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Semi-linearized compressible Navier-Stokes equations perturbed by noise *

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Abstract

In this paper, we study semi-linearized compressible barotropic Navier-Stokes equations perturbed by noise in a 2-dimensional domain. We prove the existence and uniqueness of solutions in a class of potential flows.

1 Introduction

We consider the following system of equations with a stochastic perturbation

$$\begin{aligned} \bar\rho u_t + \nabla p(\rho) = \mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u + G_t \quad \text{in } Q_T, \\ \rho_t + \operatorname{div}(\rho u) = 0 \quad \text{in } Q_T, \end{aligned} \tag{1.1}$$

where $Q_T = (0,T)\times D$, $D = (0,1)^2)$, $\bar\rho, \lambda, \mu$ are constants such that $\bar\rho > 0$, $\mu > 0$, $\mu+\lambda \geq 0$; while G is a stochastic process in a function space, which we will precise below, and u_t and G_t denote the derivative with respect to t in the distribution sense. ∇ and div are the gradient and divergence operators with respect to the space variables, Δ is the Laplace operator. The space variables are denoted by $x = (x_1, x_2)$ and the time by t.

In absence of the random perturbation G_t, (1.1) is reduced to the system

$$\begin{aligned} \bar\rho u_t + \nabla p(\rho) = \mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u, \\ \rho_t + \operatorname{div}(\rho u) = 0. \end{aligned} \tag{1.2}$$

This system can be considered as a semi-linearized approximation of the compressible Navier-Stokes equations of a barotropic viscous fluid

$$\begin{aligned} \rho\left(\partial_t u_i + (u\cdot\nabla)u_i\right) - \partial_i p(\rho) = \sum_{j=1}^{n} \partial_j \left(\mu(\partial_j u_i + \partial_i u_j)\right) + \partial_i(\lambda \operatorname{div} u), \\ \rho_t + \operatorname{div}(\rho u) = 0, \end{aligned} \tag{1.3}$$

where $i = 1,\dots,n$; $u, \rho, p(\rho)$ represent respectively the velocity vector, the density, and the pressure; while μ, λ are viscosity coefficients which according to the thermodynamic principles should satisfy the inequalities $\mu > 0$ and $3\lambda + 2\mu \geq 0$.

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System (1.3) has been investigated mostly for one-dimensional flows ($n = 1$). For many-dimensional flows, considerably less is known except for small initial data or in small time interval. A global existence theorem for the model (1.3) has been proved by P.L.Lions [9, 10] and Vaigant-Kazhikhov [16]. Notice that in the first μ and λ are considered constants while some particular requirements on the growth of the viscosity coefficient λ and the pressure as functions of the density ρ are imposed for the last result. The semi-linearized system (1.2) is studied in [15], which proves the existence and uniqueness of the strong solution. As far as the stochastic equations for incompressible viscous fluids are concerned, some existence theorems and some results on various aspects are known see [2, 5, 6] etc... But in the compressible case, the variation of the fluid density gives some difficulties. For this reason, only the two dimensional space is considered here with some other restrictions. In the one dimensional case, the full equation (1.3) subject to a perturbation is studied in [13] and [14].

We use the standard notation $W^{l,p}$ for the Sobolev spaces consisting in the functions which are integrable in power p as well as their derivatives up to the order l and $H^l = W^{l,2}$; $C([0,T];X)$ denotes the space of the continuous functions with values in a Banach space X. In this paper, we use the Orlicz space $L_\phi(D)$ associated to the convex function $\phi(r) = (1+r)\log(1+r) - r$, $r \geq 0$. We denote by $\langle .,. \rangle$ the inner product in L^2 and by $\|.\|$ the corresponding norm. We use the abbreviated notation

$$\partial_j = \frac{\partial}{\partial x_j}, \quad \partial_j^2 = \frac{\partial^2}{\partial x_j^2}\,.$$

The propose of the present paper is to prove the existence (and uniqueness) of a global solution to (1.1). The solution will be constructed in the class of periodic and potential flows as in [15], i.e., in the case where u has the form

$$u = \nabla\varphi,$$

with some function φ, which is periodic in x_1 and x_2. More precisely we suppose that every function appearing in (1.1) is periodic of period 1 in x_1 and x_2 and take the equation of state $p(\rho) = c\rho$, $c = \mathit{const} > 0$. We also suppose that the perturbation G is the gradient of a potential i.e. $G = \nabla W$. For simplicity, we assume that the constants c and $\bar\rho$ are equal to 1 and the constants λ and μ are respectively equal to 1/4 and 1/2 and impose $\int_D \varphi(t,x)\mathrm{d}x = 0$. When

$$\int_D W(t,x)\mathrm{d}x = 0$$

which will follows from the assumptions of section 2, integrating the momentum equation $(1.1)_1$, the system acquires the form

$$\begin{gathered} \mathrm{d}\varphi = (\Delta\varphi + 1 - \rho)\mathrm{d}t + \mathrm{d}W \quad \text{in } Q_T, \\ \rho_t + \operatorname{div}(\rho\nabla\varphi) = 0. \end{gathered} \tag{1.4}$$

Below, W will be a Wiener process taking values in a particular Hilbert space. The unknown functions are assumed to take prescribed values at the initial time,

$$\rho|_{t=0} = \rho_0(x) \geq 0, \quad \int_D \rho_0(x)dx = 1,$$
$$\varphi|_{t=0} = \varphi_0(x), \quad \int_D \varphi_0(x)dx = 0.$$

