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A characterization for all interval doubling schemes of the lattice of permutations

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The lattice $\mathcal{S}_n$ of all permutations on a n-element set has been shown to be *bounded* [5], which is a strong constructive property characterized by the fact that $\mathcal{S}_n$ admits what we call an *interval doubling scheme*. In this paper we characterize all interval doubling schemes of the lattice $\mathcal{S}_n$, a result that gives a nice precision on the bounded nature of the lattice of permutations. This theorem is a direct corollary of two strong properties (Proposition 3 and Theorem 3) that are also given with their proofs.

Keywords: Permutations, lattice, bounded lattice, interval doubling schemes, arrow relations, linear extension, tableaux.

1 Introduction

Permutations on a n-element set play a fundamental role in numerous fields, especially in computer science and pure or applied mathematics. The main reason for the importance of permutations in applied mathematics is that they appear like a fundamental ordinal model since the set $\mathcal{S}_n$ of all permutations on a n-element set is clearly in a bijective correspondence with the set of all linear orders on a set with the same cardinality. Guilbaud and Rosenstiehl have proved in 1963 that the set of all permutations on a finite set is a lattice [15]: indeed the transpositions which reverse neighbours in a permutation (and which give rise, as a generating system, to the symmetric group $\mathcal{S}_n$) define the cover relation of an order relation $\leq_{\mathcal{S}_n}$, the so-called *weak Bruhat order* on $\mathcal{S}_n$, which is a lattice (Björner [4]). The lattice $(\mathcal{S}_n, \leq_{\mathcal{S}_n})$ of permutations, also called *Permutoedron*, has been studied by several authors (see e.g. Barbut and Monjardet [1], Chameni-Nembua [6], Le Conte de Poly-Barbut [16, 17], Edelman and Greene [11], Markowsky [19]). Recently, Duquenne and Cherfouh [10] (and, independently and in a more general context, Le Conte de Poly-Barbut [18]) have proved that this lattice is *semidistributive*. This property has been reinforced in [5] where it is proved that the lattice of permutations is *bounded* (see Day [8] for the origins of this concept and for a number of characterizations), which means that the Permutoedron can be obtained from the two-element lattice $\mathcal{Q} = (\{0,1\}, \leq)$ by a (finite) sequence of *doublings of convex sets* (the definition of this constructive operation can be found for instance in [2] or in [14]). Another characterization of bounded lattices is expressed in terms of A-*table*, a (particularly rich lattice encoding) concept defined from the *arrow relations*. A lattice is bounded if and only if its A-table admits what we call an *interval doubling*

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scheme [14]. In [2], the authors have shown how all interval doubling schemes of a bounded lattice L are in a 1-1 correspondence with all different ways to construct L from the two-element lattice by the doubling operation mentioned above. These interval doubling schemes are also in a 1-1 correspondence with all linear extensions of the poset of join-irreducible congruences and with all linear extensions of the poset J_L with respect to the $D - rank$ defined in [12] (see this reference and also [9] for more definitions and details on this subject). The aim of this paper is to provide a simple characterization of all interval doubling schemes of the lattice of permutations (Corollary 2). This characterization is obtained as a corollary of two results: Proposition 3 which characterizes the order relation between all join-irreducible permutations (and dually between all meet-irreducible permutations) in terms of arrow relations, and Theorem 3 that generalizes our characterization result to a larger class of lattices.

In the following section, we give all necessary notions on lattices and on the arrow relations, and provide some useful preliminaries about permutations. Section 3 presents the results with their proofs.

2 Preliminaries

A partially ordered set (or poset) $(L, \leq)$ is a *lattice* if any pair $\{x, y\}$ of elements of L has a least *upper bound* (also called *join*) denoted by $x \vee y$ and a greatest *lower bound* (also called *meet*) denoted by $x \wedge y$. The order relation $\leq$ on L is the transitive closure of the *cover relation* $\prec$ of L, which is defined by $x \prec y$ if $x < y$ and there exists no $z \in L$ such that $x < z < y$. We then say that x is *covered* by y (or y *covers* x). An element j (resp. m) of L is a *join-* (resp. *meet-*) *irreducible* of L if it cannot be obtained as the join (resp. meet) of elements of L distinct from j (resp. from m). Equivalently, an element j (resp. m) of L is a non-zero (resp. non-unit) join- (resp. meet-) irreducible if it covers (resp. is covered by) a unique element in L, which is then denoted by j^- (resp. m^+). The set of non-zero join- (resp. non-unit meet-) irreducibles of a lattice L is denoted by J_L — or J if no confusion arises — (resp. M_L or M). We will also identify any lattice $(L, \leq)$ with its underlying set L and all lattices will be represented by their *diagram*, i.e. by the transitive reduction of their cover relation, directed from bottom to top.

