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Bernstein - Chebyshev inequality and Baran's radial extremal function on algebraic sets

Leokadia Bialas-Ciez[a] · Agnieszka Kowalska[b]

Dedicated to Professor Mirosław Baran on his 60th birthday

Abstract

We study a Bernstein-Chebyshev inequality and some Pleśniak type properties on polynomially determining sets and on a wide class of algebraic varieties. We show that a compact subset E of algebraic variety V satisfies a Bernstein-Chebyshev inequality if and only if a projection of E satisfies a Bernstein-Chebyshev inequality. Moreover, we give an estimate of appropriate constants. These inequalities are also studied on preimages under simple polynomial maps. Baran's radial extremal function is calculated for some compacts on algebraic sets.

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1 Introduction

For a given number $r \in [1,\infty]$ consider such a compact set $K \subset \mathbb{K}^N$ that for any $\delta > 0$ and $n \in \mathbb{N}$ the following *Bernstein-Chebyshev inequality* holds

$$\|p\|_{r,K_\delta} \le C\,\|p\|_{r,K} \tag{1}$$

where

$$K_\delta := \{x \in \mathbb{K}^N : \operatorname{dist}(x,K) \le \delta\}, \qquad \operatorname{dist}(x,K) := \inf\{\|x-y\| : y \in K\}$$

and $C = C(\delta,n,r)$ is a constant independent of the polynomial p of N variables and degree at most n. The symbol $\|\cdot\|$ denotes one of norms in $\mathbb{K}^N$, say the Euclidean one. Here and throughout, μ is a positive measure on an open neighbourhood of K and

$$\|p\|_{r,K} := \left(\int_K |p(x)|^r\,d\mu(x)\right)^{1/r}, \quad r < \infty,$$

$$\|p\|_{\infty,K} := \max\{|p(x)| : x \in K\}.$$

A pointwise version of inequality (1) is often called a *Bernstein-Walsh inequality*. Similar problems have been studied recently by Bos, Ma'u and Waldron in [10]. For notational convenience, set $\|p\|_K := \|p\|_{\infty,K}$ and $C(\delta,n) := C(\delta,n,\infty)$. Obviously, C depends also on the set K and the measure μ. Assume that $C(\delta,n,r)$ is the optimal constant in inequality (1), i.e.

$$C(\delta,n,r) = \sup\left\{\frac{\|p\|_{r,K_\delta}}{\|p\|_{r,K}} : p \not\equiv 0 \text{ on } K,\ \deg p \le n\right\} < \infty.$$

It is worth noticing that if K is a set not determining for polynomials then inequality (1) does not hold, which can be interpreted as $C(\delta,n,r) = \infty$. Our aim is to present some estimates of the constants $C(\delta,n,r)$ for selected compact sets. These estimates are linked to several concepts which we mention below. We first explain a relationship of (1) with Bernstein's and Chebyshev's results.

In the case of a segment in $\mathbb{R}$ and the sup norm, estimate (1) is usually called the *Chebyshev inequality* :

$$\|p\|_{[-1,1]_\delta} = \|p\|_{[-1-\delta,1+\delta]} \le T_n(1+\delta)\,\|p\|_{[-1,1]} \tag{2}$$

where T_n is the n-th Chebyshev polynomial of the first kind. If $K = \mathbb{D} := \{z \in \mathbb{C} : |z| \le 1\}$ then the estimate

$$\|p\|_{\mathbb{D}_\delta} = \|p\|_{(1+\delta)\mathbb{D}} \le (1+\delta)^n\,\|p\|_{\mathbb{D}}$$

is known as the *Bernstein inequality*. In the two estimates above we get equality for $p(x) = T_n(x)$ and $p(z) = z^n$, respectively. Consequently,

$$C(\delta,n) = T_n(1+\delta) \quad \text{for} \ \ K = [-1,1] \subset \mathbb{R},$$

$$C(\delta,n) = (1+\delta)^n \quad \text{for} \ \ K = \mathbb{D} \subset \mathbb{C}.$$

By means of the constants $C(\delta,n,r)$, we can state definitions of certain known properties of $K \subset \mathbb{C}^N$. We list below some of them.

[a]Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland, email: leokadia.bialas-ciez@im.uj.edu.pl

[b]Institute of Mathematics, Pedagogical University of Krakow, Podchorążych 2, 30-084 Kraków, Poland, email: agnieszka.kowalska@up.krakow.pl

1. If there exists a positive constant m such that
$$\sup_{n\in\mathbb{N}} C(\tfrac{1}{n^m}, n, r) < \infty \tag{3}$$
then inequality (1) is called *Pleśniak's property* of K in the L^r norm, see e.g. [12], [2]. For $r = \infty$, condition (3) is equivalent to Markov's property (see [12]). We say that the set K satisfies the *Markov inequality* in the L^r norm, if
$$\|\|\mathrm{grad}\, p\|\|_{r,K} \le M n^m \|p\|_{r,K} \tag{4}$$
where the constants $M, m > 0$ are independent of the polynomial p of degree at most n and grad p is the gradient of p.
2. If the following condition holds
$$\limsup_{\delta\to 0} \sup_{n\in\mathbb{N}} [C(\delta, n)]^{1/n} = 1 \tag{5}$$
then the set K is *L-regular*, i.e. the *pluricomplex Green's function* V_K of the set K is continuous in $\mathbb{C}^N$, where
$$V_K(z) := \sup\{u(z) : u \in L(\mathbb{C}^N) \text{ and } u \le 0 \text{ on } K\}, \quad z \in \mathbb{C}^N,$$
and $L(\mathbb{C}^N)$ is the *Lelong class* of plurisubharmonic functions in $\mathbb{C}^N$ of logarithmic growth at infinity.
3. Define *Baran's radial extremal function* of the set K by
$$\varphi_K^\bullet(\delta) := \sup\{\exp V_K(z+w) : z \in K, \|w\| \le \delta\} = \sup\{\exp V_K(z) : z \in K_\delta\} \quad \text{for } \delta > 0.$$
This function was introduced for the first time in the article [1]. It is strictly related to polynomials because of the *Siciak-Zakharyuta formula*:
$$V_K(z) = \log \Phi_K(z), \quad z \in \mathbb{C}^N,$$
and the definition of the *Siciak extremal function*
$$\Phi_K(z) := \sup\left(\frac{|p(z)|}{\|p\|_K}\right)^{1/\deg p}$$
where the supremum is taken over all polynomials p of N variables of degree $\deg p \ge 1$, $p \not\equiv 0$ on K. Consequently, one can easily prove that
$$\varphi_K^\bullet(\delta) = \sup_{n\in\mathbb{N}} [C(\delta, n)]^{1/n}.$$
Taking into account Siciak's theorem (see e.g. [13]):
$$\Phi_K(z) = \lim_{n\to\infty} (\Phi_n(z))^{1/n}, \quad \Phi_n(z) := \sup\left\{\frac{|p(z)|}{\|p\|_K} : p \in \mathcal{P}_n(\mathbb{C}^N), \ p \not\equiv 0 \text{ on } K\right\},$$
we can easily prove that
$$\varphi_K^\bullet(\delta) = \lim_{n\to\infty} (C(\delta, n))^{1/n}, \quad \delta > 0. \tag{6}$$
The function $\varphi_K^\bullet$ has been extended to the case of other norms by M. Baran in a presentation at the 8th European Congress of Mathematics 2021. Namely, for the L^r norm on K we have
$$\varphi_{r,K}^\bullet(\delta) := \sup\left\{\left(\frac{|p(z+w)|}{\|p\|_{r,K}}\right)^{1/\deg p} : \deg p \ge 1, \ p \not\equiv 0 \text{ on } K, \ z \in K, \ \|w\| \le \delta\right\}.$$
The above definition is equivalent to the condition
$$\varphi_{r,K}^\bullet(\delta) = \sup_{n\in\mathbb{N}} [C(\delta, n, r)]^{1/n}. \tag{7}$$
This implies that $C(\delta, n, r) \le \left[\varphi_{r,K}^\bullet(\delta)\right]^n$ but we cannot expect equality here for an arbitrary set K, see the Chebyshev inequality (2).

