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On the bialgebra of functional graphs and differential algebras

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We develop the bialgebraic structure based on the set of functional graphs, which generalize the case of the forests of rooted trees. We use noncommutative polynomials as generating monomials of the functional graphs, and we introduce circular and arborescent brackets in accordance with the decomposition in connected components of the graph of a mapping of $\{1,2,...n\}$ in itself as in the frame of the discrete dynamical systems. We give applications for differential algebras and algebras of differential operators.

Keywords: bialgebraic structure, functional graphs, noncommutative polynomials

1 Introduction

We have already described the expansion of $\Delta = \Sigma\lambda^i\partial_i$, i.e. the powers of a Lie operator in any dimension, in order to find the expression of the flow of formal nonlinear evolution equations [1–3]. In the one-dimensional case, the explicit expansion can be found first in Comtet [4], and other aspects connected to the ordinary differential equations can be found in Leroux and Viennot [5] and Bergeron and Reutenauer [6]. On the other hand, Grossman and Larson [7] introduced several Hopf algebras [8–10] of forests of rooted labeled trees to express the product of finite dimensional vector fields. In this paper we concentrate us on the bialgebra **G** of functional graphs, i.e. graphs representing mappings of finite sets in themselves [11–15]. We give only the results without proofs. In a forthcoming paper [16], we develop Hopf algebra structures, computing the antipode and giving detailed aspects and proofs.

In Sect. 1 we consider a bialgebra structure on **G** and three interesting subalgebras: **T** the set of labeled forests; **S** the set of permutation graphs; and **L** the set of well labeled forests, i.e. with strictly decreasing labels on the chains toward the roots. Recall that the graded bialgebra **L** is sufficient for the calculus of the powers of one derivation [1], and it is extendable in a Hopf algebra, the element of which is known in the computer literature as 'heap ordered trees'. This bialgebra is useful to compute products of derivations or to transform differential monomials in differential algebras [17], and it is interesting to note that the elements $\mathbf{L}_n$ (n edges) can be coded by the words (monomials) of the expansion of $Q_n \equiv q_0(q_0+q_1)\dots(q_0+q_1+\dots q_{n-1})$, where $Q^0 \equiv \{q_0,q_1,...\}$ is a noncommutative alphabet. We describe in particular the bialgebra **L**, first in the polynomial form by the 'factorial' monoid $L^0 = (L^0_n)_{n\geq 0}$, where L^0_n is the set of words in the expansion of Q_n, and second, we establish the bijective correspondence between

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L and $\mathbf{L}$. We show that the calculus are easier with L, and that the product on L can be expressed in a very natural way. For example, $(q_0)^{\cdot n} = Q_n$, hence the (exponential) generating function of all the elements of $\mathbf{L}$. We describe principally the formalism in the general case $\mathbf{G}$, and the calculus uses the fields $F_2 = \{0,1\}$ as well as characteristic zero fields K.

In Sect. 2, we describe the link with the graded differential algebra $K\{U\} = \oplus_{r\geq 0} K\{U\}_r$ and the graded algebra of differential operators $K\{U,D\} = \oplus_{r\geq 0} K\{U\}_r D^r$, where $U \equiv (u_1, u_2, ...) = (u_\beta)_{\beta\geq 1} = (u_\beta{}^\alpha)_{\alpha\geq 0,\beta\geq 1}$ is a set of indeterminates,$D \equiv \{\partial_0, \partial_1, ...\}$ and the differential indeterminates $u_\beta{}^\alpha{}_{\sigma_1 \cdots \sigma_p} = \partial_{\sigma_1} ... \partial_{\sigma_p} u_\beta{}^\alpha$ generate $K\{U\}_r$ [17]. This shows that the above Q-calculus, which is a kind of 'dissection' on functional graphs can be used as pre-calculus in differential algebras as well as in discrete dynamical systems [18].