For other definitions about lattices not recalled here, see for instance the books by Barbut and Monjardet [1], Birkhoff [3] or Davey and Priestley [7].

Let $N = \{1, 2, ..., n\}$ be a set with n elements. A *permutation* on N is a bijection α on N that we denote $\alpha = \alpha_1 ... \alpha_i ... \alpha_n$. For $i = 1, ..., n-1$, α_i and α_{i+1} are said *adjacent* in the permutation α. We denote by $\mathcal{S}_n$ the set of all permutations on N. It is clear that $\mathcal{S}_n$ is in a bijective correspondence with the set of all *linear orders* on a set with cardinality n. According to this bijection, we will freely mix these two notions, considering a permutation α as the linear order $\alpha = \{(\alpha_i, \alpha_j) : 1 \leq i < j \leq n\}$ and applying all ordinal notations and concepts to permutations. In the following, an ordered pair (α_i, α_j) will rather be written $\alpha_i \alpha_j$.

For a permutation α, we note $A(\alpha)$ the set of all *agreements* of α, i.e. the set of all ordered pairs (α_i, α_j) of α satisfying $\alpha_i < \alpha_j$. The set $D(\alpha)$ of all *disagreements* of α is defined as the set of all ordered pairs (α_i, α_j) of α satisfying $\alpha_j < \alpha_i$. An order relation between the permutations of $\mathcal{S}_n$ is defined by $\alpha \leq \beta$ if $A(\beta) \subseteq A(\alpha)$ (or equivalently if $D(\alpha) \subseteq D(\beta)$). For this order relation (which equals the weak Bruhat order on the set of permutations), the set $\mathcal{S}_n$ of permutations is a lattice with the natural order $0_{\mathcal{S}_n} = 1...k(k+1)...n$ as least element and the dual order $1_{\mathcal{S}_n} = n...k(k-1)...1$ as greatest element. The cover relation $\prec$ on $\mathcal{S}_n$ is defined by $\alpha \prec \beta$ if there exists a unique ordered pair (α_i, α_{i+1}) of elements of N that are adjacent in α and β and that satisfy $\beta = [\alpha \setminus (\alpha_i, \alpha_{i+1})] \cup (\alpha_{i+1}, \alpha_i)$. If a permutation

α admits an agreement made of two adjacent elements α_i and α_{i+1} (i.e. $\alpha = \alpha_1 \ldots \alpha_i \alpha_{i+1} \ldots \alpha_n$ with $\alpha_i < \alpha_{i+1}$), we say that α has a *increasing* (in i) and dually, if α admits a disagreement made of two adjacent elements α_i and α_{i+1} (i.e. $\alpha = \alpha_1 \ldots \alpha_i \alpha_{i+1} \ldots \alpha_n$ with $\alpha_{i+1} < \alpha_i$), we say that α has an *decreasing* (in i).

The irreducible permutations are characterized by means of the increasings and decreasings as follows: a permutation is join-irreducible (resp. meet-irreducible) if and only if it admits a unique decreasing (resp. increasing).