In addition, we impose the following natural requirement on the solution,

$$\rho(x,t) \geq 0 \quad \text{in } Q_T.$$

2 Main result

Before stating the existence results, we have to precise some conditions on the noise term appearing in (1.1). We set

$$D(A) = \big\{u \in H^2(D) : u \text{ is periodic of period 1 in } x_1 \text{ and } x_2, \int_D u\mathrm{d}x = 0\big\}.$$

and define a linear operator

$$A : D(A) \to \big\{u \in L^2(D) : u \text{ is periodic of period 1 in } x_1 \text{ and } x_2\big\},$$

as $Au = -\Delta u$. The operator A is self-adjoint with compact resolvent. We denote by $0 < \lambda_1 \leq \lambda_2 \dots$ ($\lim \lambda_j = \infty$) the eigenvalues of A and by $e_1, e_2 \dots$ the corresponding complete orthonormal system of eigenvectors. As well known, for the space of periodic functions the eigenvectors are trigonometric functions and we see easily that $\int_D e_j(x)\mathrm{d}x = 0$, $j = 1, 2 \dots$. Let

$$W(t) = \sum_{j=1}^{\infty} \sigma_j \beta_j(t) e_j(x). \tag{2.1}$$

where $\{\sigma_j\}_{j=1}^{\infty}$ is a sequence of constants satisfying the condition

$$\sum_{j=1}^{\infty} \lambda_j^{\delta+2} \sigma_j^2 < \infty, \tag{2.2}$$

with some $\delta > 0$ while $\beta_1, \beta_2, \dots$ are independent standard 1-dimensional Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We denote by $\mathbf{E}$ the expectation relative to $(\Omega, \mathcal{F}, \mathbf{P})$.

Now, we state the main theorem of this paper.

Theorem 2.1 *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and T a positive number. Suppose that W is a Wiener process satisfying (2.1) and the condition (2.2), and that ρ_0 and φ_0 are two Random variables with values respectively in $L^\infty(D)$ and $W^{1,q}(D) \cap H^2(D)$ ($q \geq 1$) satisfying respectively the conditions (1.5) and (1.6) $\mathbf{P}$-a.s. and $\inf_{x\in D} \rho_0(x) > 0$ and $\sup_{x\in D} \rho_0(x) < \infty$ $\mathbf{P}$-a.s. Then there exists a unique solution to (1.4) up to a modification. Besides ρ satisfies $\inf_{Q_T} \rho(x,t) > 0$ and $\sup_{Q_T} \rho(x,t) < \infty$ $\mathbf{P}$-a.s.*

3 Reduction of the problem via the Ornstein-Uhlenbeck equation

Let us consider an auxiliary problem, the Ornstein-Uhlenbeck equation,

$$\begin{gathered} dz(t) + Az(t)dt = dW(t), \\ z(0) = 0. \end{gathered} \tag{3.1}$$

This equation has a solution given by the process (see [4])

$$z(t) = \int_0^t e^{-(t-s)A} dW(s), \tag{3.2}$$

where e^{-tA} denotes a C_0-semigroup generated by A. The regularity of $z(t)$ depends on the regularity of $W(t)$. Indeed, we have for an arbitrary $k > 0$

$$A^k z(t) = \sum_{j=1}^{\infty} \int_0^t \lambda_j^k \mathrm{e}^{-(t-s)\lambda_j} \sigma_j \mathrm{d}\beta_j(s) e_j.$$

$$\begin{aligned} \mathbf{E}\|z(t)\|^2_{D(A^k)} &= \mathbf{E}\|A^k z(t)\|^2 \\ &= \mathbf{E}\Big(\sum_{j=1}^{\infty} \int_0^t \lambda_j^k \mathrm{e}^{-(t-s)\lambda_j} \sigma_j \mathrm{d}\beta_j(s) \Big)^2 \\ &= \sum_{j=1}^{\infty} \int_0^t |\lambda_j^k \mathrm{e}^{-(t-s)\lambda_j} \sigma_j|^2 \mathrm{d}s \\ &= \sum_{j=1}^{\infty} \frac{\lambda_j^{2k}\sigma_j^2}{2\lambda_j}(1 - \mathrm{e}^{-2t\lambda_j}). \end{aligned}$$

According to (2.2), $W(t)$ belongs in $D(A^{(\delta+2)/2})$ for some $\delta > 0$ which yields, for $k = (\delta+3)/2$ in the above equality, that $z(t)$ has continuous trajectories taking values in $D(A^{(3+\delta)/2})$ (as we will need in the next sections), i.e. $z(t) \in C([0,T]; D(A^{(3+\delta/2)}))$ $\mathbf{P}$-a.s. for some $\delta > 0$.

Following the idea of Bensoussan-Temam [2], we set

$$y(t) = \varphi(t) - z(t). \tag{3.3}$$

Using this change of variable in (1.4) and equation (3.1), one obtains the system

$$\begin{gathered} y_t - \Delta y = 1 - \rho \quad \text{in } Q_T, \\ \rho_t + \operatorname{div}(\rho\nabla(y+z)) = 0 \quad \text{in } Q_T. \end{gathered} \tag{3.4}$$

4 Reduced deterministic problem

In this section, we study the following reduced deterministic problem

$$\begin{gathered} y_t - \Delta y = 1 - \rho \quad \text{in } Q_T, \\ \rho_t + \operatorname{div}(\rho \nabla (y+z)) = 0 \quad \text{in } Q_T, \end{gathered} \tag{4.1}$$

where $z(t)$ is a continuous function taking values in $H^{3+\delta}(D)$, $\delta > 0$. For this problem, we state the following existence and uniqueness theorem.

Theorem 4.1 *Let T be positive number and suppose that $y_0 \in W^{2,s}(D)$ and $\rho_0 \in L^s(D)$, $s \geq 2$. We suppose also that $z \in C^0([0,T]; H^{3+\delta}(D))$ ($\delta > 0$)). Then there exists at least one solution (y, ρ) to Problem (4.1) which satisfies*

$$\begin{gathered} y \in L^\infty(0,T; W^{1,q}(D)) \cap L^2(0,T; H^2(D)), \\ y_t \in L^\infty(0,T; L^s(D)) \cap L^2(0,T; H^1(D)), \\ \rho \in L^\infty(0,T; L_\phi(D)) \cap L^s(Q_T), \end{gathered}$$

where $q \geq 2$. Moreover, if $\inf_{x\in D} \rho_0(x) > 0$ and $\sup_{x \in D} \rho_0(x) < \infty$, then $\inf_{Q_T} \rho(x,t) > 0$, $\sup_{Q_T} \rho(x,t) < \infty$, and (4.1) is uniquely solvable.

The proof follows the lines of Vaigant-Kazhikhov [15], of which we will use the ideas without quote them explicitly.