2 Bialgebra Based on the Semi-group of Functional Graphs

2.1 Types of Functional Graphs

In this paper, a connected functional graph will be called *excycle* [13, 15]. In the area of discrete dynamical systems, an excycle is known as a basin of attraction. Consider several graded and filtered sets of labeled functional graphs

(i) $\mathbf{E}$ (resp.$\mathbf{G}$) the set of excycles (resp. functional graphs) and designated by $\mathbf{G}_n$ (resp.$\mathbf{G}_{(n)}$), the set of functional graphs having (resp. having at most) $n+1$ nodes for $n \geq 0$.

(ii) $\mathbf{R}$ (resp.$\mathbf{T}$) the set of labeled arborescences (resp. forests).

(iii) $\mathbf{C}$ (resp.$\mathbf{S}$) the set of cycles (resp. permutation graphs).

(iv) $\mathbf{A}$ (resp.$\mathbf{L}$) the set of well labeled arborescences (resp. forests), i.e. with strictly decreasing labels on the chains toward the root(s).

As in (i), we consider for (ii)–(iv) graduations and filtrations.

2.2 Free Representation by Q-polynomials

Let G_n be the semigroup of mappings of $\{1, 2, ...n\}$ in itself ('Semigroup of endofunctions'),

- Card $G_n = n^n$, and the subsemigroups,
- $T_n \equiv \{f; f \in G_n, f^n = f^{n-1}\}$ (i.e. f acyclic and Card $T_n = (n+1)^{n-1}$),
- S_n the symmetric group and Card $S_n = n!$
- $L_n \equiv \{f; f \in G_n, f(i) \leq i\}$ (i.e. f subdiagonal and Card $L_n = n!$).

We have the well known bijections $^{[F-F]}$: $\mathbf{G}_n \approx G_n$, $\mathbf{T}_n \approx T_n$, $\mathbf{S}_n \approx S_n$, $\mathbf{L}_n \approx L_n$.

Let $Q \equiv \{q_0, q_1, ...\}$ be a noncommutative alphabet, $Q^0 \equiv \{q_0\} \cup Q$ with q_0 noncommuting with the q_i's, $Q_n \equiv \{q_1, q_2, ...q_n\}$, $Q_n^0 \equiv \{q_0\} \cup Q_n$ and Q^* (resp. Q^{0*}), the corresponding free monoids. Taking $F_2 = \{0,1\}$ as the field, consider

(i) the G_n module $F_2 Q_n$ by the F_2 linear incidence matrix action of $f \in G_n$ as $l_f q_i = q_{f(i)}$ hence $l_f l_g = l_{f \bullet g}$.

(ii) the generating monomial associated with f.

By morphism extension, denoted again by l_f, we define

$$Q_f \equiv q_{f(1)}q_{f(2)}...q_{f(n)} = l_f Q_{\iota_n} \tag{1}$$

where $Q_{\iota_n} \equiv q_1q_2...q_n$ is associated with the identity ι_n of G_n and $Q\iota_0 = 1$. One again has $l_f l_g = l_{f\bullet g}$.

For the following we consider

(iii) The graded subsets of Q^* as $G \equiv (G_n)_{n\geq 0}, T \equiv (T_n)_{n\geq 0}, S \equiv (S_n)_{n\geq 0}$, $L \equiv (L_n)_{n\geq 0}$, respectively associated with **G**,**T**,**S** and **L**, with $G_0 = T_0 = S_0 = L_0 = \{1\}$.

(iv) The corresponding graded F_2-modules: F_2G, F_2T, F_2S, F_2L, admit components of degree n which are, respectively, G_n, T_n, S_n, L_n modules, with

$$dimF_2G_n = n^n, dimF_2T_n = (n+1)^{n-1}, dimF_2S_n = dimF_2L_n = n!$$

(v) We will denote by R_n one of the above subsemi-groups of G_n (or of another category).

Similarly, let $R \equiv (R_n)_{n\geq 0}, (resp. F_2R = \oplus_{n\geq 0}F_2R_n)$ be the corresponding graded subsets of Q^* (resp. graded F_2-modules of
$F_2G = \oplus_{n\geq 0}F_2G_n)$.