Remark 1 *Any join-irreducible permutation γ can be denoted by $\gamma = A|\overline{A} = Bv|u\overline{B}$ $(u, v \in N)$, where $A = Bv$ and $\overline{A} = u\overline{B}$ are the left and right factors of γ compatible with the order $0_{\mathcal{S}_n}$, and where vu is the unique decreasing of γ. Dually, any meet-irreducible permutation μ will be written $\mu = D|\overline{D} = Cl|p\overline{C}$, with $D = Cl$ and $\overline{D} = p\overline{C}$ the left and right factors of μ compatible with the order $1_{\mathcal{S}_n}$ and where lp is the unique increasing of μ. It is clear that a join-irreducible (resp. meet-irreducible) permutation is completely determined by A or by $\overline{A}$ (resp. by D or by $\overline{D}$).*

Remark 2 *The unique lower cover of a join-irreducible permutation $\gamma = A|\overline{A} = Bv|u\overline{B}$ is the permutation $\gamma^- = Buv\overline{B}$, obtained by changing the unique decreasing vu of γ into the increasing uv. Dually, the unique upper cover of the meet-irreducible permutation $\mu = D|\overline{D} = Cl|p\overline{C}$ is the permutation $\mu^+ = Cpl\overline{C}$, obtained by changing the unique increasing lp of μ into the decreasing pl.*

The following fact is well known:

Remark 3 $|J| = |M|$ $(= 2^n - (n+1))$.

The original definition of bounded lattices can be found for instance in [12]. There exists a simple characterization of these lattices due to Day [8] and expressed in [14] in terms of *arrow relations* or equivalently, in terms of A-*table*, a very useful tool for the description of numerous properties on lattices. We start with the definition of these notions, introduced by Wille and the "Darmstadt School" (see [13] and [20]).

Definition 1 *Let j be a join-irreducible and m a meet-irreducible of a lattice L. The three arrow relations $\uparrow$, $\downarrow$ and $\updownarrow$ are defined as follows:*

(a) $j \uparrow m$ if m is maximal in $\{t \in L : j \not\leq t\}$.

(b) $j \downarrow m$ if j is minimal in $\{t \in L : t \not\leq m\}$.

(c) $j \updownarrow m$ if $j \uparrow m$ and $j \downarrow m$.

Equivalently, $j \uparrow m$ if and only if $j \not\leq m$ and $j \leq m^+$ and, dually, $j \downarrow m$ if and only if $j \not\leq m$ and $j^- \leq m$.

Moreover it is useful to define the following "strict" arrow relations:

Definition 2 *Let L, j and m be as above.*

(d) $j \uparrow_{\cdot} m$ if $j \uparrow m$ and $j \not\downarrow m$.

(e) $j \downarrow_{\cdot} m$ if $j \downarrow m$ and $j \not\uparrow m$.

It is obvious that relations $\leq, \updownarrow, \uparrow_{\cdot}$ and $\downarrow_{\cdot}$ do not intersect. This allows us to define a fundamental concept of lattice theory, the A-*table* (A for *Arrow*) of a lattice (Wille and the "Darmstadt School", 1983).

Definition 3 *Let $(L, \leq) = L$ be a lattice and J and M its sets of join-irreducibles and of meet-irreducibles respectively. The A-table of L — denoted by A_L — is equal to the tuple $A_L = (J, M, \leq, \updownarrow, \uparrow_{\cdot}, \downarrow_{\cdot})$.*

According to its definition, the A-table of a lattice L can always be described by a two-dimensional table, whose rows are indexed by the join-irreducibles of L, and whose columns are indexed by its meet-irreducibles. Each cell (j, m) in the A-table contains a cross $\times$ if $j \leq m$, and an arrow $\uparrow_{\cdot}$, $\downarrow_{\cdot}$, or $\updownarrow$ according to the adequate case. The cell (j, m) contains the symbol $\circ$ if (j, m) does not satisfy any of the previous conditions.

Obviously there exists many ways to present the A-table of a lattice L, according to the orders chosen on J and M for the indexing of rows and columns of the A-table. This naturally leads to the simple and fundamental notion of *tableau*, that we define below:

Definition 4 *A* tableau *T of lattice L is a triple $T = (A_L, \mathcal{L}_J, \mathcal{L}_M)$ where A_L is the A-table of L and $\mathcal{L}_J$ and $\mathcal{L}_M$ two linear orders respectively defined on J and M and defining (from the top to the bottom and from the left to the right) the two orders of the "rows " and of the "columns" of tableau T.*

So the A-table of L can be described by $(|J|! \times |M|!)$ equivalent such tableaux and a tableau is completely determined by the two orders $\mathcal{L}_J$ and $\mathcal{L}_M$. Figure 1 shows (the diagram of) the lattice of permutations $\mathcal{S}_4$ and its A-table, given by a tableau.