4.1 A priori estimates and existence of solutions

In this section, we obtain *a priori* estimates that permit us to prove the existence of a solution. The first energy estimate is obtained by multiplying the first equation in (4.1) by Δy and the second equation by $\log \rho$, followed by integrating over D. More precisely, we obtain

$$\frac{d}{dt} \int_D \Big(\frac{1}{2}|\nabla y|^2 + \rho \log \rho - \rho + 1\Big) dx + \int_D |\Delta y|^2 \leq \|\Delta z\|_{L^\infty}. \tag{4.2}$$

This relation implies that the solution is bounded in the norms of the spaces

$$y \in L^\infty(0,T; H^1(D)), \quad \Delta y \in L^2(0,T; L^2(D)), \quad \rho \in L^\infty(0,T; L_\phi(D)).$$

The following lemma may be derived from the second equation in system (4.1).

Lemma 4.2 *If $\rho_0 \in L^{p-1}(D)$ then there exists a constant C depending on p such that the inequality*

$$\begin{aligned} &\|\rho(t)\|^{p-1}_{L^{p-1}(D)} + \int_0^t \|\rho(\tau)\|^p_{L^p(D)} d\tau \\ &\leq C\Big(\|\rho_0\|^{p-1}_{L^{p-1}(D)} + \int_0^t \|y_\tau(\tau)\|^p_{L^p(D)} d\tau + \int_0^t \|\Delta z(\tau)\|^p_{L^p(D)} d\tau\Big) \end{aligned} \tag{4.3}$$

holds for any exponent p, $2 < p < \infty$ and any $t \in [0,T]$.

Proof: For $r > 1$ using $(4.1)_1$, the expression $(4.1)_2$ can be rewritten in the form

$$\frac{\partial \rho^r}{\partial t} + \nabla \cdot (\rho^r \nabla (y+z)) + (r-1)\rho^{r+1} = (r-1)\rho^r (y_t - 1 - \Delta z). \tag{4.4}$$

Integrating over D and estimating $\rho^r|y_t|$ and $\rho^r|\Delta z|$ by the Young inequality,

$$\begin{aligned} \rho^r|y_t| &\le \epsilon_1 \rho^{r+1} + C_1 |y_t|^{r+1}, \\ \rho^r|\Delta z| &\le \epsilon_2 \rho^{r+1} + C_2 |\Delta z|^{r+1}, \end{aligned}$$

with convenient small numbers ϵ_1, ϵ_2, we obtain (4.3) for $p = r+1$. $\square$

Lemma 4.3 *If $\rho_0 \in L^{p-1}(D)$, then there exists a constant C depending on p such that the inequality*

$$\|y\|_{W^{2,p}(Q_T)} \le C(\|\rho_0\|_{L^{p-1}(D)} + \|y_t\|_{L^p(Q_T)} + \|\Delta z\|_{L^p(Q_T)}) \tag{4.5}$$

holds for $2 < p < \infty$.

The proof of this lemma is is a consequence of (4.3) and the first equation of (4.1).

Now, to obtain additional a priori estimates, we differentiate $(4.1)_1$ with respect to x_1, x_2, t so that we obtain

$$\nabla y_t - \Delta \nabla y = -\nabla \rho, \tag{4.6}$$

$$y_{tt} - \Delta y_t = -\rho_t = \operatorname{div}(\rho \nabla (y+z)). \tag{4.7}$$

Lemma 4.4 *If $\rho_0 \in L^3(D)$, $y_0 \in W^{1,q}(D) \cap H^2(D)$, where $q \ge 4$ then there exists a constant C depending on q such that the inequality*

$$\sup_{0<\tau<t} \Big(\int_D |\nabla y|^q + \int_D |y_t|^2 \Big) + \int_0^t \int_D |\nabla y_t|^2 \le C \tag{4.8}$$

holds for all $t \in [0, T]$.

Proof: For arbitrary $q \ge 2$ and $s \ge 2$, we multiply (4.6) by $q|\nabla y|^{q-2}\nabla y$ and (4.7) by $s|y_t|^{s-2}y_t$, sum these equations and integrate over D to obtain

$$\begin{aligned} &\frac{d}{dt}\int_D (|\nabla y|^q + |y_t|^s) + q\sum_{j,k}^2 \int_D (\partial_j \partial_k y)^2 |\nabla y|^{q-2} \\ &+ q(q-2)\sum_{j,k,l}^2 \int_D (\partial_j\partial_k y)(\partial_j \partial_l y)(\partial_k y)(\partial_l y)|\nabla y|^{q-4} + s(s-1)\int_D |\nabla y_t|^2 |y_t|^{s-2} \\ &= q\int_D \rho \Delta y |\nabla y|^{q-2} + q(q-2)\sum_{j,k}^2 \int_D \rho(\partial_j y)(\partial_k y)(\partial_j\partial_k y)|\nabla y|^{q-4} \\ &\quad - s(s-1)\int_D \rho \nabla(y+z)\cdot \nabla y_t |y_t|^{s-2}. \end{aligned} \tag{4.9}$$

By taking $q=4$ and $s=2$ in (4.9) and substituting $\rho = 1+\Delta y - y_t$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_D \left(|\nabla y|^4 + |y_t|^2\right) + 4\sum_{j,k}^{2}\int_D (\partial_j\partial_k y)^2|\nabla y|^2 + 2\int_D |\nabla y_t|^2 \\
&= 4\int_D \Delta y|\nabla y|^2 + 4\int_D (\Delta y)^2|\nabla y|^2 - 4\int_D (\Delta y)y_t|\nabla y|^2 \\
&\quad + 8\sum_{j,k}^{2}\int_D (\partial_j y)(\partial_k y)(\partial_j\partial_k y) + 8\sum_{j,k}^{2}\int_D \Delta y(\partial_j y)(\partial_k y)(\partial_j\partial_k y) \\
&\quad - 8\sum_{j,k}^{2}\int_D y_t(\partial_j y)(\partial_k y)(\partial_j\partial_k y) - 2\int_D \nabla(y+z).\nabla y_t - 2\int_D \Delta y\nabla(y+z).\nabla y_t \\
&\quad + 2\int_D \nabla(y+z).\nabla y_t y_t - 8\sum_{j,k,l}^{2}\int_D (\partial_j\partial_k y)(\partial_j\partial_l y)(\partial_k y)(\partial_l y) \\
&= I_1 + \dots + I_{10}\,.
\end{aligned}
$$