2.3 Virtual Root and External Product

Let $f \in G_n, I_0$ be the set of fixed points of f and H_0 a subset of I_0, and set $[p,q] \equiv \{r; p \leq r \leq q\}$ if $p \leq q$ and $\emptyset$ otherwise.

Define $f^0 : [1,n] \to [0,n]$ such that $f^0(i) = f(i)$ if $i \notin H_0$ and $f^0(i) = 0$ if $i \in H_0$. The '0' is the label of a virtual root added to the graph representation of f, and we will say that H_0 is 'confined in 0', which is a fixed point of f^0. We call 'extended endofunctions' such functions f^0, denote by $G_n^0 \equiv [0,n]^{[1,n]}$ their set, and we consider G_n as a subset of G_n^0. Similarly, we will have $T_n^0 \supset T_n, S_n^0 \supset S_n, L_n^0 \supset L_n$. Consequently, adding q_0, we get the extended graded sets $G^0 \equiv (G_n^0)_{n\geq 0}$, the extended graded F_2 -module $F_2G^0 = \oplus_{n\geq 0}F_2G_n^0$ and their substructures F_2T^0, F_2S^0, F_2L^0.

Now let $\phi \in G_m^0, \chi_0$ be the characteristic function of $H_0 = \phi^{-1}(0)$, and write

$$Q_\phi = l_\phi Q_{\iota_m} = q_{\phi(1)}q_{\phi(2)}...q_{\phi(m)} = \prod_{i=1}^{m} q_{\phi(i)}$$

(cf. Figures 1 and 2).

With $\psi \in G_n^0$, consider the F_2-bilinear product in F_2G^0 defined by

$$Q_\phi.Q_\psi = Q_\psi \prod_{i=1}^{m}(q_{\phi(i)+n} + \chi_0(i)(q_0 + q_1 + ...q_{n-1})) \tag{2}$$

On the right-hand side we have a sum of concatened monomials, and on the right factor the substitutions $q_0 \to q_0 + q_1 + ...q_n$ and $q_h \to q_{h+n}$ when $h \neq 0$.

On the other hand, the product belongs to $F_2G^0_{m+n}$. This external product is associated with unit 1 and F_2G^0 is '.' graded. To see this consider i, j, k being $\neq 0$, three homogeneous polynomials,

$$A = A(q_0;q_i) \in F_2G^0_m,\ B = B(q_0;q_j) \in F_2G^0_n,\ C = C(q_0;q_k) \in F_2G^0_p,$$

then by (2)

$$A.B = B(q_0;q_j)A(q_0+q_1+...q_n;q_{i+n}) \tag{3}$$

and so, using deg $(B.C) = n+p$,

$$\begin{aligned}
(A(q_0;q_i).B(q_0;q_j)).C &= (B(q_0;q_j)A(q_0+q_1+...q_n;q_{i+n})).C \\
&= CB(q_0+q_1+...q_p;q_{j+p})A(q_0+q_1+...q_p+q_{1+p}+...q_{n+p};q_{i+n+p}) \\
&= A(q_0;q_i).(CB(q_0+q_1+...q_p;q_{j+p})) \\
&= A(q_0;q_i).(B(q_0;q_j)).C)
\end{aligned}$$

Moreover, because T_n, S_n, L_n are subsemi-groups of G_n, one can see that $F_2R^0 = F_2T^0, F_2S^0, F_2L^0$ are '.' graded subalgebras of F_2G^0. Hence

Proposition 1 *Let the sequence $(G^0_m)_{m\geq 1}$ of the sets of the extended endofunctions in $\{1,2,...m\}$ and $Q^0 \equiv \{q_0,q_1,...\}$ be a noncommutative alphabet. For $\phi \in G^0_m$ let $Q_\phi = \prod_{i=1}^m q_{\phi(i)}$ be the generating monomial of ϕ and the graded module F_2 -module $F_2G^0 = \oplus_{n\geq 0}F_2G^0_n$ on $F_2 \equiv \{0,1\}$ generated by all the $\phi's$.*

Then F_2G^0 is a graded algebra for the associative product with unit 1

$$Q_\phi.Q_\psi = Q_\psi \prod_{i=1}^{m}(q_{\phi(i)+n} + \chi_0(i)(q_0+q_1+...q_{n-1}))$$

where $\psi \in G^0_n$ and χ_0 is the characteristic function of $\phi^{-1}(0)$.