Remark 4 *Each row and each column of this tableau contains at least one $\updownarrow$.*

In the introduction, we have recalled that the Permutoedron is semidistributive. Let us recall that a lattice L is said *semidistributive* if for all elements $x, y, z \in L$, $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$ and $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$). The following characterization of these lattices is well known:

Proposition 1 *A finite lattice L is semidistributive if and only if the two following conditions are satisfied:*

1. *For any $j \in J$, there exists a unique $m \in M$ such that $j \updownarrow m$.*
2. *For any $m \in M$, there exists a unique $j \in J$ such that $j \updownarrow m$.*

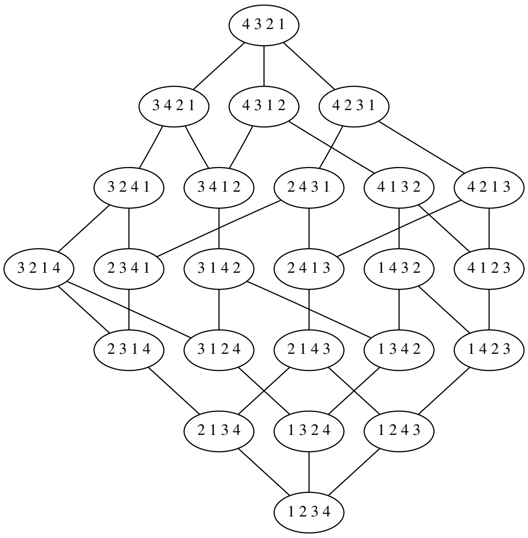
This result means that L is semidistributive if and only if the A-table of L contains a unique $\updownarrow$ on each line and on each column. According to Remark 4, this fact implies that J and M have the same cardinality and that these two sets are in a bijective correspondence induced by the relation $\updownarrow$. Theorem 2 gives the expression of this bijection in the case of the semidistributive lattice of permutations.
The characterization of bounded lattices given in Theorem 1 clearly shows that a bounded lattice is semidistributive. We begin with one definition.

Definition 5 *Let L be a semidistributive lattice and $T = (A_L, \mathcal{L}_J, \mathcal{L}_M)$ a tableau of its A-table. We say that T is an* interval doubling scheme *of L if it satisfies the two following conditions:*

1. *The $|J|$ double arrows $\updownarrow$ are on the principal diagonal of T.*
2. *All arrows $\uparrow_{\cdot}$ are below this diagonal and all arrows $\downarrow_{\cdot}$ are above.*

For an illustration of that definition see Figure 2.

Theorem 1 (Geyer, 1994) *A lattice L is bounded if and only if its A-table admits at least one interval doubling scheme.*



$J\backslash M$	3421	4231	3241	2431	4312	4213	3214	2413	4132	3142	1432
1243	$\updownarrow$	$\times$	$\downarrow.$	$\times$	$\times$	$\times$	$\downarrow.$	$\times$	$\times$	$\downarrow.$	$\times$
1324	$\times$	$\updownarrow$	$\times$	$\downarrow.$	$\times$	$\downarrow.$	$\times$	$\downarrow.$	$\times$	$\times$	$\times$
1342	$\times$	$\uparrow.$	$\updownarrow$	$\circ$	$\times$	$\circ$	$\downarrow.$	$\circ$	$\times$	$\times$	$\times$
1423	$\uparrow.$	$\times$	$\circ$	$\updownarrow$	$\times$	$\times$	$\circ$	$\downarrow.$	$\times$	$\circ$	$\times$
2134	$\times$	$\times$	$\times$	$\times$	$\updownarrow$	$\times$	$\times$	$\times$	$\downarrow.$	$\downarrow.$	$\downarrow.$
2314	$\times$	$\times$	$\times$	$\times$	$\uparrow.$	$\updownarrow$	$\times$	$\downarrow.$	$\circ$	$\circ$	$\circ$
2341	$\times$	$\times$	$\times$	$\times$	$\uparrow.$	$\uparrow.$	$\updownarrow$	$\circ$	$\circ$	$\circ$	$\circ$
2413	$\uparrow.$	$\times$	$\circ$	$\times$	$\uparrow.$	$\times$	$\circ$	$\updownarrow$	$\circ$	$\circ$	$\circ$
3124	$\times$	$\uparrow.$	$\times$	$\circ$	$\times$	$\circ$	$\times$	$\circ$	$\updownarrow$	$\times$	$\downarrow.$
3412	$\times$	$\uparrow.$	$\uparrow.$	$\circ$	$\times$	$\circ$	$\circ$	$\circ$	$\uparrow.$	$\updownarrow$	$\circ$
4123	$\uparrow.$	$\times$	$\circ$	$\uparrow.$	$\times$	$\times$	$\circ$	$\circ$	$\times$	$\circ$	$\updownarrow$