Exact formulas of $\varphi_K^\bullet$ are only known for a few selected polynomially determining sets, see [1, 2]:

- if K is a unit ball in $\mathbb{C}^N$ (with respect to a fixed complex norm), then
$$\varphi_K^\bullet(\delta) = 1 + \delta/c(K) \text{ where } c(K) \text{ is the L-capacity of } K,$$
- if K is a convex symmetric body in $\mathbb{R}^N$, then $\varphi_K^\bullet(\delta) = h(1 + \delta/(2c(K)))$ where $h(x) = x + \sqrt{x^2 - 1}$ for $x \ge 1$.

Denote by $\mathcal{P}_n(\mathbb{C}^N)$ or $\mathcal{P}_n(z)$, $z \in \mathbb{C}^N$ the space of polynomials of N variables, degree at most n and complex coefficients. Let $\mathcal{P}(\mathbb{C}^N) := \bigcup_{n\in\mathbb{N}_0} \mathcal{P}_n(\mathbb{C}^N)$ and $\mathcal{P}(z) := \bigcup_{n\in\mathbb{N}_0} \mathcal{P}_n(z)$.

In this paper we are interested in estimation of constants $C(\delta, n)$ and Baran's radial extremal function not only for polynomially determining sets but also for some subsets of algebraic varieties. Since estimate (1) does not hold if K is not determining for polynomials, we need a version of the Bernstein-Chebyshev inequality appropriate for a compact subset E of an algebraic variety V. It suffices to restrict inequality (1) to a certain space $\mathbf{W}_v \subset \mathcal{P}(\mathbb{C}^{N+1})$ as we showed it in the papers [7], [8], [9] for Markov and division inequalities. A precise construction of $\mathbf{W}_v$ for an arbitrary algebraic set V is given in [8] by means of Gröbner bases. Moreover, a simple isomorphism is constructed between $\mathbf{W}_v$ and the space

$$\mathcal{P}(V) := \{p_{|_V} \,:\, p \in \mathcal{P}(\mathbb{C}^N)\} = \mathcal{P}(\mathbb{C}^N)/I(V)^1.$$

We are interested in algebraic hypersurfaces (see Section 3) and some algebraic sets defined by two polynomial equations (Section 4). In these two cases we can easily indicate an appropriate subspace $\mathbf{W}_v$ of polynomials. Indeed, consider an algebraic hypersurface V. It is convenient to assume that V is a subset of $\mathbb{C}^{N+1}$. The variety V is defined as a zero set of a polynomial $s \in \mathcal{P}(\mathbb{C}^{N+1})$ of degree at least 1. Taking a linear invertible change of variables if necessary, we can write the polynomial s in the following form

$$s(z,y) = y^k + \sum_{j=0}^{k-1} s_j(z)\, y^j \quad \text{for } k \ge 1,\ z \in \mathbb{C}^N,\ y \in \mathbb{C} \tag{8}$$

where $s_j \in \mathcal{P}(z)$, $j = 0, ..., k-1$. Let

$$\mathbf{W}_v = \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y) := \{q \in \mathcal{P}(z,y) \,:\, q(z,y) = \sum_{j=0}^{k-1} q_j(z)\, y^j,\ \ q_0, ..., q_{k-1} \in \mathcal{P}(z)\}.$$

Observe that $\mathbf{W}_v$ is an infinite dimensional subspace of $\mathcal{P}(\mathbb{C}^{N+1})$ and is invariant under derivation. Moreover, we can easily give an isomorphism Φ between $\mathbf{W}_v$ and the quotient space $\mathcal{P}(V)$

$$\Phi \;:\; \mathbf{W}_v \ni p \;\mapsto\; p_{|_V} \in \mathcal{P}(V). \tag{9}$$

On a compact subset E of $V = V(s)$ we will consider Bernstein-Chebyshev inequalities (1) only for polynomials $p \in \mathbf{W}_v$, i.e.

$$\|p\|_{E_\delta} \le C\, \|p\|_E, \quad p \in \mathbf{W}_v \tag{10}$$

where $E_\delta := \{z \in \mathbb{C}^{N+1} \,:\, \operatorname{dist}(z, E) \le \delta\}$ and

$$C = C(\delta, n, E) = \sup\left\{ \frac{\|p\|_{E_\delta}}{\|p\|_E} \,:\, p \not\equiv 0 \text{ on } E,\ p \in \mathcal{P}_n(\mathbb{C}^{N+1}) \cap \mathbf{W}_v \right\}. \tag{11}$$

Let us emphasise that the metric hull E_δ is taken in $\mathbb{C}^{N+1}$ and not only in V. Estimate (10) is called the *Bernstein-Chebyshev inequality on* $E \subset V(s)$. In Section 3 we will give an equivalent condition for E to satisfy inequality (10).

Analogously as for a compact set K in $\mathbb{C}^N$, see (5), (3), (4), (7), we can define L-regularity, Pleśniak's property, and Baran's extremal function $\varphi_E^\bullet$ of a compact subset E of V restricting the space of polynomials to $\mathbf{W}_v$, see Section 3 for the details.

The paper is organized as follows. In Section 2 we are interested in Bernstein-Chebyshev inequalities on preimages of compact sets under simple polynomial maps. We recall a construction of a measure on preimages that allows to transfer some polynomial inequalities, cf. [6]. Section 3 concerns Bernstein-Chebyshev type inequalities on compact subsets of algebraic hypersurfaces. Some estimates of constants C give formulas of Baran's radial extremal functions. In Section 4 we show analogous results for certain algebraic sets of codimension greater than 1. Some concrete examples are presented in the last section.