Moreover, if $(R^0_m)_{m\geq 1}$ is a sequence of subsets associated with subsemi-groups of the sequence $(G^0_m)_{m\geq 1}$, then $F_2R^0 = \oplus_{n\geq 0}F_2R^0_n$ is a graded subalgebra of F_2G^0.

2.4 Splitting Operator $\delta_n \in F_2G^0$

This operator substitutes the n-coproduct Δ_n of the Leibniz–Lie type. Associate to $A \in Q^{0*}$ the left linear operator $\tau_n A$ acting on $B \in Q^{0*}$, such that, if $A \in G^0_m$, $B \in G^0_n$, then $[B](\tau_n A) \equiv BA$ if $degB = n$, and 0 otherwise, where BA is the concatenation of B and A.

(i) Now let $f \in G_m$ and H_0 as in Sect. 3, and note first that if

τ_n is viewed as acting on f, then for $i \in [1,m]$ one has $(\tau_n f)(i+n) = f(i)+n$, and by $f^0(i) = \bar{\chi}_0(i)f(i)$ one has $(\tau_n f^0)(i+n) = \bar{\chi}_0(i)(f^0(i)+n)$, where $\bar{\chi}_0 = 1-\chi_0$. According to (2), define for $\phi \in G^0_m$

$$\delta_n Q_\phi \equiv \tau_n \prod_{i=1}^{m}(q_{\phi(i)+n} + \chi_0(i)(q_0+q_1+...q_{n-1})) \tag{4}$$

If $d_0 = Card\ H_0$ the expansion (4) gives a sum of $(n+1)^{d_0}$ generating monomials of functions $\psi_{(\kappa)}$ of $[n+1, n+m]$ into $\{0\} \cup [n+1, n+m]$, and the corresponding functional graphs factorized in commutative excycles.

The operator $\delta_n A$ is left linear on F_2G^0, and (2) can be written

$$Q_\phi.Q_\psi = [Q_\psi](\delta_n Q_\phi) \tag{5}$$

(ii) Moreover, δ_p is a graded antimorphism for '.'

$$\delta_p(A.B) = (\delta_p B)(\delta_{p+n} A) \tag{6}$$

where $n = degB$ and $p \in \mathbf{N}$.

For this to compute with (5) and A, B, C as in Sect. 3, $[C]\delta_p(A.B) = (A.B).C = A.(B.C) = [B.C]\delta_{p+n}A = [C](\delta_p B)(\delta_{p+n}A)$. If $p = 0$ we recover $A.B = [B]\delta_n A$ and $[B]\delta_k A = 0$ if $k \neq degB$.

(iii) Also, δ_n is a power

$$\delta_n = \delta^n, \delta = \delta_1, \delta^0 = 1 \tag{7}$$

For this to compute, $\delta_p\delta_n A(q_0;q_i) = \delta_p\tau_n A(q_0+q_1+...q_n;q_{i+n}) = \tau_{n+p}A(q_0+q_1+...q_p+q_{1+p}+...q_{n+p};q_{i+n+p}) = \delta_{n+p}A(q_0;q_i)$.

(iv) Define the left linear operator μ in F_2G^0 by the expansion

$$\mu \equiv \sum_{n\geq 0} \delta^n \tag{8}$$

By left linear action of μA on F_2G^0, we get $A.B = [B](\mu A)$ for $A, B \in F_2G^0$ with the antimorphism property

$$\mu(A.B) = (\mu B)(\mu A) \tag{9}$$

which express the associativity of '.'.