Fig. 1: The lattice $\mathcal{S}_4$ of permutations and (a tableau of) its A-table.

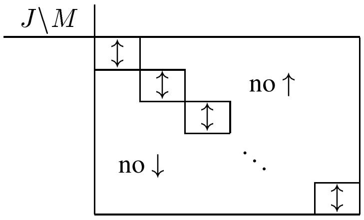


Fig. 2: An interval doubling scheme

Clearly, the tableau in Figure 1 is an interval doubling scheme of the lattice $\mathcal{S}_4$. It is completely determined by the chosen linear order on J (which is the *lexicographical* order on J) since, for this linear order, there exists a unique linear order on M allowing to get all $\updownarrow$ of the A-table on its principal diagonal. In [5], the author has shown the bounded nature of the Permutoedron by showing that the unique tableau given by the lexicographical order on J and allowing the alignment of the $\updownarrow$ on the diagonal is an interval doubling scheme of $\mathcal{S}_n$. Our aim here is to characterize all interval doubling schemes of the Permutoedron. To do so, we use the following characterization of the arrow relations in the lattice $\mathcal{S}_n$ (whose proof, easy, is left to the reader):

Proposition 2 *Let $\gamma = Bv|u\overline{B}$ be a join-irreducible and $\mu = Cl|p\overline{C}$ a meet-irreducible of the lattice $\mathcal{S}_n$.*

1. $\gamma \leq \mu \iff D(\gamma) \subseteq D(\mu) \iff A(\mu) \subseteq A(\gamma)$.
2. $\gamma \uparrow \mu \iff pl \in D(\gamma)$ *and* $D(\gamma) \subseteq D(\mu^+)$.
3. $\gamma \downarrow \mu \iff uv \in A(\mu)$ *and* $A(\mu) \subseteq A(\gamma^-)$.
4. $\gamma \updownarrow \mu \iff pl \in D(\gamma)$, $uv \in A(\mu)$, $D(\gamma) \subseteq D(\mu^+)$ *and* $A(\mu) \subseteq A(\gamma^-)$.

3 The results

We have seen that a semidistributive lattice is characterized by the existence of a unique $\updownarrow$ on each line and on each column of its A-table, this relation $\updownarrow$ inducing in the lattice a bijective correspondence between the sets J and M. Without proof, we give the following theorem and corollary which make this bijection explicit in the case of the Permutoedron. For the proof see [5].

Theorem 2 *1. Let $\gamma = Bv|u\overline{B}$ be a join-irreducible and μ a meet-irreducible of lattice $\mathcal{S}_n$. Then*

$$\gamma \updownarrow \mu \iff \mu = Cu|v\overline{C} \quad with \left\{ \begin{array}{l} C = \big(\{x \in B : u < x\} \cup \{x \in \overline{B} : v < x\}, > \big) \\ \overline{C} = \big(\{x \in B : x < u\} \cup \{x \in \overline{B} : x < v\}, > \big) \end{array} \right.$$

2. Let $\mu = Cl|p\overline{C}$ be a meet-irreducible and γ a join-irreducible of lattice $\mathcal{S}_n$. Then

$$\gamma \updownarrow \mu \iff \gamma = Bp|l\overline{B} \quad with \left\{ \begin{array}{l} B = \big(\{x \in C : x < p\} \cup \{x \in \overline{C} : x < l\}, < \big) \\ \overline{B} = \big(\{x \in C : p < x\} \cup \{x \in \overline{C} : l < x\}, < \big) \end{array} \right.$$

Corollary 1 *The relation $\updownarrow$ induces a bijection between the sets of join-irreducibles and meet-irreducibles of $\mathcal{S}_n$ (semidistributivity of $\mathcal{S}_n$).*

Definition 6 *In a semidistributive lattice, the meet-irreducible image of the join-irreducible element j in the bijection $\updownarrow$ will be denoted by m_j and conversely, the join-irreducible image of the meet-irreducible element m will be denoted by j_m.*

We shall now prove a strong property of the arrow relations in the lattice of permutations, that will be used to characterize all interval doubling schemes of $\mathcal{S}_n$.