2 Invariance under simple polynomial maps

Recall that $q = (q_1, \dots, q_N) :\ \mathbb{C}^N \to \mathbb{C}^N$ is a *simple polynomial map of degree* $m \ge 1$ if we have $LM(q_i) = z_i^m$, $i = 1, \dots, N$ where the leading monomial LM is taken with respect to the graded lexicographic order in the family of monomials $\mathbb{T}^N := \{z^\alpha \,:\, \alpha \in \mathbb{N}^N\}$. Every simple polynomial map is proper (see [6]) and therefore, $q^{-1}(E)$ is compact for any compact set $E \subset \mathbb{C}^N$. Morever, the set $q^{-1}(w)$ has exactly m^N elements counting with multiplicities and continuously depends on $w \in E$ if we assume that $\det q' \neq 0$ at any points of $q^{-1}(E)$, where q' denotes the derivative of q and $\det q'$ is its Jacobian (see [6]).

For a given positive measure μ on a compact set $E \subset \mathbb{C}^N$, a continuous function $f :\ q^{-1}(E) \to \mathbb{C}$, and a simple polynomial map $q :\ \mathbb{C}^N \to \mathbb{C}^N$ of degree $m \ge 1$, we define the positive measure $q_*\mu$ following [6]:

$$\int_{q^{-1}(E)} f\, dq_*\mu := \int_E q_*f\, d\mu$$

and

$$q_*f(w) := \frac{1}{m^N} \sum_{z \in q^{-1}(w)} f(z), \qquad w \in E$$

where any root is repeated according to its multiplicity. The function q_*f is continuous if $\det q' \neq 0$ on $q^{-1}(E)$.

[1] $I(V) = I(V(s)) = \{p \in \mathcal{P}(\mathbb{C}^{N+1}) \,:\, p|_V = 0\}$ is the ideal of polynomials vanishing on V

Proposition 2.1. *Fix $\delta \in (0,1]$ and $r \in [1,\infty]$. Let K be a compact subset of $\mathbb{C}^N$ and μ be a positive measure on K_1. Assume that $q : \mathbb{C}^N \to \mathbb{C}^N$ is a simple polynomial map of degree m with $\det q' \neq 0$ on $q^{-1}(K)$. If K satisfies a Bernstein-Chebyshev inequality (1) with a constant $C(\delta,n,r,K)$ with respect to μ on K_1, then on the preimage $q^{-1}(K)$ we have*

$$\|p\|_{r,q^{-1}(K_\delta)} \leq C_* \|p\|_{r,q^{-1}(K)} \quad \text{for} \;\; p \in \mathcal{P}_n(\mathbb{C}^N), \tag{12}$$

where $\|p\|_{r,q^{-1}(K_\delta)}$, $\|p\|_{r,q^{-1}(K)}$ are taken with respect to the measure $q_*\mu$ defined above,

$$C_* = C_*(\delta,n,r,q^{-1}(K)) = M \cdot C\left(\delta, \left\lceil \frac{n}{m} \right\rceil, r, K\right) \tag{13}$$

and M is independent of n, δ and r.

Proof. First, we consider the case $r < \infty$. Let $p \in \mathcal{P}_n(\mathbb{C}^N)$. Then $m(d-1) < \deg p \leq md$ for some $d \in \mathbb{N}$. Lemma 3 in [6] states that for any $d \in \mathbb{N}$, every polynomial $p \in \mathcal{P}_{md}(\mathbb{C}^N)$ can be written as

$$p(z) = \sum_{\alpha \in A_{m-1}} z^\alpha R_\alpha(q(z)), \; z \in \mathbb{C}^N$$

where A_{m-1} denotes the set of multi-indices $\alpha = (\alpha_1,\ldots,\alpha_N)$ such that $\max \alpha_i \leq m-1$ and $R_\alpha \in \mathcal{P}_d(\mathbb{C}^N)$ for $\alpha \in A_{m-1}$. We apply this result to obtain

$$\begin{aligned}\|p\|_{r,q^{-1}(K_\delta)} &= \left[\frac{1}{m^N} \int_{K_\delta} \sum_{z \in q^{-1}(w)} |p(z)|^r \, d\mu(w) \right]^{1/r} \\ &= \frac{1}{m^{N/r}} \left[\int_{K_\delta} \sum_{z \in q^{-1}(w)} \left| \sum_{\alpha \in A_{m-1}} z^\alpha R_\alpha(q(z)) \right|^r d\mu(w) \right]^{1/r}.\end{aligned}$$

For $z \in q^{-1}(w) \subset q^{-1}(K_\delta)$ we have $|z^\alpha| \leq \|z\|^{|\alpha|} \leq \|z\|^{N(m-1)}_{q^{-1}(K_\delta)}$. Moreover,

$$\left| \sum_{\alpha \in A_{m-1}} z^\alpha R_\alpha(q(z)) \right| \leq \sum_{\alpha \in A_{m-1}} |z^\alpha R_\alpha(q(z))| \leq \left(\sum_{\alpha \in A_{m-1}} |z^\alpha|^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} \left(\sum_{\alpha \in A_{m-1}} |R_\alpha(q(z))|^r \right)^{1/r}.$$

Consequently,

$$\begin{aligned}\|p\|_{r,q^{-1}(K_\delta)} &\leq \frac{1}{m^{N/r}} \left[\int_{K_\delta} \sum_{z \in q^{-1}(w)} \|z\|^{rN(m-1)}_{q^{-1}(K_\delta)} (Nm)^{r-1} \sum_{\alpha \in A_{m-1}} |R_\alpha(q(z))|^r \, d\mu(w) \right]^{1/r} \\ &= \|z\|^{N(m-1)}_{q^{-1}(K_\delta)} (Nm)^{\frac{r-1}{r}} \left[\sum_{\alpha \in A_{m-1}} \int_{K_\delta} |R_\alpha(w)|^r \, d\mu(w) \right]^{1/r}.\end{aligned}$$

Since K satisfies the Bernstein-Chebyshev inequality (1), we have also

$$\|p\|_{r,q^{-1}(K_\delta)} \leq \|z\|^{N(m-1)}_{q^{-1}(K_\delta)} (Nm)^{\frac{r-1}{r}} \left[\sum_{\alpha \in A_{m-1}} C(\delta,d,r,K)^r \int_K |R_\alpha(w)|^r \, d\mu(w) \right]^{1/r}.$$

Comparing it with $|R_\alpha(w)|^r \leq C(K,q)^r \|p\|^r_{q^{-1}(w)} \leq C(K,q)^r \sum_{z \in q^{-1}(w)} |p(z)|^r$ where a constant $C(K,q)$ depends only on K and q (see Proposition 7 in [6]), we obtain

$$\begin{aligned}\|p\|_{r,q^{-1}(K_\delta)} &\leq \|z\|^{N(m-1)}_{q^{-1}(K_\delta)} (Nm)^{\frac{r-1}{r}+\frac{1}{r}} C(\delta,d,r,K)\, C(K,q) \left[\int_K \sum_{z \in q^{-1}(w)} |p(z)|^r d\mu(w) \right]^{1/r} \\ &\leq \|z\|^{N(m-1)}_{q^{-1}(K_1)} Nm^{\frac{N}{r}+1} \cdot C(\delta,d,r,K) \cdot C(K,q)\, \|p\|_{r,q^{-1}(K)}.\end{aligned}$$