Proposition 2 *Let $A \in F_2G^0_m, B \in F_2G^0_n$. Then the splitting linear operator δ_p defined left linearly by $[B]\delta_p A \equiv A.B$ if $p = n$, and 0 otherwise, verifies $\delta_p = \delta^p$* with $\delta = \delta_1, \delta^0 = 1$ *and $\delta^p(A.B) = (\delta^p B)(\delta^{p+n}A)$. Moreover, $\mu \equiv \sum_{n\geq 0}\delta^n$ is an antimorphism in F_2G^0 such that $A.B = [B](\mu A)$.*

2.5 Exponential Generating Function of the Monomials of L^0

All the words of L^0_n (i.e. subdiagonals) are obtained from the expansion of $Q_n \equiv q_0(q_0+q_1)\dots(q_0+q_1+\dots q_{n-1}) \in F_2L^0$ and $Q_0 = 1$.

By equation (3), one has $Q_m.Q_n = Q_{m+n}$, and if $A \in F_2L^0_m, B \in F_2L^0_n$ we have $A.B \in F_2L^0_{m+n}$, and then we recover that F_2L^0 is stable for the product '.'. Because $Q_1 = q_0$, the associativity gives

$$Q_n = (q_0)^{.n} \tag{10}$$

With the $\mathbf{Q}$[[t]]-modules on L^0, one has the exponential generating function

$$exp_.(tq_0) = \sum_{n\geq 0}\frac{t^n}{n!}(q_0)^{.n} = \sum_{n\geq 0}\frac{t^n}{n!}Q_n \tag{11}$$

$$exp_.(sq_0).exp_.(tq_0) = exp_.((s+t)q_0)$$

2.6 Examples

Consider equations (4) and (5) for $Q_\psi = q_0^n$.

2.6.1 Rooted Trees with n=1

$\delta(q_0^2 q_1) = \tau(q_0+q_1)^2 q_2 = \tau q_0 q_0 q_2 + \tau q_0 q_1 q_2 + \tau q_1 q_0 q_2 + \tau q_1 q_1 q_2$ (Figure 3)

$(q_0^2 q_1).q_0 = q_0(q_0+q_1)^2 q_2 = q_0^3 q_2 + q_0^2 q_1 q_2 + q_0 q_1 q_0 q_2 + q_0 q_1 q_1 q_2$ (Figure 4)

2.6.2 Excycles with n=2

$\delta^2(q_3^2 q_1 q_0) = \tau^2 q_5^2 q_3(q_0+q_1+q_2) = \tau^2 q_5^2 q_3 q_0 + \tau^2 q_5^2 q_3 q_1 + \tau^2 q_5^2 q_3 q_2$ (Figure 5)

$q_3^2 q_1 q_0 . q_0^2 = q_0^2 q_5^2 q_3(q_0+q_1+q_2) = q_0^2 q_5^2 q_3 q_0 + q_0^2 q_5^2 q_3 q_1 + q_0^2 q_5^2 q_3 q_2$ (Figure 6)

3 Differential Algebra

3.1 Differential indeterminates

Let $D \equiv \{\partial_0, \partial_1, ...\}$ where $\partial_\alpha \equiv \partial/\partial\xi^\alpha$, the α^{th} canonical derivation in $\mathbf{S} \equiv K[[\xi]]$ the algebra of formal power series in $\xi \equiv \{\xi^0, \xi^1, ...\}$, where K is a characteristic zero field. If $\mathbf{S}^{N\times N} is$ the set $U \equiv (u_1, u_2, ...) = (u_\beta)_{\beta\geq 1} = (u_\beta^\alpha)_{\alpha\geq 0,\beta\geq 1}$ with $u_\beta^\alpha \in \mathbf{S}$, consider U as a set of indeterminates, $u^\alpha_{\beta\sigma_1...\sigma_p} = \partial_{\sigma_1}...\partial_{\sigma_p} u_\beta^\alpha$ as differential indeterminates, replace $\mathbf{S}^{N\times N}$ by KU, and consider the graded differential algebra $K\{U\} = \oplus_{r\geq 0} K\{U\}_r$ and the graded algebra of differential operators $K\{U,D\} = \oplus_{r\geq 0} K\{U\}_r D^r$.