Proposition 3 *Let γ be a join-irreducible of $\mathcal{S}_n$ and μ_γ its image in the bijection $\updownarrow$. The following assertions are satisfied:*

1. $\forall \gamma' \in J,\ (\gamma' \uparrow. \mu_\gamma \iff \gamma <_{\mathcal{S}_n} \gamma')$.
2. $\forall \mu' \in M,\ (\gamma_\mu \downarrow. \mu' \iff \mu' <_{\mathcal{S}_n} \mu)$.

Proof: To prove point 1, we set:

$$\begin{cases} \gamma = Bv|u\overline{B} \\ \mu_\gamma = Cu|v\overline{C} \\ \mu_\gamma^+ = Cvu\overline{C} \\ \gamma' = B's|r\overline{B}' \end{cases}$$

We then consider the following conditions:

$$\begin{cases} (a)\ vu \in D(\gamma') \\ (b)\ \forall x \in B,\ (u < x \Longrightarrow x \in B') \\ (c)\ \forall x \in \overline{B},\ (x < v \Longrightarrow x \in \overline{B}') \end{cases}$$

and we prove the equivalence of the three following conditions:

$$\begin{cases} (i)\ \gamma <_{\mathcal{S}_n} \gamma' \\ (ii)\ conditions\ (a),\ (b)\ and\ (c)\ are\ satisfied \\ (iii)\ \gamma' \uparrow. \mu_\gamma \end{cases}$$

$(i) \Longrightarrow (ii)$: $\gamma <_{\mathcal{S}_n} \gamma'$ implies $vu \in D(\gamma')$. Condition (a) is satisfied and implies $v \in B's$ and $u \in r\overline{B}'$. Let $x \in B$ be such that $u < x$. So we have $xu \in D(\gamma)$, hence $xu \in D(\gamma')$. Then if $x \in r\overline{B}'$, we have $ux \in A(\gamma')$, which is impossible. If $x = s$, then $v \in B's$ implies $v \leq s$, and so $x = s \in vu\overline{B}$, which contradicts $x \in B$. Finally, $x \in B'$ and condition (b) is satisfied. Now for condition (c), let us consider $x \in \overline{B}$ such that $x < v$. We have $vx \in D(\gamma)$ and so $vx \in D(\gamma')$. If $x \in B's$ then $xv \in A(\gamma')$, which is impossible. If $x = r$ then $u \in r\overline{B}'$ implies $r \leq u$, hence $x = r \in Bvu$, which contradicts $x \in \overline{B}$. Finally, $x \in \overline{B}'$ and condition (c) is satisfied.

$(ii) \Longrightarrow (iii)$: Let us suppose that all conditions (a), (b) and (c) are satisfied. We prove that this implies $(B's|r\overline{B}' =)\gamma' \uparrow \mu_\gamma (= Cu|v\overline{C})$, i.e. (Proposition 2) that we have $vu \in D(\gamma')$ and $A(\mu_\gamma^+) \subseteq A(\gamma')$. Condition (a) gives $vu \in D(\gamma')$. It remains to prove that $A(\mu_\gamma^+) \subseteq A(\gamma')$. Let us compute the set $A(\mu_\gamma^+)$ of all agreements of $\mu_\gamma^+ = Cvu\overline{C}$:

$A(\mu_\gamma^+) = \{xv :\ x \in C \text{ and } x < v\} \cup \{ux :\ x \in \overline{C} \text{ and } u < x\} \cup \{xy :\ x \in C,\ y \in \overline{C} \text{ and } x < y\}$. We now prove that $A(\mu_\gamma^+) \subseteq A(\gamma')$.