Since every simple map is proper, $q^{-1}(K_1)$ is compact. To emphasise that the constant C^* depends on δ and n in the same fashion as C, we write

$$C_* = C_*(\delta,n,r,q^{-1}(K)) = M \cdot C\left(\delta, \left\lceil \frac{n}{m} \right\rceil, r, K\right)$$

where M is independent of n, $\delta > 0$ and r. The proof for the case $r = \infty$ uses similar arguments. We leave it to the reader. □

On the left hand side of inequality (12) in Proposition 2.1 we have the norm on $q^{-1}(K_\delta)$. Since $\|q(z)-q(z_0)\| \le D\|z-z_0\|$ for some positive constant D depending only on q and K, we have

$$(q^{-1}(K))_\delta \subset q^{-1}(K_{\delta D}) \quad \text{ for all } \ \delta \in (0.1].$$

Therefore, (12) implies the Bernstein-Chebyshev inequality for $q^{-1}(K)$:

$$\|p\|_{r,(q^{-1}(K))_\delta} \le C_* \, \|p\|_{r,q^{-1}(K)} \quad \text{for} \ \ p \in \mathcal{P}_n(\mathbb{C}^N)$$

with respect to the measure $q_*\mu$ and $C_* = M \cdot C\left(\delta D, \left\lceil \frac{n}{m} \right\rceil, r, K\right)$.

Theorem 2.2. *Assume that K, r, μ and q are as in Proposition 2.1. If $C(\delta,n,r,K,\mu)$ is finite for all $\delta \in (0.1]$, $n \in \mathbb{N}$ then*

$$C(\delta,n,r,q^{-1}(K),q_*\mu) \le M \cdot C\left(\delta D, \left\lceil \frac{n}{m} \right\rceil, r, K, \mu\right), \quad \delta \in (0, \tfrac{1}{D}]$$

where M, D are independent of n, δ and r. In particular,

1. *If K has Pleśniak's property in the $L^r(\mu)$ norm then $q^{-1}(K)$ has it in the $L^r(q_*\mu)$ norm.*
2. *If K is L-regular then so also is $q^{-1}(K)$.*
3. *$\varphi^\bullet_{r,q^{-1}(K)}(\delta) \le M_0\left(\varphi^\bullet_{r,K}(\delta D)\right)^{1/m}$ for $\delta \in (0, \frac{1}{D}]$ where M_0, D are independent of n, δ and r.*

In the case of $r = \infty$, properties (2) and (3) (with $M_0 = 1$, because in this case we have (6)) can be also proved by Klimek's theorem, see Th. 5.3.1 in [11], because for any simple polynomial map q of degree m we have

$$\liminf_{\|z\|\to\infty} \|q(z)\|/\|z\|^m > 0.$$

Proposition 2.1 gives an estimate from below of C by C_*. We can give also a reverse estimate that is easier to prove.

Proposition 2.3. *Assume that K, r and μ are as in Proposition 2.1. Let $q : \mathbb{C}^N \to \mathbb{C}^N$ be a simple polynomial map of degree m. If inequality (12) is satisfied with a constant $C_*(\delta,n,r,q^{-1}(K),q_*\mu)$ for all $n \in \mathbb{N}$ then K satisfies Bernstein-Chebyshev inequality (1) and*

$$C(\delta,n,r,K,\mu) \le C_*(\delta,nm,r,q^{-1}(K),q_*\mu).$$

Proof. The case of $r = \infty$ is easy to show:

$$\begin{aligned}\|p\|_{K_\delta} &= \|p\circ q\|_{q^{-1}(K_\delta)} \le C_*(\delta,nm,\infty,q^{-1}(K),q_*\mu)\ \|p\circ q\|_{q^{-1}(K)}\\ &= C_*(\delta,nm,\infty,q^{-1}(K),q_*\mu)\ \|p\|_K.\end{aligned}$$

For $r < \infty$, we can observe that

$$q_*(|p\circ q|^r)(w) = \frac{1}{m^N}\sum_{z\in q^{-1}(w)} |(p\circ q)(z)|^r = \frac{1}{m^N}\sum_{z\in q^{-1}(w)} |p(w)|^r = |p(w)|^r.$$

Hence

$$\begin{aligned}\|p\|_{r,K_\delta} &= \left(\int_{K_\delta} |p|^r d\mu\right)^{1/r} = \left(\int_{K_\delta} q_*(|p\circ q|^r) d\mu\right)^{1/r}\\ &= \left(\int_{q^{-1}(K_\delta)} |p\circ q|^r \, dq_*\mu\right)^{1/r} = \|p\circ q\|_{r,q^{-1}(K_\delta)}.\end{aligned}$$

By the assumption,

$$\|p\|_{r,K_\delta} \le C_*(\delta,nm,r,q^{-1}(K),q_*\mu)\ \|p\circ q\|_{r,q^{-1}(K)} = C_*(\delta,nm,r,q^{-1}(K),q_*\mu)\ \|p\|_{r,K}$$

and the assertion follows. □

3 On general algebraic hypersurfaces

This section is devoted to the Bernstein-Chebyshev inequality and related properties of compact subsets of an algebraic hypersurface V. Usually, we will denote compact subsets of V by E, and compact subsets of $\mathbb{C}^N$ by K. Consider an algebraic hypersurface V given by a polynomial s. It is convenient to assume that V is a subset of $\mathbb{C}^{N+1}$. Then s is a polynomial of $N+1$ variables and, as explained in the Introduction, we can write s in the form (8). Every element of the quotient space $\mathcal{P}(V)$ can be represented by only one polynomial from $\mathbf{W}_v = \mathcal{P}(z)\otimes\mathcal{P}_{k-1}(y)$, $z \in \mathbb{C}^N$, $y \in \mathbb{C}$, see the isomorphism Φ in (9). We consider the projection π given by

$$\pi : V \ni (z,y) \ \mapsto \ z \in \mathbb{C}^N.$$

We say that $E \subset V$ has a *Markov property for polynomials from* $\mathbf{W}_v$ if there exist positive constants m, M such that

$$|||\mathrm{grad}\, p|||_E \le M(\deg p)^m \|p\|_E \ \text{ for every polynomial } p \in \mathbf{W}_v. \tag{14}$$

In [8] this property was also called $\mathbf{W}_v$-Markov property and we proved there the following characterization of this property.

Theorem 3.1. *Let $V = V(s) \subset \mathbb{C}^{N+1}$ be an algebraic variety defined by an irreducible polynomial s in form (8) and E be a compact subset of $V(s)$. Then E has a Markov property for polynomials from $\mathbf{W}_\nu$ if and only if $\pi(E)$ satisfies the Markov inequality (4) with $r = \infty$.*

Now, we are interested in a similar result for the Bernstein-Chebyshev inequality. We need the following property.

Definition 3.1. Let K be a compact subset of $\mathbb{C}^N$. We say that K satisfies the *division inequality* with exponent m if for any polynomial $q \not\equiv 0$ on K there exists a positive constant M such that for all polynomials $p \in \mathcal{P}(\mathbb{C}^N)$

$$\|p\|_K \le M(\deg p + \deg q)^{m\,\deg q}\|pq\|_K. \tag{15}$$

We know a lot of compact sets with this property, e.g. on any Markov set (a compact satisfying inequality (4) with $r = \infty$) a division inequality (15) holds, see [7].