To each $W \in F_2 R^0$ we associate the differential operator $W(U) \in \{U,D\}$; for example, with $W_r(U) \in K\{U\}_r$ one has

$$W(U) = W_0(U) + W_1(U)^\alpha \partial_\alpha + W_2(U)^{\alpha\beta}\partial_\alpha\partial_\beta + ... = W_0(U) + \sum_{r\geq 1}\langle W_r(U), D^r\rangle \tag{12}$$

We will use now the summation convention.

3.2 Brackets in $K\{U\}$

Define for $u, v, w... \in U$ the multilinear operations valued in $K\{U\}$.

3.2.1 Arborescent Brackets (Valued in $K\{U\}_1$)

$\langle u,v\rangle, \langle uv,w\rangle ...,\ \langle u,v\rangle^\beta \equiv u^\alpha v^\beta{}_\alpha$, hence $\langle\langle u,v\rangle, D\rangle = u^\alpha v^\beta{}_\alpha \partial_\beta$ (1 fixed point sent to '0') $\langle uv,w\rangle^\gamma \equiv u^\alpha v^\beta w^\gamma{}_{\alpha\beta}$, hence

$$\langle\langle u,v\rangle, D\rangle = u^\alpha v^\beta w^\gamma{}_{\alpha\beta}\partial_\gamma \tag{13}$$

Also, for $A \in K\{U\}_r, B \in K\{U\}_s$, $\langle A,B\rangle^{\beta_1...\beta_s} \equiv A^{\alpha_1...\alpha_r} B^{\beta_1...\beta_s}{}_{\alpha_1...\alpha_r}$.

3.2.2 Circular Brackets (Valued in $K\{U\}_0$)

$(u) \equiv u^\alpha{}_\alpha$ (1 fixed point), $(u)(v) \equiv u^\alpha{}_\alpha v^\beta{}_\beta$ (2 fixed points)

$$(u,v) \equiv u^\alpha{}_\beta v^\beta{}_\alpha (2\!-\!cycle),\ (u,v,w) \equiv u^\alpha{}_\gamma v^\beta{}_\alpha w^\gamma{}_\beta (3\!-\!cycle) \tag{14}$$

3.2.3 Mixed Brackets (Valued in $K\{U\}_0$)

Let E be a proper excycle (i.e. with no fixed point); we can write it $E = (A_{i_1}, A_{i_2}, ...A_{i_p})$, where the A_{i_k}'s are arborescences with root i_k. If in each arborescence A_{i_k} is reduced to its root i_k, we recover simply a cycle $E = (i_1, i_2, ...i_p)$.

Now let F_k be the forest under i_k, i.e. obtained by cutting the root of A_{i_k}, and defined with $F_{i_k}(U) \equiv F_{i_k}(\{u_j\}; j \in N_{i_k})$, where N_{i_k} is the set of nodes of F_{i_k}:

$$E(U) \equiv (u_{i_1}^{\langle F_{i_1}\rangle}, u_{i_2}^{\langle F_{i_2}\rangle}, ...u_{i_p}^{\langle F_{i_p}\rangle}) \equiv \langle F_{i_1}(U), {u_{i_1}}^{\alpha_1}{}_{\alpha_p}\rangle\langle F_{i_2}(U), {u_{i_2}}^{\alpha_2}{}_{\alpha_1}\rangle ... \langle F_{i_p}(U), {u_{i_p}}^{\alpha_p}{}_{\alpha_{p-1}}\rangle$$

3.3 Action of F_2R^0

Moreover, F_2R^0 operates K-linearly in $K\{U\}$ with values in $K\{U, D\}$. For this let $\phi \in G_m^0, H_0 = \phi^{-1}(0)$ for $j \in [0,m], I = [1,m]$, and $H = u_{\beta_1}u_{\beta_2}...u_{\beta_m} \in U^*$, a word on U of length m. Then the action is

$$Q_\phi = \prod_{i\in I} q_{\phi(i)} \to Q_\phi(H) = \prod_{j\in I}\left(\prod_{i\in H_j}\partial_{\alpha_i}\right) u^{\alpha_j}{}_{\beta_j}\prod_{k\in H_0}\partial_{\alpha_k} \tag{15}$$