The assumption that D is a two-dimensional region is essential for this estimate.

$$
I_2 = 4\int_D |\nabla y|^2|\Delta y|^2 \le \|\Delta y\|\|\Delta y\|_{L^4(D)}\||\nabla y|^2\|_{L^4(D)}.
$$

On the other hand

$$
\||\nabla y|^2\|_{L^4(D)} \le \||\nabla y|^2\|^{1/2}\|\nabla|\nabla y|^2\|^{1/2}.
$$

Then using Young's inequality twice, we obtain for arbitrary positive small numbers ϵ_1 and ϵ_2 such that

$$
I_2 \le \epsilon_1\|\Delta y\|\|\Delta y\|^2_{L^4(D)} + \epsilon_2\|\nabla|\nabla y|^2\|^2 + C\|\Delta y\|^2\||\nabla y|^2\|^2.
$$

Since

$$
\|\nabla|\nabla y|^2\|^2 = \int_D |\nabla|\nabla y|^2|^2 = 2\sum_{j,k}^{2}\int_D |\nabla y|^2(\partial_j\partial_k y)^2
$$

and

$$
\||\nabla y|^2\|^2 = \int_D (|\nabla y|^2)^2 = \int_D |\nabla y|^4,
$$

one obtains

$$
I_2 \le \epsilon_1\|\Delta y\|\|\Delta y\|^2_{L^4(D)} + \epsilon_2\sum_{j,k}^{2}\int_D |\nabla y|^2(\partial_j\partial_k y)^2 + C\|\nabla y\|^4_{L^4(D)}\|\Delta y\|^2.
$$

By Young's inequality and for arbitrary small positive constant ϵ, ϵ_1 and ϵ_2 we can estimate I_1, I_3, I_4, I_5 and I_6 as follows:

$$
I_1 = 4\int_D |\nabla y|^2\Delta y \le \epsilon I_2 + C\int_D |\nabla y|^2,
$$

$$\begin{aligned}
I_3 &= -4\int_D \Delta y|\nabla y|^2 y_t \\
&\le \epsilon_1 \int_D |\nabla y|^2|\Delta y|^2 + \epsilon_2\|\nabla y_t\|^2 + C\|y_t\|^2\|\nabla y\|^2\|\Delta y\|^2,
\end{aligned}$$

$$I_4 = 8\sum_{j,k}^2 \int_D (\partial_j y)(\partial_k y)(\partial_j\partial_k y) \le \epsilon \sum_{j,k}^2 \int_D (\partial_k y)^2(\partial_j\partial_k y)^2 + C\int_D |\nabla y|^2,$$

$$I_5 = 8\sum_{j,k}^2 \int_D \Delta y(\partial_j y)(\partial_k y)(\partial_j\partial_k y) \le \epsilon \sum_{j,k}^2 \int_D (\partial_k y)^2(\partial_j\partial_k y)^2 + CI_2.$$

$$\begin{aligned}
I_6 &= -8\sum_{j,k}^2 \int_D y_t(\partial_j y)(\partial_k y)(\partial_j\partial_k y) \\
&\le \epsilon_1 \sum_{j,k}^2 \int_D (\partial_k y)^2(\partial_j\partial_k y)^2 + \epsilon_2\|\nabla y_t\|^2 + C\sum_{j,k}^2 \|y_t\|^2\|\nabla y\|^2\|\partial_j\partial_k y\|^2.
\end{aligned}$$

$$I_7 = -2\int_D \nabla(y+z)\cdot\nabla y_t \le \epsilon\|\nabla y_t\|^2 + C\int_D |\nabla(y+z)|^2.$$

$$I_8 = -2\int_D \Delta y\nabla(y+z)\cdot\nabla y_t \le \epsilon\|\nabla y_t\|^2 + CI_2 + \|\nabla z\|^2_{L^\infty(D)}\int_D |\Delta y|^2.$$

$$I_9 = 2\int_D \nabla(y+z)\cdot\nabla y_t y_t \le \epsilon\|\nabla y_t\|^2 + C\|y_t\|^2\|\Delta(y+z)\|^2\|\nabla(y+z)\|^2.$$

$$\begin{aligned}
I_{10} \le& \sum_{j,k,l}^2 \Big(\int_D (\partial_j\partial_k y)^2\Big)^{1/2}\Big(\int_D (\partial_j\partial_l y)^4\Big)^{1/4}\Big(\int_D (\partial_k y)^8\Big)^{1/8}\Big(\int_D (\partial_l y)^8\Big)^{1/8} \\
\le& \epsilon_1 \sum_{j,k,l}^2 \|\partial_j\partial_k y\|\|\partial_j\partial_l y\|^2_{L^4(D)} + \epsilon_2\|\nabla|\nabla y|^2\|^2 \\
&+ C_{\epsilon_1,\epsilon_2}\sum_{j,k} \||\nabla y|^2\|^2\|\partial_j\partial_k y\|^2.
\end{aligned}$$

We set

$$\begin{gathered}
\alpha(t) = \sup_{0<t<T}\int_D |\nabla y|^4 + |y_t|^2, \\
\beta(t) = \sum_{j,k}^2 \int_D (\partial_j\partial_k y)^2|\nabla y|^2 + \int_D |\nabla y_t|^2.
\end{gathered}$$

On the other hand from (4.5) in particular for $p=4$ and $\rho_0 \in L^3(D)$, we have

$$\|y\|^2_{W^{2,4}(D)} \le C\big(\|\rho_0\|_{L^3(D)} + \|y_t\|^2_{L^4(D)} + \|\Delta z\|^2_{L^4(D)}\big).$$

Consequently, by using Cauchy's inequality and the energy estimate (4.2) we obtain

$$\epsilon \int_0^t \|\Delta y\| \|\Delta y\|_{L^4(D)}^2 \leq C\epsilon \Big(\int_0^t \|\Delta y\|_{L^4(D)}^4 \Big)^{1/2}.$$

But in the two-dimensional space we have

$$\|y_t\|_{L^4(D)}^4 \leq C \|y_t\|^2 \|\nabla y_t\|^2, \tag{4.10}$$

so that

$$\epsilon \int_0^t \|\Delta y\| \|\Delta y\|_{L^4(D)}^2 \leq C\epsilon \Big(\alpha(t)^{1/2} \Big(\int_0^t \beta(\tau) d\tau \Big)^{1/2} + \|\Delta z\|_{L^4(Q_T)}^2 \Big).$$

It is the same for $\epsilon_1 \sum_{j,k,l}^2 \int_0^t \|\partial_j \partial_k y\| \|\partial_j \partial_l y\|_{L^4(D)}^2$.