The proposition given below will be crucial in the proof of the main result of this section.

Proposition 3.2. *(see [8, 9]) Let $V(s) \subset \mathbb{C}^{N+1}$ be an algebraic variety defined by an irreducible polynomial s in form (8), $K \subset \mathbb{C}^N$ be a compact set and $E := \pi^{-1}(K) \subset V(s)$. If K satisfies the division inequality with exponent m then*

$$\|[p_0,\dots,p_{k-1}]\|_K := \max_{j=0,\dots,k-1}\{\|p_j\|_K\} \le\ M_0\,(\deg p)^{m_0}\|p\|_E \tag{16}$$

for any polynomial p written in the form $p(z,y) = \sum_{j=0}^{k-1} p_j(z)\,y^j$ *on $V(s)$ where M_0, $m_0 \ge 0$ are constants independent of $p_0,\dots,p_{k-1}$ and* $m_0 = m(k-1)\deg s$.

Proposition 3.3. *Let $V = V(s) \subset \mathbb{C}^{N+1}$ be an algebraic variety defined by an irreducible polynomial s of form (8). Assume that $E \subset V$ is a compact set such that $\pi(E) \subset \mathbb{C}^N$ satisfies the division inequality and $E = \pi^{-1}(\pi(E))$. Then E satisfies a Bernstein-Chebyshev inequality (10) if and only if $\pi(E)$ satisfies a Bernstein-Chebyshev inequality (1) with $r = \infty$. Moreover, if $\pi(E)$ satisfies (1) with $r = \infty$ and $C = C(\delta,n,\pi(E))$, then $C(\delta,n,E) \le \tilde{M}(\delta)\,n^{m_0}C(\delta,n,\pi(E))$ in (10). On the other hand, $C(\delta,n,\pi(E)) \le C(\delta,n,E)$.*

Proof. Let $K = \pi(E)$. For $r = \infty$ and any polynomial $p \in \mathbf{W}_\nu$ written in the form $p(z,y) = \sum_{j=0}^{k-1} p_j(z)\,y^j$ with $\deg p \le n$ we get

$$\|p\|_{E_\delta} \le \sum_{j=0}^{k-1}\|p_j\|_{K_\delta}\,\|y\|^j_{E_\delta}.$$

It follows from (1) that

$$\|p\|_{E_\delta} \le C(\delta,n,K)\sum_{j=0}^{k-1}\|p_j\|_K\,\|y\|^j_{E_\delta}.$$

From Proposition 3.2 we have

$$\|p\|_{E_\delta} \le C(\delta,n,K)\,M_0\,n^{m_0}\|p\|_E\sum_{j=0}^{k-1}\|y\|^j_{E_\delta}$$

and this finishes the proof. □

Let us now introduce Pleśniak's property, the L-regularity and Baran's radial extremal function for compact subsets of algebraic sets. As noted before, we shall do this by restricting the space of polynomials to $\mathbf{W}_\nu$. More precisely, for a compact subset E of an algebraic set V we can use $C(\delta,n,E)$ defined by (11).

We say that a compact set $E \subset V$ has *Pleśniak's property for polynomials from* $\mathbf{W}_\nu$ with exponent m if

$$\|p\|_{E_\delta} \le C\,\|p\|_E, \quad \text{for } p \in \mathcal{P}_n(\mathbb{C}^{N+1}) \cap \mathbf{W}_\nu \ \text{ where } \delta = n^{-m} \tag{17}$$

and $C = \sup_{n\in\mathbb{N}} C(n^{-m},n,E) < \infty$.

A compact set $E \subset V$ is said to be *L-regular* if

$$\limsup_{\delta\to 0}\ \sup_{n\in\mathbb{N}}[C(\delta,n,E)]^{1/n} = 1. \tag{18}$$

Similarly as in (6), *Baran's radial extremal function* of the set $E \subset V$ is defined by

$$\varphi^\bullet_E(\delta) = \lim_{n\to\infty}[C(\delta,n,E)]^{1/n}, \delta > 0. \tag{19}$$

As an immediate consequence of Proposition 3.3 we have the following

Theorem 3.4. *Let $V = V(s) \subset \mathbb{C}^{N+1}$ be an algebraic variety defined by an irreducible polynomial s of form (8). Assume that $E \subset V$ is a compact such that $\pi(E)$ satysfies the division inequality and $\pi^{-1}(\pi(E)) = E$. Then*

1. *E has Pleśniak's property for polynomials from $\mathbf{W}_\nu$ with exponent m if and only if $\pi(E)$ has Pleśniak's property with exponent m.*
2. *E is L-regular if and only if $\pi(E)$ is L-regular.*

3. $\varphi_E^{\bullet} = \varphi_{\pi(E)}^{\bullet}$.

It is well known that for $r = \infty$ Pleśniak's property (3) is equivalent to Markov inequality (4) for any compact set in $\mathbb{C}^N$ (see [12]). Similarly to [12], using Taylor's formula and Cauchy's inequality one can prove that for compact subsets of $V(s)$ the Markov property (14) is equivalent to Pleśniak's property (17).

In the real case some generalizations of Pleśniak's property (3) were used in characterizations of semialgebraic curves in the class of compact, piecewise $\mathcal{C}^1$ curves and compact subsets with Zariski dimension σ in the class of compact sets with an analytic parametrization of order σ (see [4] and [5]). Another generalization of Pleśniak's property for certain compact subsets of algebraic sets in $\mathbb{R}^n$ were introduced in [3]. In a similar fashion we can define Pleśniak's property for compact subsets of algebraic sets in the complex space $\mathbb{C}^N$. For a compact subset E of an algebraic set $V \subset \mathbb{C}^N$, $\delta > 0$ and $p \in \mathcal{P}(\mathbb{C}^N)$ set

$$|||p|||_{E_\delta} \;:=\; \inf\{\|f_p\|_{E_\delta} \;:\; f_p \in \mathcal{P}(\mathbb{C}^N),\; f_p \equiv p \text{ on } V\}, \quad \text{the quotient norm.}$$

Definition 3.2. A compact subset E of an algebraic set $V \subset \mathbb{C}^N$ is said to have a *V-Pleśniak property with exponent m* if there exists a constant $M > 0$ such that for all polynomials $p \in \mathcal{P}(V)$ we have

$$|||p|||_{E_\delta} \;\le\; M\,\|p\|_E \;\text{ where } \delta = (\deg_V p)^{-m} \tag{20}$$

with $\deg_V p := \min\{\deg q \;:\; q \in \mathcal{P}(\mathbb{C}^N),\; q_{|_V} \equiv p\}$.

The above $\mathbb{C}^N$-Pleśniak property is equivalent to Pleśniak's property (3) with $r = \infty$.