- $\{xv : x \in C \text{ and } x < v\}$: we have $u < x$ since $x \in C$ and on the other hand, $x \in C$ and $x < v$ imply (Theorem 2) $x \in B$. Thus condition (b) implies $x \in B'$. Since $v \in B's$ (by condition (a)) and $x < v$, we obtain $xv \in A(\gamma')$.
- $\{ux : x \in \overline{C} \text{ and } u < x\}$: we have $x < v$ since $x \in \overline{C}$ and on the other hand, $x \in \overline{C}$ and $u < x$ imply $x \in \overline{B}$ (Theorem 2). Thus condition (c) gives $x \in \overline{B}'$ and, since $u \in r\overline{B}'$ by condition (a), we have $ux \in A(\gamma')$.

- $\{xy : x \in C,\ y \in \overline{C} \text{ and } x < y\}$: $x \in C,\ y \in \overline{C}$ and $x < y$ imply $u < x < y < v$. So we have on one hand $x \in C$ and $x < v$ that lead to $x \in B$ (Theorem 2) and since $u < x$, we obtain $x \in B'$ according to (b). On the other hand, we have $y \in \overline{C}$ and $u < y$ that lead to $y \in \overline{B}$ and since $y < v$, we obtain $y \in \overline{B}'$ by condition (c). Finally, $x \in B',\ y \in \overline{B}'$ and $x < y$ and so $xy \in A(\gamma')$.

$(iii) \Longrightarrow (i)$: This fact trivially holds in any semidistributive lattice.

We have just given the proof of point 1. of the proposition. The proof of point 2. is dual. ■

Before we establish the characterization result for all interval doubling schemes of the Permutoedron (Corollary 2) describing all possible constructions of $\mathcal{S}_n$ by doublings of convex sets starting from the two-element lattice (details on this subject are given in [2]), we first prove the more general theorem below.

For a semidistributive lattice L and a linear order $\mathcal{L}_J$ on J, we denote by $\mathcal{L}^*_M$ the unique linear order on M such that the tableau $T = (A_L, \mathcal{L}_J, \mathcal{L}^*_M)$ has all $\updownarrow$ on the principal diagonal.

Theorem 3 *Let L be a semidistributive lattice satisfying conditions 1. and 2. of Proposition 3 and let $\mathcal{L}_J$ be a linear order on the join-irreducible elements of L. The following conditions are equivalent:*

1. *$T = (A_L, \mathcal{L}_J, \mathcal{L}^*_M)$ is an interval doubling scheme of lattice L.*
2. *$\mathcal{L}_J$ is a linear extension of $(J, \leq_L)$ and $\mathcal{L}^*_M$ is a linear extension of $(M, \geq_L)$.*

Proof: 1. $\Longrightarrow$ 2: Let $\mathcal{L}_J$ be a linear order on J. The tableau $T = (A_L, \mathcal{L}_J, \mathcal{L}^*_M)$ satisfies the alignment of the $\updownarrow$ on the principal diagonal. If $\mathcal{L}_J$ is not a linear extension of $(J, \leq_L)$, there exists $j,\ j' \in J$ such that $j\mathcal{L}_J j'$ and $j' <_L j$. But $j' <_L j$ implies $j \uparrow_{\cdot} m_{j'}$ (Proposition 3). Thus, j and j' satisfy $j\mathcal{L}_J j'$ and $j \uparrow_{\cdot} m_{j'}$, and T is not an interval doubling scheme.

Dually, we would show that a tableau $T = (A_L, \mathcal{L}_J, \mathcal{L}^*_M)$ is not an interval doubling scheme as soon as $\mathcal{L}^*_M$ is not a linear extension of $(M, \geq_L)$.