Set $\Lambda(t) = \|\Delta y(t)\|^2$. Since this is integrable on $[0, T]$, we have

$$\alpha(t) + \int_0^t \beta(\tau) d\tau \leq \alpha(0) + \epsilon \alpha(t)^{1/2} \big(\int_0^t \beta(\tau) d\tau \big)^{1/2} + C \big(1 + \int_0^t (\Lambda(\tau) + 1) \alpha(\tau) d\tau \big).$$

Using Gronwall's lemma we obtain,

$$\sup_{0<\tau<t)} \int_D \big(|\nabla y|^4 + |y_\tau|^2 \big) + \int_0^t \int_D |\nabla y_\tau|^2 \leq C. \tag{4.11}$$

According to this equation,

$$\int_0^T \int_D |y_t|^4 \leq \int_0^T \|y_t\|^2 \|\nabla y_t\|^2 \leq C. \tag{4.12}$$

Hence, using this inequality in (4.3) and (4.5) for $p = 4$, it follows that

$$\int_0^t \int_D \rho^4 + \int_0^t \int_D |\Delta y|^4 \leq C. \tag{4.13}$$

Now, let us consider the equation (4.9) for $q \geq 4$ and $s = 2$. We obtain the equality

$$\begin{aligned} &\frac{d}{dt} \int_D \big(|\nabla y|^q + |y_t|^2 \big) + q \sum_{j,k}^2 \int_D (\partial_j \partial_k y)^2 |\nabla y|^{q-2} + 2 \int_D |\nabla y_t|^2 \\ &= -2 \int_D \rho \nabla (y+z) \cdot \nabla y_t + q \int_D \rho \Delta y |\nabla y|^{q-2} \\ &\quad + q(q-2) \sum_{j,k}^2 \int_D \rho (\partial_j y)(\partial_k y)(\partial_k \partial_j y) |\nabla y|^{q-4} \\ &\quad - q(q-2) \sum_{j,k,l}^2 \int_D (\partial_j \partial_k y)(\partial_j \partial_l y)(\partial_k y)(\partial_l y) |\nabla y|^{q-4}. \end{aligned} \tag{4.14}$$

The first term on the right hand side of (4.14) by Young's inequality is bounded by

$$\int_0^T\int_D \rho\nabla(y+z)\cdot\nabla y_t \le \epsilon\int_0^T \|\nabla y_t\|^2 + C\int_0^T \|\rho\|^2_{L^4(D)}\|\nabla(y+z)\|^2_{L^4(D)}.$$

In view of (4.11) and (4.13), ρ and ∇y are bounded respectively in $L^4(Q_T)$ and in $L^\infty(0,T;L^4(D))$. Hence

$$\int_0^T\int_D \rho\nabla(y+z)\cdot\nabla y_t \le \epsilon\int_0^T \|\nabla y_t\|^2 + C\int_0^T \|\nabla z\|^4_{L^4(D)}.$$

Using the Young and the Hölder's inequality for the second term on the right hand side of (4.14), one obtains

$$\begin{aligned}\int_D \rho\Delta y|\nabla y|^{q-2} &\le \epsilon\int_D |\Delta y|^2|\nabla y|^{q-2} + C\int_D \rho^2|\nabla y|^{q-2}\\ &\le \epsilon\int_D |\Delta y|^2|\nabla y|^{q-2} + C\|\rho\|^2_{L^4(D)}\Big(\int_D |\nabla y|^{2(q-2)}\Big)^{1/2}.\end{aligned}$$

We write the last integral in the above inequality as

$$\Big(\int_D |\nabla y|^{2(q-2)}\Big)^{1/2} = \||\nabla y|^{q/2}\|^{2(q-2)/q}_{L^{4(q-2)/q}}.$$

Using the embedding inequality (see [8], pp 62)

$$\|f\|_{L^{4(q-2)/q}} \le C\|\nabla f\|^a\|f\|^{1-a}, \tag{4.15}$$

for the function $f=|\nabla u|^{q/2}$, where $a=(q-4)/(2(q-2))$, yields

$$\begin{aligned}&\int_D \rho\Delta y|\nabla y|^{q-2}\\ &\le C\|\rho\|^2_{L^4}\|\nabla(|\nabla y|^{q/2}|)\|^{(q-4)/q}\||\nabla y|^{q/2}\| + \epsilon\int_D |\Delta y|^2|\nabla y|^{q-2}.\end{aligned}$$

Since $\||\nabla y|^{q/2}\| = \Big(\int_D |\nabla y|^q\Big)^{1/2}$, and

$$\|\nabla(|\nabla y|^{q/2})\| = \frac{q}{2}\Big(\sum_{j,k}\int_D (\partial_j\partial_k)^2|\nabla y|^{q-2}\Big)^{1/2},$$

and in virtue of the Young's inequality with $p=2q/(q-4)$ and $p'=2q/(q+4)$ we obtain

$$\int_D \rho\Delta y|\nabla y|^{q-2} \le \epsilon\sum_{j,k}\int_D (\partial_j\partial_k)^2|\nabla y|^{q-2} + C\|\rho\|^{4q/(q+4)}_{L^4(D)}\Big(\int_D |\nabla y|^q\Big)^{q/(q+4)}.$$