The following result establishes an important relationship between the degrees of polynomials (see [9, Theorem 2.3])

Theorem 3.5. *Let $V = V(s) \subset \mathbb{C}^{N+1}$ be an algebraic variety defined by a polynomial s of form (8) and $d = \deg s$. For all polynomials $p \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ we have*

$$\deg_V p \le \deg p \le d \cdot \deg_V p.$$

Moreover, if $d = k$ then $\deg_V p = \deg p$.

In [7] we gave a better estimate of $\deg p$ in a specific case.

Theorem 3.6. *If $V = \{(z, y) \in \mathbb{C}^N \times \mathbb{C} \;:\; y^k + s_0(z) = 0\}$ and $d = \deg(y^k + s_0(z))$ then for any polynomial $p \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ we have*

$$\deg_V p \le \deg p \le \frac{d}{k}\, \deg_V p.$$

Theorem 3.7. *Let $V(s) \subset \mathbb{C}^{N+1}$ be an algebraic variety defined by an irreducible polynomial s of form (8) and $E \subset V(s)$. If E has the Markov property (14) with exponent m then E has V-Pleśniak property (20) with the same exponent m.*

Proof. Let $p \in \mathcal{P}(V)$, $\delta = (\deg_V p)^{-m}$ and Φ be an isomorphism given by (9). Fix $v \in E_\delta$ and $v_0 \in E$ such that $\operatorname{dist}(v, E) = \|v - v_0\|$. From Markov property (14) we get

$$\begin{aligned} |\Phi^{-1}(p)(v)| \;&\le\; \sum_\alpha \tfrac{1}{\alpha!} |D^\alpha \Phi^{-1}(p)(v_0)|\, \|v - v_0\|^{|\alpha|} \\ &\le\; \sum_\alpha \tfrac{1}{\alpha!}\, M^{|\alpha|} (\deg \Phi^{-1}(p))^{m|\alpha|} \|\Phi^{-1}(p)\|_E\, \|v - v_0\|^{|\alpha|}. \end{aligned}$$

Since $\sum_{|\alpha|=j} \frac{1}{\alpha!} = N^j/j!$, we get

$$|||p|||_{E_\delta} \le \|\Phi^{-1}(p)\|_{E_\delta} \le \sum_{j=0}^{\deg \Phi^{-1}(p)} \tfrac{N^j}{j!}\, M^j (\deg \Phi^{-1}(p))^{mj}\, \delta^j\, \|p\|_E.$$

Appling Theorem 3.5 we see that

$$\begin{aligned} |||p|||_{E_\delta} &\le \exp(NM(\deg \Phi^{-1}(p))^m \delta)\, \|p\|_E \le \exp(NM(d \deg_V p)^m \delta)\, \|p\|_E \\ &\le \exp(NMd^m)\, \|p\|_E. \end{aligned}$$

□

4 Algebraic sets of codimension greater than one

One can also consider an algebraic set

$$V(s_1, s_2) = \{(z, y_1, y_2) \;:\; s_1(z, y_1) = 0,\; s_2(z, y_1, y_2) = 0\} \subset \mathbb{C}^{N+2}$$

given by two polynomials s_1 and s_2 in the following forms

$$s_1(z, y_1) \;=\; y_1^{k_1} + \sum_{j=0}^{k_1-1} s_{1,j}(z)\, y_1^j, \tag{21}$$

$$s_2(z, y_1, y_2) \;=\; y_2^{k_2} + \sum_{l=0}^{k_2-1} s_{2,l}(z, y_1)\, y_2^l \tag{22}$$

with $z \in \mathbb{C}^N$, $y_1, y_2 \in \mathbb{C}$, $k_1 \geq 1$, $k_2 \geq 1$, $s_{1,j} \in \mathcal{P}(\mathbb{C}^N)$ for $j \in \{0,\dots,k_1-1\}$ and $s_{2,l} \in \mathcal{P}(\mathbb{C}^{N+1})$ for $l \in \{0,\dots,k_2-1\}$. Observe that in this case $\dim V = N$ and the space

$$\mathbf{W}_v = \mathcal{P}(z) \otimes \mathcal{P}_{k_1-1}(y_1) \otimes \mathcal{P}_{k_2-1}(y_2)$$

is an appropriate space of representatives of $\mathcal{P}(V(s_1,s_2))$ that is isomorphic to $\mathcal{P}(V(s_1,s_2))$. We denote by d_1 the degree of s_1 and by d_2 the degree of s_2. We now define three projections

$$\pi_1 : V(s_1) \ni (z,y_1) \mapsto z \in \mathbb{C}^N, \quad \pi_2 : V(s_1,s_2) \ni (z,y_1,y_2) \mapsto (z,y_1) \in V(s_1), \quad \tilde{\pi} := \pi_1 \circ \pi_2.$$

For a compact subset K of $\mathbb{C}^N$ we consider the set $F := \pi_1^{-1}(K) \subset V(s_1)$ and the set $E := \tilde{\pi}^{-1}(K) \subset V(s_1,s_2)$ that can be written in the form $E = \pi_2^{-1}(F)$. The assumption of the division inequality is crucial in Proposition 3.2, this suggests a question: under what conditions is it true that F satisfies the division inequality? The answer can be found in [9]

Theorem 4.1. *Let $V = V(s) \subset \mathbb{C}^{N+1}$ be an algebraic hypersurface given by an irreducible polynomial s in form (8) with $\deg s = d$ and K be a compact subset of $\mathbb{C}^N$ and $F := \{(z,y) \in V \ : \ z \in K\}$. If K is determining for polynomials from $\mathcal{P}(z)$ and satisfies the division inequality then F also satisfies the division inequality.*

From this theorem we obtain a division inequality for $F \subset V(s_1)$. In order to obtain the main result for sets of codimension greater than 1, we also need the definition presented in [8]

Definition 4.1. Let $\mathcal{V} \subset \mathbb{C}^N$ be an algebraic set and $q,s \in \mathcal{P}(z,y)$ where $z \in \mathbb{C}^N$, $y \in \mathbb{C}$. We say that polynomials q and s are *coprime (or relatively prime) on $\mathcal{V}$* if $\operatorname{Res}_y(q,s) \not\equiv 0$ on $\mathcal{V}$ ($\operatorname{Res}_y(q,s)$ is the resultant of q and s in y). The polynomial s is said to be *irreducible on $\mathcal{V}$* if it is relatively prime on $\mathcal{V}$ with any polynomial $q \in \mathcal{P}(z,y)$, $q \not\equiv 0$ on $\mathcal{V}$.