The differential monomial $Q_\phi(H)$ is such that u_{β_j} is associated with j in the domain I of ϕ. If d_j is the degree in q_j (in-degree of the node labeled by 'j'), then $u^{\alpha_j}{}_{\beta_j}$ is derived d_j times and the indices of derivation are related to the places of the q_j's in the word. Similarly, the differential operator D^r is characterized by the number r (degree of the root) of the q_0's and their places. So we can summarize: *In a word $A \in R^0$ where q_j is at the place (i), then in $A(H)$ the j^{th} letter of H is derived according to i, i.e. ∂_{α_i} acts.* One has, in particular, taking $H = u_1u_2, ...$:

Arborescent brackets
$1(U) = 1$
$q_0(U) = {u_1}^{\alpha_1}\partial_{\alpha_1} = \langle u_1, D\rangle$
$q_0q_0(U) = {u_1}^{\alpha_1}{u_2}^{\alpha_2}\partial_{\alpha_1}\partial_{\alpha_2} = \langle u_1u_2, D^2\rangle$
$q_0q_1(U) = {u_1}^{\alpha_1}{}_{\alpha_2}{u_2}^{\alpha_2}\partial_{\alpha_1} = \langle\langle u_2, u_1\rangle, D\rangle$
$q_3q_3q_0(U) = {u_1}^{\alpha_1}{u_2}^{\alpha_2}{u_3}^{\alpha_3}{}_{\alpha_1\alpha_2}\partial_{\alpha_3} = \langle\langle u_1u_2, u_3\rangle, D\rangle$
$q_0q_0q_2q_2(U) = {u_1}^{\alpha_1}{u_2}^{\alpha_2}{}_{\alpha_3\alpha_4}u_3^{\alpha_3}u_4^{\alpha_4}\partial_{\alpha_1}\partial_{\alpha_2} = \langle u_1\langle u_3u_4, u_2\rangle, D^2\rangle$
Circular brackets
$q_1(U) = {u_1}^{\alpha_1}{}_{\alpha_1} = (u_1)$
$q_1q_2(U) = {u_1}^{\alpha_1}{}_{\alpha_1}{u_2}^{\alpha_2}{}_{\alpha_2} = (u_1)(u_2)$
$q_2q_1(U) = {u_1}^{\alpha_1}{}_{\alpha_2}{u_2}^{\alpha_2}{}_{\alpha_1} = (u_1, u_2)$
$q_3q_1q_2(U) = {u_1}^{\alpha_1}{}_{\alpha_3}{u_2}^{\alpha_2}{}_{\alpha_1}{u_3}^{\alpha_3}{}_{\alpha_2} = (u_1, u_2, u_3)$

3.4 Product of Differential Operators

The product (2) on words with correspondence (15) gives the product of differential operators. We state, without proof,

Proposition 3 *Let the graded differential algebra $K\{U\} = \oplus_{r\geq 0}K\{U\}_r$ and the graded algebra of differential operators $K\{U,D\} = \oplus_{r\geq 0}K\{U\}_rD^r$. Let $\phi \in G_m^0$, $I = [1,m], H_j = \phi^{-1}(j)$ for $j \in [0,m]$ and $H = u_{\beta_1}u_{\beta_2}...u_{\beta_m}$ a word on U of length m. Then the mapping of F_2G^0 into $K\{U,D\}$ which associates to the generating monomial $Q_\phi = \prod_{i\in I} q_{\phi(i)}$ of ϕ the differential operator $Q_\phi(H) = \prod_{j\in I}(\prod_{i\in H_j}\partial_{\alpha_i})u^{\alpha_j}{}_{\beta_j}\prod_{k\in H_0}\partial_{\alpha_k}$*

is a morphism, such that if $\psi \in G_n^0$ *and* K *is a word on* U *of length* n*, one has* $Q_\phi(H)Q_\psi(K) = (Q_\phi . Q_\psi)(KH)$*, where* KH *is the concatenation of* K *and* H*.*