By using the same argument for the third term on the right hand side of (4.14) we obtain, for an arbitrary ϵ,

$$\begin{aligned}&\sum_{j,k}^{2}\int_D \rho(\partial_j y)(\partial_k y)(\partial_k\partial_j y)|\nabla y|^{q-4}\\ &\leq \epsilon\sum_{j,k}^{2}\int_D |\nabla y|^{q-2}(\partial_j\partial_k y)+\|\rho\|_{L^4(D)}^{4q/(q+4)}\Big(\int_D|\nabla y|^q\Big)^{q/(q+4)}.\end{aligned}$$

For the last term on the right hand side of (4.14), we use the same arguments as before, we have

$$\begin{aligned}&\sum_{j,k,l}^{2}\int_D (\partial_j\partial_k y)(\partial_j\partial_l y)(\partial_k y)(\partial_l y)|\nabla y|^{q-4}\\ &\leq \epsilon\sum_{j,k,l}^{2}\int_D |\nabla y|^{q-2}(\partial_j\partial_k y)^2+C_\epsilon\sum_{j,l}^{2}\|\partial_j\partial_l y\|_{L^4(D)}^{4q/(q+4)}\Big(\int_D|\nabla y|^q\Big)^{q/(q+4)}\end{aligned}$$

Consequently, using (4.5) for $p=4$ and (*), we have

$$\begin{aligned}&\int_D\big(|\nabla y|^q+|y_t|^2\big)+\int_0^T\int_D|\nabla y_t|^2\\ &\leq C\big(\|\nabla y_0\|_{L^q(D)}^q+\|\rho_0\|^2+\|\Delta y_0\|^2+\|\Delta z\|_{L^4(Q_T)}^2\big)\\ &\quad+\Big(\int_0^T\|\rho\|_{L^4(D)}^{4q/(q+4)}+\|y_t\|^{2q/(q+4)}\|\nabla y_t\|^{2q/(q+4)}\\ &\quad+\|\Delta z\|_{L^4(Q_T)}^{4q/(q+4)}+\|\rho_0\|_{L^3(D)}^{4q/(q+4)}\Big)\Big(\int_D|\nabla y|^q\Big)^{q/(q+4)}\end{aligned}$$

Now using Gronwall's lemma, (4.12) and (4.13), we obtain (4.8). $\square$

Lemma 4.5 *If $\rho_0\in L^s(D)$, $y_0\in W^{2,s}(D)$ then the inequality*

$$\sup_{0<\tau<t}\int_D|y_\tau|^s dx\leq C \tag{4.16}$$

holds for $s>2$ and $t\in[0,T]$.

Proof: We multiply (4.7) by $s|y_t|^{s-2}y_t$, $s>2$ and then we integrate over D, to obtain

$$\frac{d}{dt}\int_D|y_t|^s+s(s-1)\int_D|y_t|^{s-2}|\nabla y_t|^2=-s(s-1)\int_D\rho\nabla(y+z)\cdot\nabla y_t|y_t|^{s-2}. \tag{4.17}$$

Cauchy's inequality applied to the right hand side of the above quality gives

$$\Big|\int_D\rho\nabla(y+z)\cdot\nabla y_t|y_t|^{s-2}\Big|\leq\epsilon\int_D|y_t|^{s-2}|\nabla y_t|^2+C\int_D\rho^2|\nabla(y+z)|^2|y_t|^{s-2}.$$

According to $(4.1)_1$, with $\rho = 1 + \Delta y - y_t$, (4.17) yields

$$\frac{d}{dt}\int_D |y_t|^s dx + \int_D |y_t|^{s-2}|\nabla y_t|^2 \\ \le C\Big(\int_D |\nabla(y+z)|^2|y_t|^{s-2} + \|\nabla(y+z)\|^2_{C(D)}\int_D \big(|y_t|^s + |\Delta y|^2|y_t|^{s-2}\big)\Big). \tag{4.18}$$

The first term on the right hand side of (4.18), using Young's inequality with $p = s/2$ and $p' = s/(s-2)$, is estimated as

$$\int_D |\nabla(y+z)|^2|y_t|^{s-2} \le \big(\|\nabla y\|^2_{L^s(D)} + \|\nabla z\|^2_{L^s(D)}\big)\|y_t\|^{s-2}_{L^s(D)}.$$

According to (4.5)

$$\int_D |\nabla(y+z)|^2|y_t|^{s-2} \le C\big(\|y_t\|^s_{L^s(D)} + \|y_t\|^{s-2}_{L^s(D)}\|\nabla z\|^2_{L^s(D)}\big).$$

Using Hölder's inequality, the third term on the right hand side of (4.18) yields

$$\int_D |\Delta y|^2|y_t|^{s-2} \le \|y_t\|^{s-2}_{L^s(D)}\|\Delta y\|^2_{L^s(D)}.$$

On the other hand, from the Gagliardo-Nirenberg's inequality it follows

$$\|\nabla y\|_{C(D)} \le \|\Delta y\|^b_{L^4(D)}\|\nabla y\|^{1-b}_{L^q(D)}.$$

For $b = 4/(q+4)$ and by (4.8) and (4.13), we obtain

$$\int_0^t \|\nabla y\|^{q+4}_{C(D)} \le \int_0^t \|\Delta y\|^4_{L^4(D)}\|\nabla y\|^q_{L^q(D)} \le C. \tag{4.19}$$

We set $\beta_s(t) = \sup_{0<\tau<t}\|y_\tau\|^s_{L^s}$. Then the expression on the right-hand side of (4.18), after integrating on (0,t), can be estimated as

$$\begin{aligned} &\|y_\tau\|^s_{L^s(D)} + \int_0^t\int_D |y_\tau|^{s-2}|\nabla y_\tau|^2 \\ &\le C(\|y_\tau(0)\|^s_{L^s(D)} + \int_0^t \|y_\tau\|^{s-2}_{L^s(D)}\|\nabla z\|^2_{L^s(D)} \\ &\quad + \int_0^t \Big(1 + \|\nabla(y+z)\|^2_{C(D)}\Big)\|y_\tau\|^s_{L^s(D)} \\ &\quad + \int_0^t \|\nabla(y+z)\|^2_{C(D)}\|y_\tau\|^{s-2}_{L^s(D)}\|\Delta y\|^s_{L^s(D)}). \end{aligned} \tag{4.20}$$