Theorem 4.2. *Let $V = V(s_1,s_2) \subset \mathbb{C}^{N+2}$ be an algebraic set given by polynomials s_1 and s_2 of forms (21-22). Assume that s_1 is an irreducible polynomial and s_2 is irreducible on $V(s_1)$ and $K \subset \mathbb{C}^N$ be a compact set satisfying the division inequality. If K satisfies the Bernstein-Chebyshev inequality (1) with $r = \infty$ then $E = \tilde{\pi}^{-1}(K) \subset V(s_1,s_2)$ satisfies the Bernstein-Chebyshev inequality for polynomials from $\mathbf{W}_v$. Moreover, if K satisfies (1) with $r = \infty$ and $C = C(\delta,n)$, then $C(\delta,n,V) \leq \tilde{M}(\delta)n^{m_0}C(\delta,n)$ for E in (10).*

Proof. For any polynomial $p \in \mathbf{W}_v$ written in the form $p(z,y_1,y_2) = \sum_{j=0}^{k_1-1}\sum_{l=0}^{k_2-1} p_{jl}(z)\, y_1^j y_2^l$ with $\deg p \leq n$ we get

$$\|p\|_{E_\delta} \leq \sum_{j=0}^{k_1-1}\sum_{l=0}^{k_2-1} \|p_{jl}\|_{K_\delta}\, \|y_1\|_{E_\delta}^j \|y_2\|_{E_\delta}^l.$$

It follows from (1) that

$$\|p\|_{E_\delta} \leq C(\delta,n) \sum_{j=0}^{k_1-1}\sum_{l=0}^{k_2-1} \|p_{jl}\|_K\, \|y_1\|_{E_\delta}^j \|y_2\|_{E_\delta}^l.$$

Since K satisfies the division inequality with some exponent m, from Proposition 3.2 we obtain

$$\|p\|_{E_\delta} \leq C(\delta,n)\, M_1\, n^{m_1} \sum_{j=0}^{k_1-1}\sum_{l=0}^{k_2-1} \|\tilde{p}_l\|_F\, \|y_1\|_{E_\delta}^j \|y_2\|_{E_\delta}^l$$

where $\tilde{p}_l(z,y_1) = \sum_{j=0}^{k_1-1} p_{jl}(z)\, y_1^j$ on $V(s_1)$ and $m_1 = m(k_1-1)d_1$.

From Theorem 4.1 the set F satisfies the division inequality with exponent $\tilde{m}$ and from Proposition 3.2 we have

$$\|p\|_{E_\delta} \leq C(\delta,n)\, M_1 \cdot M_2\, n^{m_2+m_1} \|p\|_E \sum_{j=0}^{k_1-1}\sum_{l=0}^{k_2-1} \|y_1\|_{E_\delta}^j \|y_2\|_{E_\delta}^l$$

where $m_2 = \tilde{m}(k_2-1)d_2$. $\square$

Analogously to Theorem 3.4, we may also get

Corollary 4.3. *Let $V = V(s_1,s_2) \subset \mathbb{C}^{N+2}$ be an algebraic set given by polynomials s_1 and s_2 of forms (21-22). Assume that s_1 is an irreducible polynomial and s_2 is irreducible on $V(s_1)$ and $E \subset V(s)$ such that $\tilde{\pi}(E)$ satysfies the division inequality and $\tilde{\pi}^{-1}(\tilde{\pi}(E)) = E$.*

1. *E has Pleśniak's property with exponent m for polynomials from $\mathbf{W}_v$ if and only if $\tilde{\pi}(E)$ has Pleśniak's property with exponent m.*
2. *E is L-regular if and only if $\tilde{\pi}(E)$ is L-regular.*
3. *$\varphi_E^\bullet(\delta) = \varphi_{\tilde{\pi}(E)}^\bullet(\delta)$*

In [9] we also proved the following relationship between the degrees of polynomials

Theorem 4.4. *For all polynomials $p \in \mathcal{P}(z) \otimes \mathcal{P}_{k_1-1}(y_1) \otimes \mathcal{P}_{k_2-1}(y_2)$*

$$\deg_V p \leq \deg p \leq d_1\, d_2\, \deg_V p.$$

Moreover, if $d_1 = k_1$ and $d_2 = k_2$ then $\deg_V p = \deg p$.

Using Theorem 4.4 we can prove

Proposition 4.5. *Let $V = V(s_1,s_2) \subset \mathbb{C}^{N+2}$ be an algebraic set given by polynomials s_1 and s_2 of forms (21-22). Assume that s_1 is an irreducible polynomial and s_2 is irreducible on $V(s_1)$. If $E \subset V(s_1,s_2)$ satisfies Markov property with exponent m for polynomials from $\mathbf{W}_v$ then E satisfies a V-Pleśniak property with exponent m.*

5 Examples

We now give some examples of the application of theorems proved in the previous sections. The computer system software Singular is used to determine the reduced Groebner basis of ideals $I(V)$ in these examples.

Example 5.1. Consider

$$V = V((x-1)^2 + y^2 + 9z^2 - 1, x^2 + y^2 + z^2 - 1) \subset \mathbb{C}^3$$

and the lexicographical ordering in the family of monomials $\{(x_1, x_2, x_3)^\alpha\}$ where $x_1 = y$, $x_2 = z$, $x_3 = x$, so we have an ordering $\preceq$ such that $x \preceq z \preceq y$. The reduced Gröbner basis of $I(V)$ is the set $G = \{y^2 + x^2 + \frac{1}{4}x - \frac{9}{8}, z^2 - \frac{1}{4}x + \frac{1}{8}\}$. Consequently, $\mathcal{P}(V)$ is isomorphic to $\mathbf{W}_v = \mathcal{P}(x) \otimes \mathcal{P}_1(z) \otimes \mathcal{P}_1(y)$ for $V = V(s_1, s_2)$ with $s_1(x, y) = y^2 + x^2 + \frac{1}{4}x - \frac{9}{8}$ and $s_2(x, y, z) = z^2 - \frac{1}{4}x + \frac{1}{8}$. Polynomials s_1 and s_2 are of forms (21-22).

We consider the set $E = \{(x, y, z) \in V \ : \ x \in [\frac{1}{2}, \frac{\sqrt{73}-1}{8}]\}$. Observe that $E \subset \mathbb{R}^3$ is the real curve given by the intersection of the two surfaces in Figure 1.

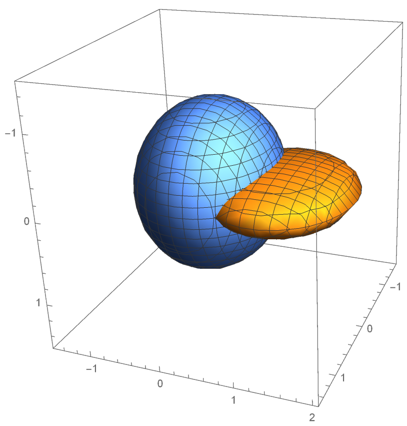

Figure 1: The set E, Example 5.1.

From Theorem 4.2 the set E satisfies the Bernstein-Chebyshev inequality

$$\|p\|_{E_\delta} \le M_0(\delta) T_n \left(\frac{(\sqrt{73}+5)\delta}{3} + 1 \right) n^{2(m+\tilde{m})} \|p\|_E, \quad p \in \mathcal{P}(x) \otimes \mathcal{P}_1(z) \otimes \mathcal{P}_1(y).$$

Moreover, Corollary 4.3 yields that E has Pleśniak's property and is L-regular. We have also the Baran's radial extremal function of the set E

$$\varphi_E^\bullet(\delta) = \varphi^\bullet_{\left[\frac{1}{2}, \frac{\sqrt{73}-1}{8}\right]}(\delta) = h\left(1 + \frac{(5+\sqrt{73})\delta}{3}\right)$$

where $h(x) = x + \sqrt{x^2 - 1}$ for $x \ge 1$.