Example

$$A = q_0, B = q_2q_1q_0, H = u_4, K = u_1u_2u_3$$

$$A^{\cdot}B = q_2q_1q_0(q_0+q_1+q_2+q_3) = q_2q_1q_0q_0+q_2q_1q_0q_1+q_2q_1q_0q_2+q_2q_1q_0q_3$$

(Figure 7)

$$A(H) = \langle u_1, D\rangle, B(K) = (u_1,u_2)\langle u_3, D\rangle,$$

$$A(H)B(K) = (u_1,u_2)\langle u_3u_4, D^2\rangle + \langle u_4,(u_1,u_2)\rangle\langle u_3,D\rangle + (u_1,u_2)\langle\langle u_4,u_3\rangle, D\rangle$$

Observe that:

$$\langle u_4,(u_1,u_2)\rangle\langle u_3,D\rangle = {u_1}^{\alpha_1}{}_{\alpha_2\alpha_4}{u_2}^{\alpha_2}{}_{\alpha_1}{u_3}^{\alpha_3}{u_4}^{\alpha_4}\partial_{\alpha_3} + {u_1}^{\alpha_1}{}_{\alpha_2}{u_2}^{\alpha_2}{}_{\alpha_1\alpha_4}{u_3}^{\alpha_3}{u_4}^{\alpha_4}\partial_{\alpha_3}$$

which corresponds to $q_2q_1q_0(q_1+q_2)$, i.e. the second and third terms in the graph expansion.

Appendix

To view Figures 1–7, click here. To return to the main paper, click on the red box.

References

[1] Ginocchio, M. (1995). Universal expansion of the powers of a derivation, *Letters in Math. Phys.* **34**(4), 343–364.

[2] Ginocchio, M. and Irac-Astaud, M. (1985). A recursive linearization process for evolution equations. *Reports on Math. Phys.* **21**, 245–265.

[3] Steeb, W. H. and Euler, N. (1988). *Nonlinear Evolution Equations and Painlevé Test*. World Scientific.

[4] Comtet, L. (1973). Une formule explicite pour les puissances successives de l'opérateur de dérivation de Lie. *Comm. Roy. Acad. Sci.* **276 A**, 165–168.

[5] Leroux, P. and Viennot, G. (1986). Combinatorial resolution of systems of differential equations I: ordinary differential equations. *Actes du colloque de combinatoire énumérative*, Montréal. *Lecture Notes in Mathematics 1234*, pp. 210–245. Springer-Verlag.

[6] Bergeron, F. and Reutenauer, C. (1987). Une interprétation combinatoire des puissances d'un opérateur différentiel linéaire. *Ann. Sci. Math. Quebec* **11**, 269–278.

[7] Grossman, R. and Larson, R. G. (1989). Hopf-algebraic structures of families of trees. *J. Algebra* **126**, 184–210.

[8] Joni, A. A. and Rota, G. C. (1979). Coalgebras and bialgebras in combinatorics. *Studies. in Appl. Math.* **61**, 93–139.

[9] Nichols, W. and Sweedler, M. E. (1980). Hopf algebras and combinatorics, in 'Umbral calculus and Hopf algebras'. *Contemp. Math.* **6**.

[10] Sweedler, M. E. (1969). *Hopf Algebras*. Benjamin.

[11] Berge, C. (1983). *Graphes*. Gauthier-Villars.

[12] Comtet, L. (1974). *Advanced Combinatorics*. Reidel.

[13] Denes, J. (1968). On transformations, transformation-semigroups and graphs. In Erdös-Katona, editor, *Theory of Graphs*. Academic Press, pp. 65–75.

[14] Foata, D. and Fuchs, A. (1970). Réarrangements de fonctions et dénombrements. *J. Comb. Theory* **8**, 361–375.

[15] Harary, F. (1959). The number of functional digraphs. *Math. Annalen* **138**, 203–210.

[16] Ginocchio, M. On the Hopf algebra of functional graphs and differential algebras. *Discr. Math.* To appear.

[17] Kaplansky, I. (1976). *Introduction to Differential Algebras*. Springer-Verlag.

[18] Robert, F. (1995). *Les systèmes Dynamiques Discrets*. Springer-Verlag.