By Young's inequality ($p = s/(s-2), p' = s/2$), the last term on the right-hand side of the above inequality can be estimated as

$$
\begin{aligned}
&\int_0^t \|\nabla(y+z)\|^2_{C(D)} \|y_\tau\|^{s-2}_{L^s(D)} \|\Delta y\|^2_{L^s(D)} \\
&\le \sup_{0<\tau<t} \|y_\tau\|^{s-2}_{L^s(D)} \int_0^t \|\nabla(y+z)\|^2_{C(D)} \|\Delta y\|^2_{L^s(D)} \\
&\le (\beta_s(t))^{(s-2)/s} \Big(\int_0^t \|\nabla(y+z)\|^{2s/(s-2)}_{C(D)} \Big)^{(s-2)/s} \Big(\int_0^t \|\Delta y\|^s_{L^s(D)} \Big)^{2/s}.
\end{aligned}
$$

From (4.5), we have

$$
\int_0^t \|\Delta y\|^s_{L^s(D)} \le C\Big(1 + \int_0^t \beta_s(\tau) d\tau\Big).
$$

In view of (4.19) and Young's inequality it follows that

$$
\int_0^t \|\nabla(y+z)\|^2_{C(D)} \|y_\tau\|^{s-2}_{L^s(D)} \|\Delta y\|^2_{L^s(D)} d\tau \le \epsilon \beta_s(t) + \Big(1 + \int_0^t \beta_s(\tau) d\tau\Big).
$$

On the other hand the second term of (4.20) can be estimated as

$$
\begin{aligned}
\int_0^t \|y_\tau\|^{s-2}_{L^s(D)} \|\nabla z\|^2_{L^s(D)} &\le \beta_s(t)^{(s-2)/2} \int_0^t \|\nabla z\|^2_{L^s(D)} \\
&\le \epsilon \beta_s(t) + C_\epsilon \Big(\int_0^t \|\nabla z\|^2_{L^s(D)} \Big)^{s/2}.
\end{aligned}
$$

We conclude (4.16) by using Gronwall's lemma. □

These estimates are sufficient for proving the existence of solutions. One can use the scheme of constructing solutions given in [1]. According to this scheme, approximate solutions (y_k, ρ_k) are found by the Galerkin method; y_k is sought as a finite sum of basis and ρ_k is determined from the transport equation. In particular ∇y_k is compact in $L^2(Q_T)$. Thus, the passage to the limit in the nonlinear terms in equations (4.1) is justified.

4.2 Upper and lower bounds for the density, and uniqueness of the solution

The estimates obtained in the preceding section permit us to establish that the density ρ is bounded provided that the initial density ρ_0 is bounded. To this end, we write out a special equation for $\log(\rho)$. This idea has been used in [8].

Lemma 4.6 *If $y_0 \in W^{2,s}(D)$, $s \ge 2$ and $\rho_0 \in L^\infty(D)$ then*

$$
\|\rho(t)\|_{L^\infty(D)} \le M, \quad \forall t \in [0,T]. \tag{4.21}
$$

Proof: Assuming that $\rho(x,t)>0$, let us rewrite the second equation in (4.1) in the form

$$\frac{\partial \log\rho}{\partial t}+\nabla(y+z)\cdot\nabla\log\rho+\Delta(y+z)=0.$$

By adding the above equation to $(4.1)_1$, we obtain

$$\frac{\partial}{\partial t}(\log\rho+y)+\nabla(y+z)\cdot\nabla(\log\rho+y)+\rho=1-\Delta z+\nabla y\cdot\nabla(y+z). \tag{4.22}$$

Set $\gamma=\log\rho+y$, $\gamma_+=\max\{0,\gamma(x,t)\}$. Considering (4.22) as the transport equation for γ and taking into account the fact that ρ is nonnegative, we conclude that

$$\gamma_+(x,t)\le\|\gamma_+|_{t=0}\|_{L^\infty(D)}+\int_0^t\Big(1+\|\Delta z\|^2_{C(D)}+\|\nabla y\|^2_{C(D)}\Big)\,. \tag{4.23}$$

According to (4.8), $\|y\|_{L^\infty(Q_T)}$ is bounded; indeed

$$\|y\|_{L^\infty(Q_T)}\le C\sup_{0<t<T}\|\nabla y\|_{L^q(D)}\le C,\quad q>2. \tag{4.24}$$

Hence (4.21) follows by the hypotheses of the lemma and from (4.24) and (4.19) with the constant

$$\begin{aligned}M=\exp\Big(&\|y\|_{L^\infty(Q_T)}+\|\gamma_+|_{t=0}\|_{L^\infty(D)}\\&+\int_0^t\big(1+\|\Delta z\|^2_{C(D)}+\|\nabla y\|^2_{C(D)}\big)\Big).\end{aligned}$$

□

Lemma 4.7 *If the initial density $\rho_0(x)$ is strictly positive under the hypotheses of the lemma 4.6, then $\rho(x,t)$ remains a strictly positive function in Q_T i.e.*

$$\rho(x,t)\ge m>0\quad \text{a.e. in } Q_T. \tag{4.25}$$

Proof: Let us change the sign in equation (4.22) and rewrite it for γ. We wish to find an upper bound for the function $\gamma_-=\max\{0,-\gamma\}$. By analogy, we obtain

$$\gamma_-(x,t)\le\|\gamma_-|_{t=0}\|_{L^\infty(D)}+\int_0^t\Big(\|\Delta z\|^2_{L^\infty(D)}+\|\nabla y\|^2_{C(D)}\|\rho\|_{L^\infty(D)}\Big)\,. \tag{4.26}$$

Hence (4.25) follows with the constant

$$\begin{aligned}m=\exp\Big(&-(\|y\|_{L^\infty(D)}+\|\gamma_-|_{t=0}\|_{L^\infty(D)}\\&+\int_0^t\big(\|\Delta z\|^2_\infty+\|\nabla y\|^2_{C(D)}\big)+MT)\Big).\end{aligned}$$

