\sqrt{c}(2\sqrt{c}-b)}\Big(\frac{1}{p+1}\Big)^{\frac{2}{p-1}}\Big][M_c(1)]^{\frac{p+1}{p-1}} \\ &\equiv K_0(b)[M_c(1)]^{\frac{p+1}{p-1}}.\end{aligned}$$

Therefore, if we choose b small such that $K_0(b) < 0$ then we obtain from the last inequality that $\langle S''(\varphi)B_\psi(\varphi), B_\psi(\varphi)\rangle < 0$. This finishes the proof. □

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References

[1] C.J. Amick and J.F. Toland, *Homoclinic orbits in the dynamic phase-space analogy of an elastic strut*, European J. Appl. Math., Vol. 3(2), (1992), 97-114.

[2] J. Angulo, *On the instability of solitary waves solutions of the generalized Benjamin equation*, To appear in Advances in Differential Equations, 2003.

[3] T. B. Benjamin, *A new kind of solitary waves*, J. Fluid Mechanics, Vol. 254 (1992), 401–411

[4] T. B. Benjamin, *Solitary and periodic waves of a new kind*, Phil. Trans. Roy. Soc. London Ser. A, Vol. 354 (1996), 1775–1806.

[5] D. J. Benney, *A general theory for interactions between short and long waves*, Stud. Appl. Math., Vol. 56 (1977), 81–94.

[6] J. L. Bona, P. E. Souganidis and W. A. Strauss, *Stability and instability of solitary waves of Korteweg- de Vries type*, Proc. Royal. Soc. London Ser. A, Vol. 411 (1987), 395–412.

[7] T. J. Bridges and G. Derks, *Linear instability of solitary wave solutions of the Kawahara equation and its generalization*, SIAM J. Math. Anal., Vol. 33 (2002), 1356–1378.

[8] A. R. Champneys and M. D. Groves, *A global investigation of solitary-wave solutions to a two-parameter model for water waves*, Fluid Mech., Vol. 342 (1996), 199–229.

[9] W. Craig and, M.D. Groves, *Hamiltonian long-wave approxiamtions to the water-wave problem*, Wave Motion, Vol. 19 (1994), 367–389.

[10] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Tables of integral transform*, Vol. v. 2, McGraw-Hill, New York, 1954.

[11] J. M. Gonçalves Ribeiro, *Instability of symmetric stationary states for some nonlinear Schrödinger equations with an external magnetic field*, Ann. Inst. H. Poincaré; Phys. Théor. Vol. 54 (1992), 403–433.

[12] K.A. Gorshkov, L.A. Ostrovsky and V.V. Papko, *Hamiltonian and non-Hamiltonian models for water waves*, Lecture Notes in Physics, Vol. 195, Springer, Berlin (1984), 273–290.

[13] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I.*, J. Funct. Anal. Vol. 74 (1987), 160–197.

[14] J. Hunter, J. Scheurle*Existence of perturbed solitary wave solutions to a model equation for water waves*, Physica D Vol. 32 (1988), 253–268.

[15] T. Kato, *Quasilinear equations of evolution, with applications to Partial Differential Equations*, Lectures Notes in Math., Vol. 448 (1975) 25–70.

[16] T. Kawahara, *Oscillatory solitary waves in dispersive media*, J. Phys. Soc. Jpn., Vol. 33 (1972) 260-264.

[17] C. Kenig, G. Ponce and L. Vega, *Higher-order nonlinear dispersive equations*, Proc. Amer. Math. Soc., Vol. 122 (1), 1994, 157-166.

[18] S. Kichenassamy, *Existence of solitary waves for water-wave models*, Nonlinearity, Vol. 10(1), 1997, 133-151.

[19] S. Kichenassamy and P. Olver, *Existence and nonexistence of solitary wave solutions to higher-order model evolution equations*, SIAM J. Math. Anal., Vol. 23 (1992), 1141-1166.

[20] S. P. Levandosky, *A stability analysis of fifth-order water wave models*, Physica D, Vol. 125 (1999), 222-240.

[21] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1*, Ann. Inst. H. Poincaré, Anal. Non linéare, Vol. 1 (1984), 109–145.

[22] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 2*, Ann. Inst. H. Poincaré, Anal. Non linéare Vol. 4 (1984), 223–283.

[23] P. J. McKenna and W. Walter, *Traveling waves in a suspension bridge*, SIAM, J. Appl. Math., Vol. 50 (1990), 703–715

[24] P. J. Olver, *Hamiltonian and non-Hamiltonian models for water waves*, Lecture Notes in Physics, Vol. 195, Springer, Berlin, 1984, 273–290.

[25] G. Ponce, *Lax pairs and higher order models for water waves*, J. Differential Equations, Vol. 102(2), 1993, 360–381.

[26] J. C. Saut, *Quelques généralisations de l'équation de Korteweg-de Vries, II*, J. differential Equations, Vol. 33 (1979), 320–335.

[27] J. Shatah and W. A. Strauss, *Instability of nonlinear bound states*, Comm. Math. Phys., Vol. 100 (1985), 173–190.

[28] P. E. Souganidis and W. A. Strauss, *Instability of a class of dispersive solitary waves*, Proc. Royal. Soc. Eding., Vol. 114A (1990), 195–212.

[29] M. Weinstein, *Existence and dynamic stability of solitary wave solutions of equations in long wave propagation*, Comm. P.D.E., Vol. 12 (1987), 1133–1173

[30] J. Zufiria, *Symmetric breaking in periodic and solitary gravity-capillary waves on water of finite depth*, J. Fluid Mech., Vol. 184 (1987), 183–206.

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