2. $\Longrightarrow$ 1: Let $\mathcal{L}_J$ be a linear extension of $(J, \leq_L)$. Let us assume that the tableau $T = (A_L, \mathcal{L}_J, \mathcal{L}^*_M)$ is not an interval doubling scheme. So there exists $j,\ j' \in J$ such that $j\mathcal{L}_J j'$ and ($j \uparrow_{\cdot} m_{j'}$ or $j' \downarrow_{\cdot} m_j$). In this case, $\mathcal{L}^*_M$ is not a linear extension of $(M, \geq_L)$. Indeed, $j\mathcal{L}_J j'$ implies $j' \not<_L j$ by hypothesis. But this last condition implies $j \not\uparrow_{\cdot} m_{j'}$ (Proposition 3). So we have $j' \downarrow_{\cdot} m_j$, which, according to the same proposition, implies $m_j <_L m_{j'}$ and, since $j\mathcal{L}_J j'$ implies $m_j\mathcal{L}^*_M m_{j'}$, $\mathcal{L}^*_M$ is not a linear extension of $(M, \geq_L)$. ■

The announced result is now a direct corollary of Proposition 3 and Theorem 3.

Corollary 2 (Characterization) *Let $\mathcal{L}_J$ be a linear order on the join-irreducible permutations of the Permutoedron. The following conditions are equivalent:*

1. *$T = (A_{\mathcal{S}_n}, \mathcal{L}_J, \mathcal{L}^*_M)$ is an interval doubling scheme of lattice $\mathcal{S}_n$.*
2. *$\mathcal{L}_J$ is a linear extension of $(J, \leq_{\mathcal{S}_n})$ and $\mathcal{L}^*_M$ is a linear extension of $(M, \geq_{\mathcal{S}_n})$.*

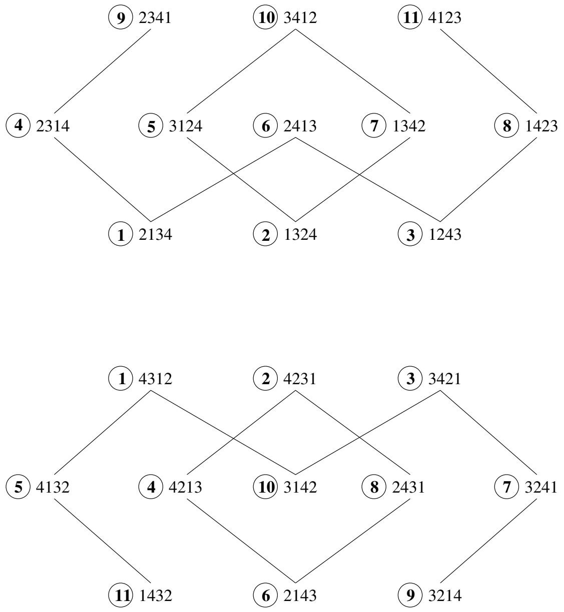


Fig. 3: A linear extension $\mathcal{L}_J$ of $(J, \leq_{\mathcal{S}_4})$ for which the linear order $\mathcal{L}^*_M$ on M is not a linear extension of $(M, \geq_{\mathcal{S}_4})$.

Figure 3 shows that for a given linear extension $\mathcal{L}_J$ of the join-irreducibles of lattice $\mathcal{S}_4$, the linear order $\mathcal{L}^*_M$ on the meet-irreducibles is not necessarily a linear extension of $(M, \geq_{\mathcal{S}_n})$ and so the theorem is not trivial. Indeed, if we numerate the join-irreducible permutations from 1 to 11 as shown on the figure, a linear extension of $(J, \leq_{\mathcal{S}_4})$ is then: $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < 11$, whereas the corresponding linear order $\mathcal{L}^*_M$ on M is not a linear extension of $(M, \geq_{\mathcal{S}_n})$ (observe permutations 6 and 8).

4 Conclusion

We have recalled that all interval doubling schemes of a bounded lattice L are in bijection with all different ways to construct L starting from the two-element lattice by doublings of convex sets. The result of characterization of all interval doubling schemes of the Permutoedron appears therefore like a significant step towards a better understanding of the constructive properties of this lattice. Moreover, in [5], the author has conjectured that all (finite) *Coxeter* lattices (containing the infinite family of lattices $(\mathcal{S}_n)_{n \geq 2}$) could themselves be bounded lattices. In this case, the question would arise to know if there exists a similar characterization result about the interval doubling schemes of these lattices.

At last, we have proved Proposition 3 in a particular class of lattices (semidistributive lattices satisfying both conditions of Proposition 3) and we are interested to determine if this class is maximal for this property on the arrow relations.

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