$\square$

If the density ρ is bounded, then the solution of (4.1) is unique. Indeed, if (ρ', y') and (ρ'', y'') are two solutions, then their difference $\rho = \rho' - \rho''$, $y = y' - y''$ is a solution to the linear problem

$$\begin{gathered} y_t - \Delta y = -\rho, \\ \rho_t + \operatorname{div}(\rho \nabla(y' + z)) + \operatorname{div}(\rho'' \nabla y) = 0, \end{gathered} \tag{4.27}$$

with the zero initial conditions. Let us introduce an auxiliary function ψ, which is a solution to periodic Neumann problem

$$\Delta \psi = \rho, \quad \int_D \psi dx = 0. \tag{4.28}$$

Since ρ is bounded, $|\nabla \psi|$ is also bounded i.e.

$$|\nabla \psi| \leq M \quad \text{a.e. in } Q_T. \tag{4.29}$$

We multiply $(4.27)_1$ by y and $(4.27)_2$ by ψ, we sum these equations and integrate over D, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|y\|^2 + \|\nabla \psi\|^2 \right) + \|\nabla y\|^2 \\ &= \langle \nabla y, \nabla \psi \rangle - \langle \rho'' \nabla y, \nabla \psi \rangle - \langle \Delta \psi \nabla(y' + z), \nabla \psi \rangle. \end{aligned}$$

The first two terms on the right-hand side of this equation are bounded by $2^{-1}\|\nabla y\|_2^2 + C\|\nabla \psi\|_2^2$ in view of the Cauchy's inequality, since ρ' is bounded. The last term can be trasformed by integrating by parts as follows

$$\langle \Delta \psi \nabla(y' + z), \nabla \psi \rangle = \langle (\nabla \psi \nabla) \cdot \nabla(y' + z), \nabla \psi \rangle - \frac{1}{2} \langle \Delta(y' + z), |\nabla \psi|^2 \rangle.$$

For the other part, by Hölder's inequality

$$\int_D |\nabla \psi|^2 D^2(y' + z) \leq \Big(\int_D |D^2(y' + z)|^p \Big)^{1/p} \Big(\int_D |\nabla \psi|^{2p'} \Big)^{1/p'},$$

with $p = \epsilon^{-1} > 1$ and $p' = 1/(1 - \epsilon)$ where $0 < \epsilon < 1$ is arbitrary.

$$\int_D |\nabla \psi|^2 D^2(y' + z) \leq M^{2\epsilon} \|\nabla \psi\|^{2(1-\epsilon)} \|D^2(y' + z)\|_{L^{1/\epsilon}}.$$

Thus, for the nonnegative function

$$Y(t) = \|y\|^2 + \|\nabla \psi\|^2, \quad Y(0) = 0\,,$$

expression (4.29) yields the inequality

$$\frac{dY(t)}{dt} \leq C_1 Y(t) + M^{2\epsilon} \|D^2(y' + z)\|_{L^{1/\epsilon}} Y(t)^{1-\epsilon}.$$

Consequently,

$$\begin{aligned}Y^\epsilon(t) &\le \epsilon M^{2\epsilon} \mathrm{e}^{(C_1+C)\epsilon t} \int_0^t \mathrm{e}^{-C\epsilon\tau} \|D^2(y'+z)\|_{L^{1/\epsilon}} d\tau \\ &\le \epsilon C M^{2\epsilon} \mathrm{e}^{(C_1+C)\epsilon t} \int_0^t \mathrm{e}^{-C\epsilon\tau} \left(\|D^2 y'\|_{L^{1/\epsilon}} + 1\right) d\tau\end{aligned}$$

We apply Hölder inequality to the integral with the exponent $p = 1/\epsilon$ and $p' = 1/(1-\epsilon)$; we obtain

$$\begin{aligned}Y^\epsilon(t) &\le \epsilon C M^{2\epsilon} \mathrm{e}^{(C_1+C)\epsilon t} \Big(\int_0^t (\mathrm{e}^{-C\epsilon\tau})^{p'} \Big)^{1/p'} \Big(\int_0^t (\|D^2 y'\|_{L^{1/\epsilon}} + 1)^p d\tau \Big)^p \\ &\le \epsilon C M^{2\epsilon} \Big(\|D^2 y'\|_{L^{1/\epsilon}(Q_T)} + 1 \Big) \Big(\frac{1-\epsilon}{C\epsilon} \big(\mathrm{e}^{C\epsilon t/(1-\epsilon)} - 1 \big) \Big)^{1-\epsilon}.\end{aligned} \tag{4.30}$$

Considering y' as a solution to the parabolic equation $y'_t - \Delta y' = 1 - \rho'$ with bounded right-hand side and using estimates for the higher derivatives in the $L^{1/\epsilon}$-norm (see [9]), we obtain

$$\|D^2 y'\|_{L^{1/\epsilon}(Q_T)} \le C\epsilon^{-1} \|1-\rho'\|_{L^{1/\epsilon}(Q_T)} \le C\epsilon^{-1}.$$

Therefore, from (4.30) follows that

$$Y(t) \le C^{1/\epsilon} M^2 \Big(\frac{1-\epsilon}{C\epsilon} \big(\mathrm{e}^{C\epsilon t/(1-\epsilon)} - 1 \big) \Big)^{(1-\epsilon)/\epsilon}.$$

It is easy to check that if $t \in [0,\tau]$, where $C\tau < 1$ on $[0,\tau]$, then the right-hand side of the last inequality vanishes as $\epsilon \to 0$. Hence $Y(t) = 0$ on $[0,\tau]$. Repeating the argument for the interval $[\tau, 2\tau]$ and so on, we obtain $Y(t) = 0$, which proves the uniqueness of the solution.

Conclusion

The solution of deterministic system of equations (4.1) will allow us constructing the solution of the stochastic system (1.4).

Proof of Theorem 2.1 We can apply Theorem 4.1 to obtain existence and uniqueness of solution for problem (3.4) for fixed ω. The measurability is an obvious fact using the uniqueness of the solution (see [17]). As a consequence, using (3.3) and the properties (measurability) of z, this theorem is proved. □

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