Now we take the algebraic hypersurface $V_1 = V(z^2 - \frac{1}{4}x + \frac{1}{8}) \subset \mathbb{C}^2$ and the set $F = \left\{(x, z) \in V_1 \ : \ x \in \left[\frac{1}{2}, \frac{\sqrt{73}-1}{8}\right]\right\} \subset \mathbb{R}^2$.

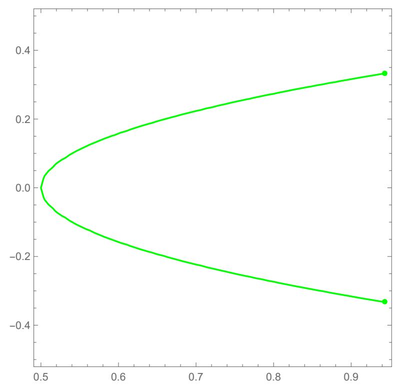

Figure 2: The set F, Example 5.1.

Then $\mathcal{P}(V_1)$ is isomorphic to $\mathcal{P}(x) \otimes \mathcal{P}_1(z)$. Polynomial $s(x, z) = z^2 - \frac{1}{4}x + \frac{1}{8}$ is of forms (8). From Proposition 3.3 we have that F satisfies the following Bernstein-Chebyshev inequality

$$\|p\|_{F_\delta} \le \tilde{M}(\delta) T_n \left(\frac{(\sqrt{73}+5)\delta}{3} + 1 \right) n^{2m} \|p\|_F, \quad p \in \mathcal{P}(x) \otimes \mathcal{P}_1(z).$$

Applying Theorem 3.4 we obtain that F has Pleśniak's property, is L-regular and $\varphi_F^\bullet(\delta)=\varphi_E^\bullet(\delta)$.

Example 5.2. Let $V=V(y^3-z^2+1,\ x^2-y^4-xz)\subset\mathbb{C}^3$ and $E=\{(x,y,z)\in V\ :\ y\in\mathbb{D}$ where $\mathbb{D}:=\{w\in\mathbb{C}\ :\ |w|\le 1\}$. For the reverse lexicographical ordering of $x_1=y,\ x_2=z,\ x_3=x$ the above two polynomials form the reduced Gröbner basis. Regarding the lexicographical ordering of x,y,z, we get $\{y^3-z^2+1, x^2-xz-yz^2+y\}$ as the reduced Gröbner basis and $W_\nu=\mathcal{P}_1(x)\otimes\mathcal{P}_2(y)\otimes\mathcal{P}(z)$ that is isomorphic to $\mathcal{P}(V)$. Eliminating the variable x leads us to the set

$$F:=\{(y,z)\in\mathbb{C}^2\ :\ y^3=z^2-1,\ y\in\mathbb{D}\}$$

that is the projection of E into the space of (y,z). By Example 23 in [7], the set F satisfies the Markov inequality for polynomials from $\mathcal{P}_2(y)\otimes\mathcal{P}(z)$ because the projection of F into the z-plane is the Bernoulli lemniscate that satisfies Markov inequality (4) in $\mathbb{C}$ with $\gamma=\infty$. It follows from Theorem 4.2 that the set E satisfies the Bernstein-Chebyshev inequality for polynomials from W_ν and is L-regular. We have the Baran's radial extremal function of the set E

$$\varphi_E^\bullet(\delta)=\varphi_{\mathbb{D}}^\bullet(\delta)=1+\delta.$$

Example 5.3. Consider the algebraic variety $V=V(p_1,p_2)$ defined by $p_1(x,y,z)=z-yx$ and $p_2(x,y,z)=z^2-y^2-25$. Regarding the lexicographical ordering or the degree lexicographical ordering or the degree reverse lexicographical ordering, we obtain the reduced Gröbner basis with three elements: $-p_1$, $-p_2$, p_3 where $p_3(x,y,z)=xz^2-25x-yz$. For the reverse lexicographical ordering we get the following Gröbner basis of $I(V)$:

$$q_1(x,y,z)=z-xy \text{ and } q_2(x,y,z)=x^2y^2-y^2-25.$$

These polynomials do not have the form as in Theorem 4.2, but if we take the linear invertible change of variables $(x_1,y_1,z_1)=(x+y,y,z)$ we obtain the algebraic variety $V'=\{(x,y,z)\in\mathbb{C}^3\ :\ z^2-y^2=25 \text{ and } z=y^2+xy\}$.

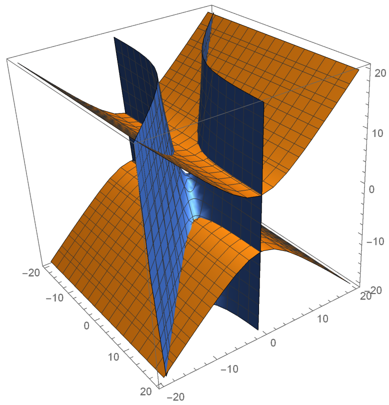

Figure 3: Example 5.3

For the reverse lexicographical ordering we obtain the following Gröbner basis of $I(V')$:

$$q_1'(x,y,z)=z-y^2-xy \quad\text{and}\quad q_2'(x,y,z)=y^4+2xy^3+x^2y^2-y^2-25$$

and $W_{\nu'}=\mathcal{P}(x)\otimes\mathcal{P}_3(y)$ is isomorphic to $\mathcal{P}(V')$. We consider the set $E=\{(x,y,z)\in V'\ :\ x\in[5,10]\}$. Observe that $E\subset\mathbb{R}^3$ is

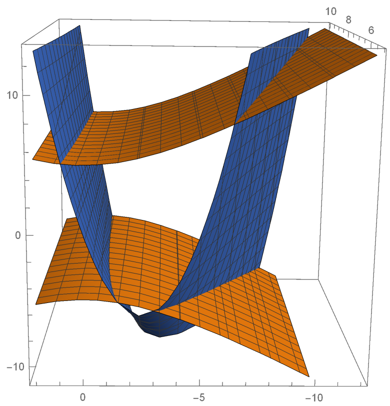

Figure 4: The set E, Example 5.3.

the real curve given by the surface intersections shown in Figure 4. From Theorem 4.2 the set E satisfies the Bernstein-Chebyshev inequality

$$\|p\|_{E_\delta} \le M_0(\delta) T_n\left(\frac{2\delta}{5}+1\right) n^{12m}\, \|p\|_E, \quad p \in \mathcal{P}(x) \otimes \mathcal{P}_3(y).$$

By Corollary 4.3 the E has Pleśniak's property and is L-regular. Moreover, we have the Baran's radial extremal function of the set E

$$\varphi_E^{\bullet}(\delta) = \varphi_{[5,10]}^{\bullet}(\delta) = h\left(1+\frac{2\delta}{5}\right) = 1 + \frac{2}{5}\left(\delta + \sqrt{5\delta+\delta^2}\right).$$

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